Homogenization on supercritical percolation cluster

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The model

Model: percolation on the hypercubic lattice

- lattice $\mathbb{Z}^d$, $d \geq 2$.
- V set of vertices: $V := \mathbb{Z}^d$.
- E_d set of edges: $E_d := \left\{ (x, y) \ : \ x, y \in \mathbb{Z}^d, \, |x - y|_1 = 1 \right\}$.
- Fix $p \in [0, 1]$.
- $(\mathbf{a}_e)_{e \in E_d}$ be a sequence of i.i.d Bernouilli random variables s.t

$$\mathbb{P}(\mathbf{a}_e = 1) = 1 - \mathbb{P}(\mathbf{a}_e = 0) = p.$$

- We assume $p > p_c(d)$ so that

$$\mathbb{P}(\text{There exists an infinite cluster}) = 1.$$

Harmonic functions

Definition

Given a function $u : \mathcal{C}_\infty \to \mathbb{R}$, we denote by

$$\Delta_{\mathcal{C}_\infty} u(x) = \sum_{y \sim x} \mathbf{a}_{(x,y)} \left(u(y) - u(x) \right).$$

Remark: This operator is the generator of the continous random walk on the infinite percolation cluster.

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- Barlow 2003: Estimates on the transition kernel.
- Sidoravicius-Sznitman 2004: Invariance principle in $d \geq 4$
- Berger-Biskup 2007 and Mathieu-Piatnitski 2007: Invariance principle in every dimension.

Some notation

- A notation for integrability. Let $X \geq 0$ be a random variable, a constant $C > 0$ and an exponent $s > 0$, we write

$$\mathcal{X} \leq \mathcal{O}_s(C) \text{ if and only if } \mathbb{E}\left[\exp\left(\left(\frac{\mathcal{X}}{C}\right)^s\right)\right] \leq 2.$$

Homogenization on percolation cluster

Theorem

Fix $p > 2$. There exist two exponents $\alpha > 0$, $s > 0$, a constant $C < \infty$ and a random variable $\mathcal{X}$ satisfying

$$\mathcal{X} \leq \mathcal{O}_s(C)$$

such that for each $u : \mathcal{C}_\infty \to \mathbb{R}$ solution of

$$\Delta_{\mathcal{C}_\infty} u = 0$$

and each $R \geq \mathcal{X}$, there exists an harmonic function u_{hom} such that

$$\| u - u_{\mathrm{hom}} \|_{L^2(B_R)} \leq C R^{1-\alpha} \| \nabla u \|_{L^p(B_R)}.$$

Two sets of interest

For each $k \in \mathbb{N}$, we define

$$\mathcal{A}_k := \left\{ u : \mathcal{C}_\infty \to \mathbb{R} \ : \ \Delta_{\mathcal{C}_\infty} u = 0 \text{ and } \lim_{R \to \infty} \frac{1}{R^{d/2+k+1}} \left\| u \right\|_{L^2(B_R)} = 0 \right\}$$

and

$$\overline{\mathcal{A}}_k := \{\text{Harmonic polynomials of degree } k\}.$$

Regularity theory

Theorem

There exist an exponent $\alpha > 0$ and a random variable $\mathcal{X}$ satisfying

$$\mathcal{X} \leq \mathcal{O}_s(C)$$

such that

- for each $k \in \mathbb{N}$ and each $u \in \mathcal{A}_k$ there exists $p \in \overline{\mathcal{A}}_k$ such that for each $R \geq \mathcal{X}$
$$\left\| u - p \right\|_{L^2(\mathcal{C}_\infty \cap B_R)} \leq CR^{-\alpha} \left\| u \right\|_{L^2(\mathcal{C}_\infty \cap B_R)} .$$
- Conversly, for each $k \in \mathbb{N}$ and each $p \in \overline{\mathcal{A}}_k$ there exists $u \in \mathcal{A}_k$ such that for each $R \geq \mathcal{X}$
$$\left\| u - p \right\|_{L^2(\mathcal{C}_\infty \cap B_R)} \leq CR^{-\alpha} \left\| p \right\|_{L^2(\mathcal{C}_\infty \cap B_R)} .$$

Regularity theory

Corollary

- For each $k \in \mathbb{N}$,
$$\dim\left(\mathcal{A}_k\right) = \dim\left(\overline{\mathcal{A}}_k\right).$$
- With $k = 0$, we obtain the following Liouville-type theorem
$$\dim\left(\mathcal{A}_0\right) = 1 \implies \mathcal{A}_0 = \{constant\}.$$

Regularity theory

Corollary

With $k = 1$, we obtain that each $u \in \mathcal{A}_1$ can be written, for some $p \in \mathbb{R}^d$,

$$u = c + p \cdot x + \chi_p(x)$$

where the function χ_p is the corrector and satisfies, for each $R \geq \mathcal{X}$

$$\operatorname*{osc}_{B_R} \chi_p \leq CR^{1-\alpha}$$

Optimal bounds for the corrector

Theorem

There exist an exponent $s > 0$ and a constant $C < \infty$ such that for each $x, y \in \mathbb{R}^d$ and each $p \in \mathbb{R}^d$

$$\left|\chi_p(x) - \chi_p(y)\right| \mathbf{1}_{\{x,y \in \mathcal{C}_\infty\}} \leq \begin{cases} \mathcal{O}_s\left(C|p| \log^{\frac{1}{2}} |x-y|\right) & \text{if } d = 2, \\ \mathcal{O}_s\left(C|p|\right) & \text{if } d \geq 3. \end{cases}$$

A renormalization structure for the infinite cluster

Definition

We say that a cube $\square$ is good if

- There exists a unique crossing cluster in $\square$, denoted by $\mathcal{C}(\square)$.
- All open paths of size larger than $\frac{\text{size}(\square)}{10}$ is connected to $\mathcal{C}(\square)$ within $\square$.

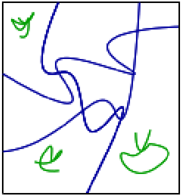

Figure 1: A good box

A notion of good cubes

Theorem (Penrose-Pisztora, 1996)

Let $\square$ be a cube in $\mathbb{Z}^d$, then there exists a constant $C := C(d,p) < \infty$,

$$\mathbb{P}(\square \text{ is a good cube}) \geq 1 - C \exp\left(-C^{-1} \operatorname{size}(\square)\right).$$

A partition of good cubes

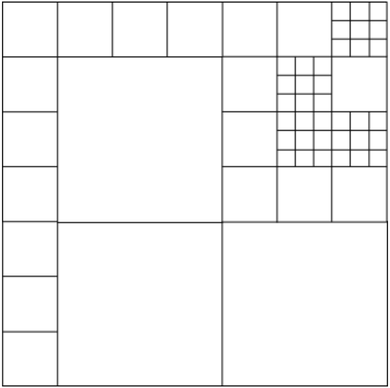

Figure 2: A partition of good cubes.

Thank you!