| k | 1 4 |
|---|---|
| P$(X = k)$ | $\frac{1}{3}$ $\frac{2}{3}$ |

$$\begin{aligned}
E(X) &= (1)\frac{1}{3} + (4)\frac{2}{3} = 3. \\
E(X^2) &= (1^2)\frac{1}{3} + (4^2)\frac{2}{3} = 11. \\
Var(X) &= E(X^2) - (E(X))^2 = 11 - 9 = 2.
\end{aligned}$$

26. **QUESTION**:
For two standard dice all 36 outcomes of a throw are equally likely. Find $P(X_1 + X_2 = j)$ for all $j$ and calculate $E(X_1 + X_2)$. Confirm that $E(X_1) + E(X_2) = E(X_1 + X_2)$.

**SOLUTION**:
The possible totals are $j = 2, 3, \ldots, 12$ and $P(X_1 + X_2 = j) = (j - 1)/36$, $j = 2, \ldots, 7$ and $P(X_1 + X_2 = j) = (13 - j)/36$, $j = 8, \ldots, 12$. For each of the dice $E(X_i) = 21/6 = 7/2$ while for the total

$$E(X_1 + X_2) = \tfrac{1}{36}(2 \cdot 1 + 3 \cdot 2 + \cdots + 7 \cdot 6 + \cdots + 12 \cdot 1) = 252/36 = 7$$

27. **QUESTION**:
$X$ takes values 1, 2, 3, 4 each with probability 1/4 and $Y$ takes values 1, 2, 4, 8 with probabilities 1/2, 1/4, 1/8 and 1/8 respectively. Write out a table of probabilities for the 16 paired outcomes which is consistent with the distributions of $X$ and $Y$. From this find the possible values and matching probabilities for the total $X + Y$ and confirm that $E(X + Y) = E(X) + E(Y)$.

**SOLUTION**:
There are 16 pairs and infinitely many ways to allocate the probabilities. Selecting one, say $p(1, 4) = p(1, 8) = 1/8$, $p(2, 2) = 1/4$, $p(3, 1) = p(4, 1) = 1/4$ we see this satisfies $\sum_j p(i, j) = P(X = i)$ and $\sum_i (i, j) = P(Y = j)$. The possible values and probabilities are

| $t$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_t$ | 0 | 0 | 1/2 | 3/8 | 0 | 0 | 0 | 1/8 | 0 | 0 | 0 |

where for instance $P(X + Y = 4) = p(2, 2) + p(3, 1) = 1/2$. From the table, $E(X + Y) = (4 \cdot 4 + 5 \cdot 3 + 9 \cdot 1)/8 = 5$. As $E(X) = 5/2$ and $E(Y) = 5/2$ the required equality holds.

28. **QUESTION**:
Calculation practice for the binomial distribution. Find $P(X = 2)$, $P(X < 2)$, $P(X > 2)$ when

(a) $n = 4$, $p = 0.2$; (b) $n = 8$, $p = 0.1$;
(c) $n = 16$, $p = 0.05$; (d) $n = 64$, $p = 0.0125$.

**SOLUTION**:
(a) $P(X = 2) = 6 \cdot 0.2^2 \cdot 0.8^2 = 0.1536$, $P(X < 2) = P(X = 0) + P(X = 1) = 0.4096 + 0.4096 = 0.8192$, $P(X > 2) = 1 - P(X \leq 2) = 1 - 0.8192 - 0.1536 = 0.0272$.
(b) $P(X = 2) = 0.1488$, $P(X < 2) = 0.8131$, $P(X > 2) = 0.0381$.
(c) $P(X = 2) = 0.1463$, $P(X < 2) = 0.8108$, $P(X > 2) = 0.0429$.
(d) $P(X = 2) = 0.1444$, $P(X < 2) = 0.8093$, $P(X > 2) = 0.0463$.

29. **QUESTION**:
A wholesaler supplies products to 10 retail stores, each of which will independently make an order on a given day with chance 0.35. What is the probability of getting exactly 2 orders? Find the most probable number of orders per day and the probability of this number of orders. Find the expected number of orders per day.

**SOLUTION**:
Using the independence of orders the chance that only the first two stores place orders is $0.35^2 \cdot 0.65^8$. As there are $10 \times 9/2 = 45$ distinct pairs of stores that could order we have

$$P(X = 2) = 450.35^2 0.65^8 = 0.1757$$

A similar argument works for any number of orders. We say that the number of orders placed has the $\text{Bin}(10, 0.35)$ distribution. The formula for $x$ orders is

$$P(X = x) = \binom{10}{x} 0.35^x 0.65^{10-x}$$

The most probable number of orders is 3 (either calculate $P(X = x)$ for a few different $x$ values or look at binomial tables in a textbook) and $P(X = 3) = 120(0.35)^3(0.65)^7 \approx 0.2522$. The expected number of orders is

$$\sum_0^{10} x \cdot P(X = x) = 1 \cdot 0.0725 + 2 \cdot 0.1757 + 3 \cdot 0.2522 + 4 \cdot 0.2377 + \cdots$$

which (barring numericals errors) will give the same answer as the formula $E(X) = np = 10 \times 0.35 = 3.5$.

(The problem of which number is most likely for general $n$ and $p$ was not set but is not all that hard – show that $P(X = x + 1) < P(X = x) \Leftrightarrow x + 1 > (n + 1)p$ and think about what that means)

30. **QUESTION**:
A machine produces items of which 1% at random are defective. How many items can be packed in a box while keeping the chance of one or more defectives in the box to be no more than 0.5? What are the expected value and standard deviation of the number of defectives in a box of that size?

**SOLUTION**:
Let $X$ be the number of defectives when $n$ items are packed into a box. $P(X = 0) = 0.99^n$ so that $P(X \geq 1) = 1 - 0.99^n$. To ensure $1 - 0.99^n < 0.5$ we must take $n < \log 0.5 / \log 0.99 = 68.97$ so $n = 68$. The expected value and standard deviation of $X$ when $n = 68$ are 0.68 and $\sqrt{0.68 \times 0.99} = 0.8205$.

31. **QUESTION**:
Suppose that 0.3% of bolts made by a machine are defective, the defectives occurring at random during production. If the bolts are packaged in boxes of 100, what is the Poisson approximation that a given box will contain $x$ defectives? Suppose you buy 8 boxes of bolts. What is the distribution of the number of boxes with no defective bolts? What is the expected number of boxes with no defective bolts?

**SOLUTION**:
$D$ is $\text{Bin}(100, 0.003)$ which is approximately Poisson with parameter $100 \times 0.003 = 0.3$. Hence $P(D = x) \approx e^{-0.3}(0.3)^x / x!\,,\ x = 0, 1, \ldots$. In particular, $P(D = 0) \approx 0.7408$. Finally, $N$, the no. of boxes with no defectives is $\text{Bin}(8, 0.7408)$ and so $E(N) = 8 \times 0.7408 = 5.926$.

32. **QUESTION**:
Events which occur randomly at rate $r$ are counted over a time period of length $s$ so the event count $X$ is Poisson. Find $P(X = 2)$, $P(X < 2)$ and $P(X > 2)$ when

(a) $r = 0.8$, $s = 1$; (b) $r = 0.1$, $s = 8$; (c) $r = 0.01$, $s = 200$; (d) $r = 0.05$, $s = 200$.

**SOLUTION**:
(a) and (b) $\lambda = rs = 0.8$ so that $P(X = 2) = e^{-0.8}0.8^2/2 = 0.1438$, $P(X < 2) = 0.4493 + 0.3695 = 0.8088$ and $P(X > 2) = 1 - 0.8088 - 0.1438 = 0.0474$. (c) $\lambda = 2$ so that $P(X = 2) = 0.2707$, $P(X < 2) = 0.4060$, $P(X > 2) = 0.3233$. (d) $\lambda = 10$ so that $P(X = 2) = 0.00227$, $P(X < 2) = 0.00050$, $P(X > 2) = 0.9972$.

33. **QUESTION**:
Given that 0.04% of vehicles break down when driving through a certain tunnel find the probability of (a) no (b) at least two breakdowns in an hour when 2,000 vehicles enter the tunnel.

**SOLUTION**:
The number of breakdowns $X$ has a binomial distribution which can be approximated by the $\text{Pn}(\lambda)$ distribution with $\lambda = 2000 \times 0.0004 = 0.8$. Hence $P(X = 0) \approx 0.4493$ and $P(X \geq 2) = 1 - P(X \leq 1) \approx 1 - 0.8088 = 0.1912$.

34. **QUESTION**:
Experiments by Rutherford and Geiger in 1910 showed that the number of alpha particles emitted per unit time in a radioactive process is a random variable having a Poisson distribution. Let $X$ denote the count over one second and suppose it has mean 5. What is the probability of observing fewer than two particles during any given second? What is the $P(X \geq 10)$? Let $Y$ denote the count over a separate period of 1.5 seconds. What is $P(Y \geq 10)$? What is $P(X + Y \geq 10)$?

**SOLUTION**:
$P(X \leq 1) = e^{-5}(1 + 5) = 0.0404$. $P(X \geq 10) = 0.0398$. $Y$ is $\text{Pn}(7.5)$ and so $P(Y \geq 10) = 1 - P(Y \leq 9) = 1 - \sum_0^9 P(Y = i) = 0.5113$. $X + Y$ is $\text{Pn}(12.5)$ and we find $P(X + Y \geq 10) = 0.7986$.

35. **QUESTION**:
A process for putting chocolate chips into cookies is random and the number of choc chips in a cookie has a Poisson distribution with mean $\lambda$. Find an expression for the probability that a cookie contains less than 3 choc chips.

**SOLUTION**:
The Poisson distribution gives probabilities for each possible number of choc chips but as a cookie can't contain two different numbers simultaneously we *add* the probabilities for the possible values 0, 1 and 2. Hence

$$P(X < 3) = e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right)$$

(if you wonder what happened to $s$, it equals 1 as we're only looking at a single cookie – for 10 cookies we take $s = 10$ etc).

36. **QUESTION**:
Let $X$ have the density $f(x) = 2x$ if $0 \leq x \leq 1$ and $f(x) = 0$ otherwise. Show that $X$ has the mean 2/3 and the variance 1/18. Find the mean and the variance of the random variable $Y = -2X + 3$.

**SOLUTION**:
To find expected values for continuous random variables we integrate e.g.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x \cdot 2x\, dx = 2/3$$

and similarly

$$\text{Var}(X) = \int_0^1 (x - 2/3)^2 \cdot 2x\, dx = 1/2 - 8/9 + 4/9 = 1/18$$

You could also use the formula $\text{Var}(X) = E(X^2) - E(X)^2$ where

$$E(X^2) = \int_0^1 x^2 \cdot 2x\, dx = 1/2$$

so that $\text{Var}(X) = 1/2 - (2/3)^2 = 1/18$. Use the linearity of expectation for the last bits to get

$$E(Y) = -2 \times 2/3 + 3 = 5/3 \quad \text{and} \quad \text{Var}(Y) = (-2)^2 \times 1/18 = 2/9$$

37. **QUESTION**:
Let the random variable $X$ have the density $f(x) = kx$ if $0 \leq x \leq 3$. Find $k$. Find $x_1$ and $x_2$ such that $P(X \leq x_1) = 0.1$ and $P(X \leq x_2) = 0.95$. Find $P(|X - 1.8| < 0.6)$.

**SOLUTION**:
To make $\int_0^3 f(x)dx = 1$ we must take $k = 2/9$. For any $c$ in $(0, 3)$, $P(X \leq c) = c^2/9$ so $x_1 = \sqrt{0.9} = 0.9487$ and $x_2 = \sqrt{8.55} = 2.9240$. $P(|X - 1.8| < 0.6) = P(1.2 < X < 2.4) = (2.4^2 - 1.2^2)/9 = 0.48$.

38. **QUESTION**:
A small petrol station is supplied with petrol once a week. Assume that its volume $X$ of potential sales (in units of $10,000$ litres) has the probability density function $f(x) = 6(x-2)(3-x)$ for $2 \leq x \leq 3$ and $f(x) = 0$ otherwise. Determine the mean and the variance of this distribution. What capacity must the tank have for the probability that the tank will be emptied in a given week to be 5%?

**SOLUTION**:
Proceed as usual, $E(X) = 5/2$ and $\mathrm{Var}(X) = 1/20$. Let $T$ denote the capacity of the tank. We need to solve $0.05 = \int_T^3 6(x-2)(3-x)dx = \int_{T-2}^1 6y(1-y)\,dy = 1 - 3(T-2)^2 + 2(T-2)^3$ and doing this numerically (iterate the equation $T - 2 = \sqrt{0.95}(7 - 2T)^{-1/2}$) we find $T \approx 2.86465$ i.e. the tank should hold approximately 28,650 litres.

39. **QUESTION**:
Find the probability that none of the three bulbs in a set of traffic lights will have to be replaced during the first 1200 hours of operation if the lifetime $X$ of a bulb (in thousands of hours) is a random variable with probability density function $f(x) = 6[0.25 - (x - 1.5)^2]$ when $1 \leq x \leq 2$ and $f(x) = 0$ otherwise. You should assume that the lifetimes of different bulbs are independent.

**SOLUTION**:
For a single bulb, $P(X > 1.2) = 6[\frac{3}{2}x^2 - \frac{1}{3}x^3 - 2x]_{1.2}^2 = 0.8960$. Hence $P(\text{no bulbs replaced}) = 0.8960^3 = 0.7193$.

40. **QUESTION**:
Suppose $X$ is $\mathrm{N}(10, 1)$. Find (i) $P[X > 10.5]$, (ii) $P[9.5 < X < 11]$, (iii) $x$ such that $P[X < x] = 0.95$. *You will need to use Standard Normal tables.*

**SOLUTION**:
*Let $Z$ denote a $N(0, 1)$ random variable from now on and let $\Phi$ denote its cdf.*

(i) $P[X > 10.5] = P[X - 10 > 0.5] = 1 - \Phi(0.5) = 0.3085$; (ii) $P[9.5 < X < 11] = \Phi(1) - \Phi(-0.5) = 0.5328$; (iii) $P[X < x] = P[Z < x - 10] = 0.95$ when $x - 10 = 1.645$ i.e. $x = 11.645$.

41. **QUESTION**:
Suppose $X$ is $\mathrm{N}(-1, 4)$. Find
(a) $P(X < 0)$; (b) $P(X > 1)$; (c) $P(-2 < X < 3)$; (d) $P(|X + 1| < 1)$.

**SOLUTION**:
As $X = 2Z + 1$ we have (a) $P(X < 0) = P(2Z + 1 < 0) = \Phi(-1/2) = 1 - \Phi(1/2) = 0.3085$; (b) $P(X > 1) = \Phi(0) = 1/2$; (c) $P(-2 < X < 3) = \Phi(1) - \Phi(-3/2) = 0.7745$; (d) $P(|X + 1| < 1) = P(-2 < X < 0) = \Phi(-1/2) - \Phi(-3/2) = 0.2417$.

42. **QUESTION**:
Suppose $X$ is $\mathrm{N}(\mu, \sigma^2)$. For $a = 1, 2, 3$ find $P(|X - \mu| < a\sigma)$.

**SOLUTION**:
$P(|X - \mu| < a\sigma) = P(-a < Z < a) = 2\Phi(a) - 1$ so the required values are approximately 0.682, 0.954 and 0.998 respectively.

43. **QUESTION**:
The height of a randomly selected man from a population is normal with $\mu = 178$cm and $\sigma = 8$cm. What proportion of men from this population are over 185cm tall? There are 2.54cm to an inch. What is their height distribution in inches? The heights of the women in this population are normal with $\mu = 165$ cm and $\sigma = 7$cm. What proportion of the women are taller than half of the men?

**SOLUTION**:
Let $M$ denote the height of a man and $W$ the height of a woman in centimetres. We want to know $P(M > 185) = P(\frac{M-178}{8} > \frac{185-178}{8}) = P(Z > 0.875) \approx 0.19$. Let $H$ denote the height of a man in inches. Then $H = M/2.54$ so that $H$ is $N(70.1\,,(3.15)^2\,)$. Finally $P(M < h) = 0.5$ when $h = 178$ and so $P(W > 178) = P(Z > \frac{13}{7}) = 0.032$ so 3.2% of the women are taller than half of the men.

44. **QUESTION**:
$N$ independent trials are to be conducted, each with "success" probability $p$. Let $X_i = 1$ if trial $i$ is a success and $X_i = 0$ if it is not. What is the distribution of the random variable $X = X_1+X_2+\ldots+X_N$? Express $P[a \leq X \leq b]$ as a sum (where $a \leq b$ and these are integers between 0 and $N$). Use the central limit theorem to provide an approximation to this probability. Compare your approximation with the limit theorem of De Moivre and Laplace on p1189 of Kreyszig.

**SOLUTION**:
As $X$ is the total number of successes in $N$ independent trials $X$ is $\mathrm{Bin}(N, p)$. Thus

$$P[a \leq X \leq b] = \sum_{x=a}^{b} \binom{N}{x} p^x (1-p)^{N-x}.$$

As each $X_i$ has $E(X_i) = p$ and $\mathrm{Var}(X_i) = p(1-p)$ *(you should confirm this)* the central limit theorem says that $X$ is approximately $\mathrm{Normal}\big(Np, Np(1-p)\big)$. Let $\bar{a} = (a - Np)/\sqrt{Np(1-p)}$ and $\bar{b} = (b - Np)/\sqrt{Np(1-p)}$. The approximation is $P[a \leq X \leq b] \approx P[\bar{a} < Z < \bar{b}]$ where $Z$ has the standard Normal distribution. The reason for the small correction in the version of this result in Kreyszig is that $P[a \leq X \leq b] = P[a - \delta < X < b + \delta]$ for any $\delta \in (0, 1)$ while the approximation varies with $\delta$ – the choice $\delta = 0.5$ is arbitrary but generally sensible.

45. **QUESTION**:
Suppose that of $1,000,000$ live births in Paris over some period, $508,000$ are boys. Suppose $X$ is $\mathrm{Bin}(10^6, 0.5)$ and calculate approximately $P[X \geq 508,000]$. Does it seem reasonable to you that the proportion of males among Parisian babies conceived soon after the above period will be 50%. (Laplace developed his limit theorem in the late 1700's to deal with a question similar to this.)

**SOLUTION**:
From q17, $\bar{b} = 8,000/500 = 16$ and from the formulæ sheet $1 - \Phi(16) \approx 6.4 \times 10^{-58}$. It seems entirely unreasonable that births at that time and place should be modelled with $p = 0.5$ chance of each sex. Standard practice would be to say that the proportion is within some small distance of $0.508(= 508,000/10^6\,)$.

46. **QUESTION**:
An airfreight company has various classes of freight. In one of these classes the average weight of packages is 10kg and the variance of the weight distribution is $9\text{kg}^2$. Assuming that the package weights are independent (it is not the case that a single company is sending a large number of identical packages, for instance), estimate the probability that 100 packages will have total weight more than 1020kg.

**SOLUTION**:
The central limit theorem says $\sum W_i$ is approximately $N(1,000\,,30^2\,)$ so that $P(\sum W_i > 1,020) \approx P\big[Z > (1,020 - 1,000)/30 = 0.67\big] = 0.251$.