Single Maths B Probability: Exercises

1. Describe the sample space and all 16 events for a trial in which two coins are thrown and each shows either a *head* or a *tail*.

2. A fair coin is tossed, and a fair die is thrown. Write down sample spaces for
   (a) the toss of the coin;
   (b) the throw of the die;
   (c) the combination of these experiments.

   Let A be the event that a head is tossed, and B be the event that an odd number is thrown. Directly from the sample space, calculate $\mathrm{P}[A \cap B]$ and $\mathrm{P}[A \cup B]$.

3. Let A, B and C be any three events. Draw Venn diagrams to deduce that
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
   (b) $(A \cap B)^c = A^c \cup B^c$;
   (c) $(A \cup B)^c = A^c \cap B^c$.

4. A bag contains fifteen balls distinguishable only by their colours; ten are blue and five are red. I reach into the bag with both hands and pull out two balls (one with each hand) and record their colours.
   (a) What is the *random phenomenon*?
   (b) What is the *sample space*?
   (c) Express the *event* that the ball in my left hand is red as a subset of the sample space.

5. M&M sweets are of varying colours and the different colours occur in different proportions. The table below gives the probability that a randomly chosen M&M has each colour, but the value for tan candies is missing.

Colour	Brown	Red	Yellow	Green	Orange	Tan
Probability	0.3	0.2	0.2	0.1	0.1	?

   (a) What value must the missing probability be?
   (b) You draw an M&M at random from a packet. What is the probability of each of the following events?
      i. You get a brown one or a red one.
      ii. You don't get a yellow one.
      iii. You don't get either an orange one or a tan one.
      iv. You get one that is brown or red or yellow or green or orange or tan.

6. You consult Joe the bookie as to the form in the 2.30 at Ayr. He tells you that, of 16 runners, the favourite has probability 0.3 of winning, two other horses each have probability 0.20 of winning, and the remainder each have probability 0.05 of winning, excepting Desert Pansy, which has a worse than no chance of winning. What do you think of Joe's advice?

7. Not all dice are fair. In order to describe an unfair die properly, we must specify the probability for each of the six possible outcomes. The following table gives answers for each of 4 different dice.

	Probabilities			
Outcome	Die 1	Die 2	Die 3	Die 4
1	1/3	1/6	1/7	1/3
2	0	1/6	1/7	1/3
3	1/6	1/6	1/7	-1/6
4	0	1/6	1/7	-1/6
5	1/6	1/6	1/7	1/3
6	1/3	1/7	2/7	1/3

Which of the four dice have validly specified probabilities and which do not? In the case of an invalidly described die, explain why the probabilities are invalid.

8. A six-sided die has four green and two red faces and is balanced so that each face is equally likely to come up. The die will be rolled several times. You must choose one of the following three sequences of colours; you will win £25 if the first rolls of the die give the sequence that you have chosen.

R	G	R	R	R	
R	G	R	R	R	G
G	R	R	R	R	R

Without making any calculations, explain which sequence you choose. (In a psychological experiment, 63% of 260 students who had not studied probability chose the second sequence. This is evidence that our intuitive understanding of probability is not very accurate. This and other similar experiments are reported by A. Tversky and D. Kahneman, *Extensional versus intuitive reasoning: The conjunction fallacy in probability judgment*, Psychological Review **90** (1983), pp. 293–315.)

9. Two sisters maintain that they can communicate telepathically. To test this assertion, you place the sisters in separate rooms and show sister A a series of cards. Each card is equally likely to depict either a circle or a star or a square. For each card presented to sister A, sister B writes down 'circle', or 'star' or 'square', depending on what she believes sister A to be looking at. If ten cards are shown, what is the probability that sister B correctly matches at least one?

10. An examination consists of multiple-choice questions, each having five possible answers. Suppose you are a student taking the exam. and that you reckon you have probability 0.75 of knowing the answer to any question that may be asked and that, if you do not know, you intend to guess an answer with probability 1/5 of being correct. What is the probability you will give the correct answer to a question?

11. Consider the following experiment. You draw a square, of width 1 foot, on the floor. Inside the square, you inscribe a circle of diameter 1 foot. The circle will just fit inside the square.
You then throw a dart at the square in such a way that it is equally likely to fall on any point of the square.

    What is the probability that the dart falls inside the circle? (Think about area!)
How might this process be used to estimate the value of π?

12. I have in my pocket ten coins. Nine of them are ordinary coins with equal chances of coming up head and tail when tossed and the tenth has two heads.
    (a) If I take one of the coins at random from my pocket, what is the probability that it is the coin with two heads ?
    (b) If I toss the coin and it comes up heads, what is the probability that it is the coin with two heads ?
    (c) If I toss the coin one further time and it comes up tails, what is the probability that it is one of the nine ordinary coins ?

13. A certain person considers that he can drink and drive: usually he believes he has a negligible chance of being involved in an accident, whereas he believes that if he drinks two pints of beer, his chance of being involved in an accident on the way home is only one in five hundred. Assuming that he drives home from the same pub every night, having drunk two pints of beer, what is the chance that he is involved in at least one accident in one year? Are there any assumptions that you make in answering the question?

14. Two events A and B are such that $\mathrm{P}[A] = 0.5$, $\mathrm{P}[B] = 0.3$ and $\mathrm{P}[A \cap B] = 0.1$. Calculate
    (a) $\mathrm{P}[A|B]$;
    (b) $\mathrm{P}[B|A]$;
    (c) $\mathrm{P}[A|A \cup B]$;
    (d) $\mathrm{P}[A|A \cap B]$;
    (e) $\mathrm{P}[A \cap B|A \cup B]$.

15. Three babies are given a weekly health check at a clinic, and then returned randomly to their mothers. What is the probability that at least one baby goes to the right mother?

16. In a certain town, 30% of the people are Conservatives; 50% Socialists; and 20% Liberals. In this town at the last election, 65% of Conservatives voted, as did 82% of the Socialists and 50% of the Liberals. A person from the town is selected at random, and states that she voted at the last election. What is the probability that she is a Socialist?

17. Three prisoners, A, B, and C, are held in separate cells. Two are to be executed. The warder knows specifically who is to be executed, and who is to be freed, whereas the prisoners know only that two are to be executed. Prisoner A reasons as follows: my probability of being freed is clearly $\frac{1}{3}$ until I receive further information. However, it is clear that at least one of B and C will be executed, so I will ask the warder to name one prisoner *other than myself* who is to be executed. Once I know which of B and C is to be executed, either I will go free or the other, unnamed, prisoner will go free, with equal probability. Hence, by asking the name of another prisoner to be executed, I raise my chances of survival from $\frac{1}{3}$ to $\frac{1}{2}$. Investigate A's reasoning. [Hint: find the conditional probability that A is freed, given that the warder names B to be executed.]

18. You're playing duplicate bridge. Your partner has bid two spades, and you have to decide whether to pass or to bid game in spades, namely to bid four spades. You reckon that there is a good chance, 40%, that four spades will make. Otherwise, you think three spades will make about 40% of time, and two spades the rest of the time. Suppose there are no doubles (by the opposition, for penalties). The gains and losses depend on whether you are *vulnerable* or not. The possible outcomes and scores are as follows:

	Not vulnerable Score if you make			Vulnerable Score if you make		
You bid	2 spades	3 spades	4 spades	2 spades	3 spades	4 spades
2 spades	110	140	170	110	140	170
3 spades	-50	140	170	-100	140	170
4 spades	-100	-50	420	-200	-100	620

What should you bid when not vulnerable? What should you bid when vulnerable? Calculate the variation in score for one of the bids.

19. Tay-Sachs disease is a rare fatal genetic disease occurring chiefly in children, especially of Jewish or Slavic extraction. Suppose that we limit ourselves to families which have (a) exactly three children, and (b) which have both parents carrying the Tay-Sachs disease. For such parents, each child has independent probability $\frac{1}{4}$ of getting the disease.
Write X to be the random variable representing the number of children that will have the disease.

    (a) Show **(without using any knowledge you might have about the binomial distribution!)** that the probability distribution for X is as follows:

k	0	1	2	3
$\mathrm{P}[X=k]$	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

    (b) Show that $\mathrm{E}[X] = \frac{3}{4}$ and that $\mathrm{Var}[X] = \frac{9}{16}$.

20. A six-sided die has four green and two red faces and is balanced so that each face is equally likely to come up. The die will be rolled several times. Suppose that we score 4 if the die is rolled and comes up green, and 1 if it comes up red. Define the random variable X to be this score. Write down the distribution of probability for X and calculate the expectation and variance for X.

21. For two standard dice all 36 outcomes of a throw are equally likely. Find $P(X_1 + X_2 = j)$ for all j and calculate $E(X_1 + X_2)$. Confirm that $E(X_1) + E(X_2) = E(X_1 + X_2)$.

22. X takes values 1, 2, 3, 4 each with probability 1/4 and Y takes values 1, 2, 4, 8 with probabilities 1/2, 1/4, 1/8 and 1/8 respectively. Write out a table of probabilities for the 16 paired outcomes which is consistent with the distributions of X and Y. From this find the possible values and matching probabilities for the total $X + Y$ and confirm that $E(X + Y) = E(X) + E(Y)$.

23. Let X have the density $f(x) = 2x$ if $0 \le x \le 1$ and $f(x) = 0$ otherwise. Show that X has the mean $2/3$ and the variance $1/18$. Find the mean and the variance of the random variable $Y = -2X + 3$.

24. Let the random variable X have the density $f(x) = kx$ if $0 \le x \le 3$. Find k. Find x_1 and x_2 such that $P(X \le x_1) = 0.1$ and $P(X \le x_2) = 0.95$. Find $P(|X - 1.8| < 0.6)$.

25. A small petrol station is supplied with petrol once a week. Assume that its volume X of potential sales (in units of $10,000$ litres) has the probability density function $f(x) = 6(x-2)(3-x)$ for $2 \le x \le 3$ and $f(x) = 0$ otherwise. Determine the mean and the variance of this distribution. What capacity must the tank have for the probability that the tank will be emptied in a given week to be 5%?

26. Find the probability that none of the three bulbs in a set of traffic lights will have to be replaced during the first 1200 hours of operation if the lifetime X of a bulb (in thousands of hours) is a random variable with probability density function $f(x) = 6[0.25 - (x - 1.5)^2]$ when $1 \le x \le 2$ and $f(x) = 0$ otherwise. You should assume that the lifetimes of different bulbs are independent.

27. Calculation practice for the binomial distribution. Find $P(X = 2)$, $P(X < 2)$, $P(X > 2)$ when

    (a) $n = 4$, $p = 0.2$; (b) $n = 8$, $p = 0.1$;
    (c) $n = 16$, $p = 0.05$; (d) $n = 64$, $p = 0.0125$.

28. A wholesaler supplies products to 10 retail stores, each of which will independently make an order on a given day with chance 0.35. What is the probability of getting exactly 2 orders? Find the most probable number of orders per day and the probability of this number of orders. Find the expected number of orders per day.

29. A machine produces items of which 1% at random are defective. How many items can be packed in a box while keeping the chance of one or more defectives in the box to be no more than 0.5? What are the expected value and standard deviation of the number of defectives in a box of that size?

30. Suppose that 0.3% of bolts made by a machine are defective, the defectives occurring at random during production. If the bolts are packaged in boxes of 100, what is the Poisson approximation that a given box will contain x defectives? Suppose you buy 8 boxes of bolts. What is the distribution of the number of boxes with no defective bolts? What is the expected number of boxes with no defective bolts?

31. Events which occur randomly at rate r are counted over a time period of length s so the event count X is Poisson. Find $P(X = 2)$, $P(X < 2)$ and $P(X > 2)$ when

    (a) $r = 0.8$, $s = 1$; (b) $r = 0.1$, $s = 8$; (c) $r = 0.01$, $s = 200$; (d) $r = 0.05$, $s = 200$.

32. Given that 0.04% of vehicles break down when driving through a certain tunnel find the probability of (a) no (b) at least two breakdowns in an hour when 2,000 vehicles enter the tunnel.

33. Experiments by Rutherford and Geiger in 1910 showed that the number of alpha particles emitted per unit time in a radioactive process is a random variable having a Poisson distribution. Let X denote the count over one second and suppose it has mean 5. What is the probability of observing fewer than two particles during any given second? What is the $P(X \ge 10)$? Let Y denote the count over a separate period of 1.5 seconds. What is $P(Y \ge 10)$? What is $P(X + Y \ge 10)$?

34. A process for putting chocolate chips into cookies is random and the number of choc chips in a cookie has a Poisson distribution with mean λ. Find an expression for the probability that a cookie contains less than 3 choc chips.

35. Suppose X is $\mathrm{N}(10, 1)$. Find (i) $P[X > 10.5]$, (ii) $P[9.5 < X < 11]$, (iii) x such that $P[X < x] = 0.95$. *You will need to use Standard Normal tables.*

36. Suppose X is $\mathrm{N}(-1, 4)$. Find

    (a) $P(X < 0)$; (b) $P(X > 1)$; (c) $P(-2 < X < 3)$; (d) $P(|X + 1| < 1)$.

37. Suppose X is $\mathrm{N}(\mu, \sigma^2)$. For $a = 1, 2, 3$ find $P(|X - \mu| < a\sigma)$.

38. The height of a randomly selected man from a population is normal with $\mu = 178$cm and $\sigma = 8$cm. What proportion of men from this population are over 185cm tall? There are 2.54cm to an inch. What is their height distribution in inches? The heights of the women in this population are normal with $\mu = 165$ cm and $\sigma = 7$cm. What proportion of the women are taller than half of the men?