Nonperiodicity in homogenization problems: Toward a theory of defects

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based on a series of joint works with

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and with

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Durham, August 2018

Practical problems are not periodic.

The periodic setting is convenient theoretically and computationally inexpensive. However, only a few practical problems are periodic.

The random setting is one generalization. It is theoretically attractive (huge recent mathematical progress) but computationally expensive (if not prohibitively expensive). It is not the only possible generalization and it does not solve all issues.

$\Rightarrow$ Explore the room between the periodic setting and the random setting.

A mathematical endeavor initiated in another context

1990s-2000s: Series of works Catto/LB/Lions, and next Blanc/LB/Lions on the Thermodynamic (bulk) limit problem for atomistic systems, for classical or quantum models.

Define the energy per unit particle

$$\frac{1}{N}\sum_{1\leq i\neq j\leq N} V(X_i - X_j)$$

in the limit $N \longrightarrow +\infty$ of an infinite number of particles. Easy if the $\{X_k\}$ are assumed periodic.

Further, identify the limit of the optimal configuration $\{X_k\}$ that minimizes the energy: Crystal problem.

$\Rightarrow$ Give a rigorous meaning to models for condensed phase systems that are not necessarily periodic.

Consider a set of points $\{X_i\}_{i\in\mathbb{N}}$ such that

(H1) $\sup_{x\in\mathbb{R}^3} \#\{i \in \mathbb{N} \quad / \quad |x - X_i| < 1\} < +\infty$ (no infinite cluster)

(H2) $\exists R_0 > 0, \inf_{x\in\mathbb{R}^3} \#\{i \in \mathbb{N}, |x - X_i| < R_0\} > 0$ (no infinite hole)

(H3) the following limit exists in $L^\infty(\mathbb{R}^n)$: (approximate correlations)

$$\lim_{R\to\infty} \frac{1}{|B_R|} \#\Big\{(i_0, i_1, \dots, i_n) \in \mathbb{N}^{n+1},$$

$$|X_{i_0}| \leq \delta^{per} R, \quad |X_{i_0} - X_{i_1} - h_1| \leq \delta_1, \dots, |X_{i_0} - X_{i_n} - h_n| \leq \delta_n\Big\}.$$

Then, it is possible to define the energy of the infinite system, for many models [Blanc/LB/Lions, Comm. PDE 2003].

≈ 2000: The questions arises to know whether this generalization can be useful for homogenization theory ($a(\frac{x}{\varepsilon})$, $\varepsilon = \frac{1}{N}$)

What is the most general property that allows homogenization while keeping formulae explicit and staying deterministic?

(Recall) Homogenization 1.0.1: the periodic setting

$$-\text{div}\left[A_{\text{per}}\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right] = f \quad \text{in} \quad \mathcal{D}, \qquad u^\varepsilon = 0 \text{ on } \partial\mathcal{D},$$

with A_{per} symmetric and $\mathbb{Z}^d$-periodic: $A_{\text{per}}(x+k) = A_{\text{per}}(x)$ for any $k \in \mathbb{Z}^d$. When $\varepsilon \to 0$, u^ε converges to $u^\star$ solution to

$$-\text{div}\left[A^\star \nabla u^\star\right] = f \quad \text{in} \quad \mathcal{D}, \qquad u^\star = 0 \text{ on } \partial\mathcal{D}.$$

The effective matrix $A^\star$ is given by

$$[A^\star]_{ij} = \int_Q (e_i + \nabla w_{e_i}(y))^T A_{\text{per}}(y)\, e_j \, dy, \qquad Q = \text{unit cube} = (0,1)^d$$

with, for any $p \in \mathbb{R}^d$, w_p solves the so-called corrector problem:

$$-\text{div}\left[A_{\text{per}}(y)\,(p + \nabla w_p)\right] = 0 \quad \text{in} \quad \mathbb{R}^d, \qquad w_p \text{ is } \mathbb{Z}^d\text{-periodic}.$$

Note that $u_p(y) = p \cdot y + w_p(y)$ satisfies $\langle \nabla u_p \rangle = p$.

$\Rightarrow$ Computationally: Solve d PDEs (for $p = e_i$, $1 \leq i \leq d$) on the bounded domain Q $\rightarrow$easy!

- in the 0D case (remove differential operators):

$$-a_{\mathrm{per}}\left(\frac{x}{\varepsilon}\right) u^{\varepsilon}(x) = f(x)$$

Then $u^{\varepsilon}(x) = -f(x)\, a_{\mathrm{per}}^{-1}\left(\frac{x}{\varepsilon}\right) \rightharpoonup -f(x)\, \langle a^{-1}\rangle$, (a rescaled periodic function weakly converges to its average). Hence $u^{\varepsilon} \rightharpoonup u^{\star}$ with

$$-a^{\star}\, u^{\star}(x) = f(x) \quad \text{with} \quad a^{\star} = \langle a_{\mathrm{per}}^{-1}\rangle^{-1} \quad \text{(harmonic average)}$$

- in the 1D case: analytical expression for u^{ε}, pass to the limit,

$$-\frac{d}{dx}\left[a^{\star}\frac{du^{\star}}{dx}\right] = f \quad \text{with again} \quad a^{\star} = \langle a_{\mathrm{per}}^{-1}\rangle^{-1}$$

$\Rightarrow$ emphasizes that existence of averages is a prerequisite.

Adaptation of our general assumptions - 1

For homogenization for rescaled coefficients $a(\frac{x}{\varepsilon})$, our (H3) (approximate correlations), which also reads as $\forall n \in \mathbb{N}$, $\exists$

$$\lim_{R\to\infty} \frac{1}{|B_R|} \sum_{X_{i_0}\in B_R} \cdots \sum_{X_{i_n}\in B_R} \delta_{(X_{i_0}-X_{i_1},\dots X_{i_0}-X_{i_n})}(h_1,\dots,h_n) = l^n(h_1,\dots,h_n),$$

non-negative uniformly locally bounded measure, has to be strenghtened into [(H3')]: for any $n \in \mathbb{N}$, there exists

$$\lim_{\varepsilon\to 0} \mu^n\left(\frac{x}{\varepsilon}, h_1,\dots,h_n\right) = \nu^n(h_1,\dots h_n),$$

$$\mu^n(y, h_1,\dots,h_n) = \sum_{i_0\in\mathbb{Z}^d} \sum_{i_1\in\mathbb{Z}^d} \cdots \sum_{i_n\in\mathbb{Z}^d} \delta_{(X_{i_0}, X_{i_0}-X_{i_1},\dots X_{i_0}-X_{i_n})}(y, h_1, h_2,\dots h_n).$$

Adaptation of our general assumptions - 2

(H3') differs from (H3) in the sense it allows for averages on balls not only centered at 0 (or at a point bounded independently of the radius R of the ball) but also at all points $|x_R| = O(R)$.

(H3') allows to have a weak-* limit of all rescaled functions, which in addition is constant:

$$f\left(\frac{x}{\varepsilon}\right) \underset{\varepsilon \to 0}{\overset{*}{\rightharpoonup}} \langle f \rangle.$$

Possible adaptation for non necessarily constant weak-* limits.

Adaptation of our general assumptions - 3

With $\{X_i\}$ as above, we introduce the functions

$$f(x) = \sum_{i \in \mathbb{N}} \varphi(x - X_i), \quad \varphi \in \mathcal{D}(\mathbb{R}^3),$$

consider the closed algebra they generate, and homogenize

$$-\text{div}\left(a(\frac{x}{\varepsilon})\,\nabla u^\varepsilon\right) = f,$$

where a is a function of this algebra. Caution: Rescaling!
Homogenization holds (H-convergence). But the issue is the existence of an explicit expression for the limit ($\Rightarrow$ Computations!)
A general theory by N'Guetseng exists, but the formulae are not sufficiently explicit (averages, etc...):

$$\forall v \in \mathcal{A}, \quad \langle A(\nabla w_p + p)\nabla v\rangle = 0.$$

Examples of sets $\{X_i\}$

Compactly perturbed periodic systems: $\{X_i\}_{i\in\mathbb{N}}$ is a periodic set, except for a finite number of points. For instance, $\mathbb{Z}^3 \setminus \{0\}$. The algebra consists of periodic functions up to local perturbations.
$\Rightarrow$ Local defects

Two semi-crystals: $\{X_i\}_{i\in\mathbb{N},i_1<0}$ and $\{X_i\}_{i\in\mathbb{N},i_1>0}$ are two, different, "half" periodic sets, and the algebra consists of the sum of functions that converge to periodic functions on the left and on the right.
$\Rightarrow$ Twin-boundaries

To date, we have been unable to obtain a general theory of existence of the corrector and homogenization in the completely generic case defined above.

Difficulty : show that the corrector problem is well posed in the algebra, that is , if $a \in \mathcal{A}$ then the corrector problem

$$-\text{div}\,(a(y)\,(p + \nabla w_p)) = 0$$

is uniquely solvable for $\nabla w_p \in \mathcal{A}$ and $< \nabla w_p >= 0$.

We have succeeded, however, for the two particular cases mentioned above: local defects and twin-boundaries.

Today: local defects.

A simple case - 1

$$-\frac{d}{dx}\left((a^{per}(x/\varepsilon)) + \tilde{a}(x/\varepsilon))\ \frac{d}{dx}\, u_\varepsilon\right) = f$$

Using the corrector $w'_{per}(y) = -1 + a^*_{per}\ (a^{per})^{-1}(y)$ solution to

$$-\frac{d}{dx}\left(a^{per}(y)\ \left(1 + \frac{d}{dy}\, w_{per}(y)\right)\right) = 0$$

we have

$$\begin{aligned}\left[u'_\varepsilon - (1 + w'_{per}(./\varepsilon))\,(u^*)'\right](x) \quad &= \quad \left[(a^{per} + \tilde{a})^{-1} - (a^{per})^{-1}\right](x/\varepsilon)\ (F(x) + c_\varepsilon) \\ &\quad + (a^{per})^{-1}(x/\varepsilon)\ (c_\varepsilon - c^*)\end{aligned}$$

Consider εx instead of x (that is, micro instead of macro scale), the R.H.S does not vanish, while it does in the periodic case...

Consider now:

$$-\frac{d}{dy}\left((a^{per}+\tilde{a})(y)\left(1+\frac{d}{dy}w(y)\right)\right)=0$$

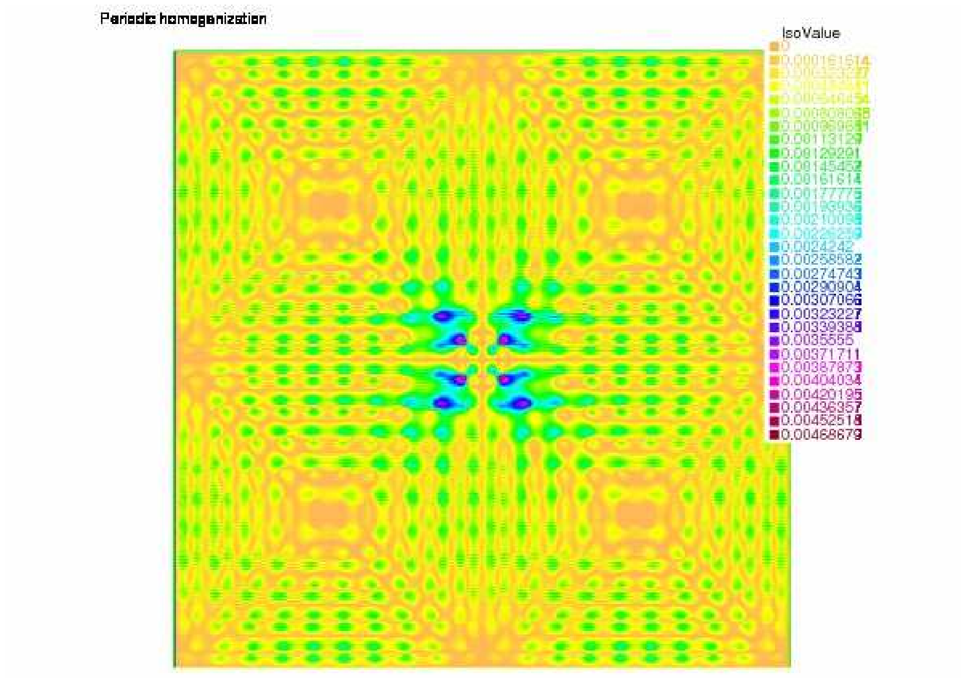
that is $w'(y)=-1+a^*_{per}\,(a^{per}+\tilde{a})^{-1}(y)$, then:

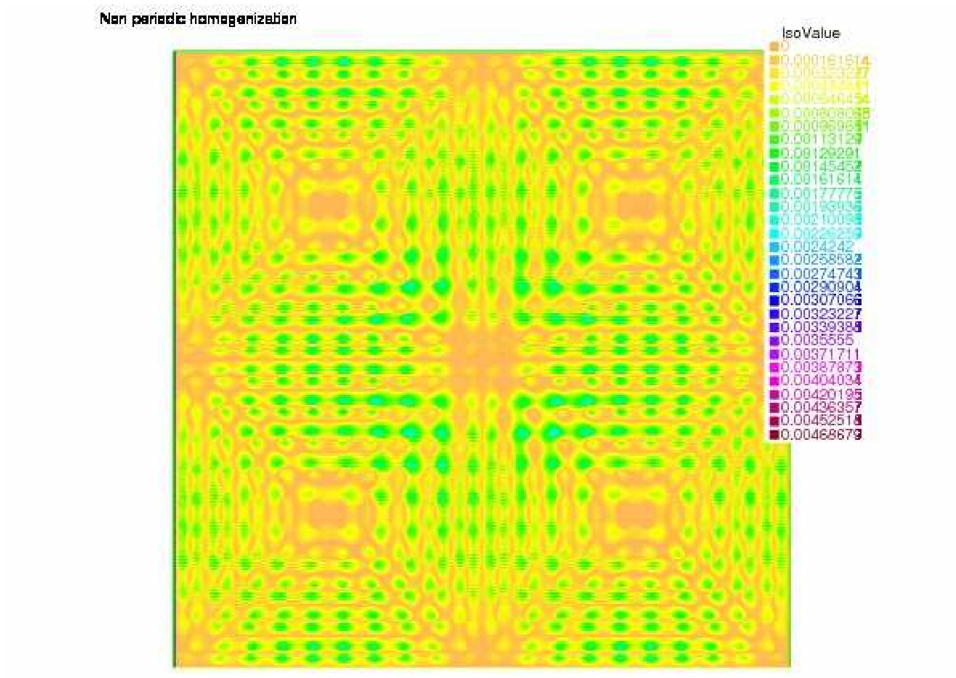
$$[u'_\varepsilon-(1+w'(./\varepsilon))\,(u^*)']\,(x)=(a^{per}+\tilde{a})^{-1}(x/\varepsilon)\;(c_\varepsilon-c^*)$$

Bingo!
The "quality" of the approximation is identical to that obtained in the perfect periodic case: one can accurately approximate u^ε close to the defects.

Goal: Obtain the existence of a corrector in higher dimensions, for suitable local defects $\tilde{a}$.

Numerical illustration





Blanc/LB/Lions, Milan Journal of Maths, 2012.

MsFEM approach

$$-\text{div}(A^\varepsilon(x)\nabla u^\varepsilon(x)) = f(x) \ \text{ in } \ \mathcal{D}, \qquad u^\varepsilon = 0 \ \text{ on } \ \partial\mathcal{D}.$$

where A^ε is not necessarily periodic.

Variational formulation: find u^ε such that

$$\forall v \in H^1_0(\mathcal{D}), \quad \mathcal{A}_\varepsilon(u^\varepsilon, v) = b(v),$$

where

$$\mathcal{A}_\varepsilon(u,v) = \int_{\mathcal{D}} (\nabla v)^T A^\varepsilon \nabla u \quad \text{and} \quad b(v) = \int_{\mathcal{D}} f \, v \, dx.$$

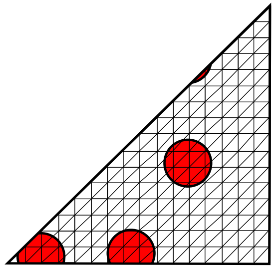
Idea: introduce an approximation with suitably chosen basis functions.

We introduce a classical P1 discretization of the domain $\mathcal{D}$, with L nodes, and denote ϕ_i^0 the basis functions.

MsFEM approach

- Coarse mesh with a P1 Finite Element basis functions ϕ_i^0.

- MsFEM basis



$$\begin{cases} -\text{div}(A^\varepsilon(x)\nabla\phi_i^{\varepsilon,\mathbf{K}}) = 0 & \text{in } \mathbf{K} \\ \phi_i^{\varepsilon,\mathbf{K}} = \phi_i^0|_{\mathbf{K}} & \text{on } \partial\mathbf{K} \end{cases}$$

  and glue them together: ϕ_i^ε such that $\phi_i^\varepsilon|_{\mathbf{K}} = \phi_i^{\varepsilon,\mathbf{K}}$ for all $\mathbf{K}$. The MsFEM functions are computed independently (in parallel) over each $\mathbf{K}$.

- Solve the macro problem with MsFEM basis functions ϕ_i^ε. This only involves the usual number of degrees of freedom !

The numerical local problems are similar in nature to the theoretical corrector problem we have introduced: they involve the actual, possibly perturbed, coefficient. ($\Rightarrow$ Numerical Analysis!)

Existence of correctors for nonperiodic homogenization problems

Joint works with X. Blanc and PL. Lions

Based on

X. Blanc/CLB/PL. Lions, Cr. Acad. Sc., Série I, vol. 353, pp 203-208 (2015)

X. Blanc/CLB/PL. Lions, Comm. PDE, vol 40, 12, pp 2173-2236 (2015)

X. Blanc/CLB/PL. Lions, Comm. PDE, in press (2018)

X. Blanc/CLB/PL. Lions, J. Maths P. & App., in press (2018)

Consider $a = a^{per} + \tilde{a}$ where a^{per} denotes the (unperturbed) background, and $\tilde{a}$ the perturbation. Assume $0 < \underline{\mu} \leq a^{per}(x) + \tilde{a}(x)$, $a^{per} \in L^\infty(\mathbb{R}^d)$, $\tilde{a} \in L^\infty(\mathbb{R}^d)$. Assume also

$$\begin{cases} -\operatorname{div}\left(a^{per}\left(p + \nabla w_{p,per}\right)\right) = 0, \\ \\ \dfrac{w_{p,per}(x)}{1+|x|} \quad \stackrel{|x|\to\infty}{\longrightarrow} \quad 0, \end{cases}$$

admits a solution $w_{p,per}$, unique up to the addition of a constant, s.t. $\nabla w_{p,per} \in L^\infty(\mathbb{R}^d)$. In many circumstances, we will have to assume a^{per}, $\tilde{a} \in C^{0,\alpha}$. We want to solve

$$-\operatorname{div}\left(\left(a^{per} + \tilde{a}\right)\left(p + \nabla w_p\right)\right) = 0$$

in the appropriate functional class, with $w_p(x) = o(1 + |x|)$.

Case considered: perturbation of a periodic background a^{per}, that is $a = a^{per} + \tilde{a}$. We would like to address the case $\tilde{a}(x) \stackrel{|x|\to\infty}{\longrightarrow} 0$, but are only able to treat

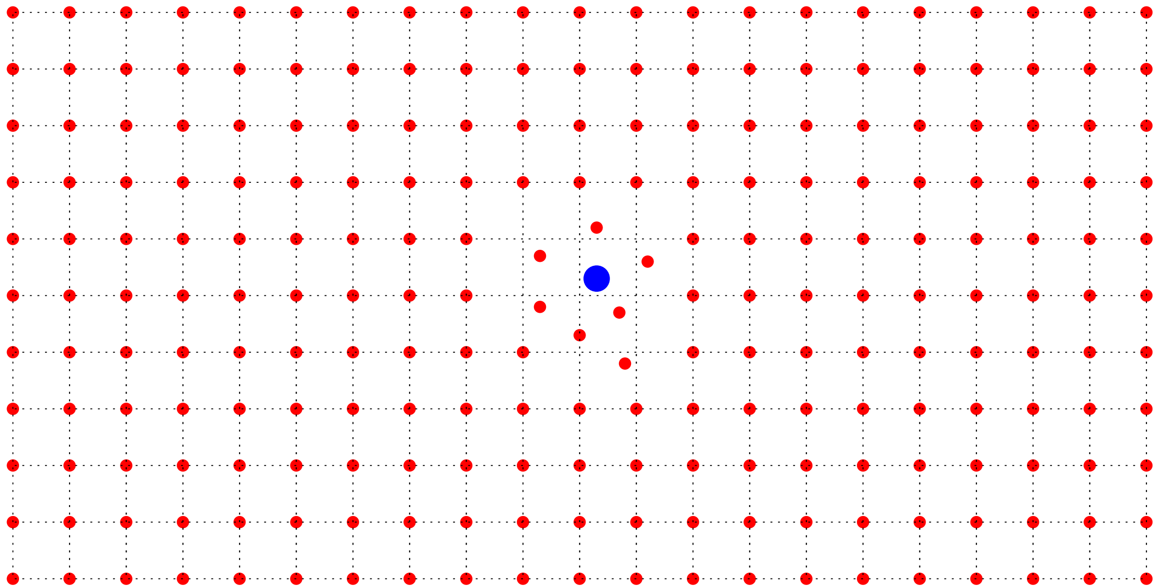
$$\tilde{a} \in L^r(\mathbb{R}^d), \quad \text{for some} \quad 1 \leq r < +\infty.$$

Second case (not considered today): twin-boundaries, that is two different periodic structures separated by a flat interface.

$$a^{per}(x) = a_{per,1,2}(x) = \begin{cases} a_{per,1}(x) & \text{when} \quad x_1 \leq 0, \\ a_{per,2}(x) & \text{when} \quad x_1 > 0, \end{cases}$$

(plus possibly a perturbation $\tilde{a}$ on top of that).

Local perturbation of a periodic structure



An easy case: [Blanc/LB/Lions, Milan Journal of Maths, 2012.]

Theorem (L^2-perturbation of periodic): Assume $\tilde{a} \in L^2(\mathbb{R}^d)$. Then, the corrector problem $-\operatorname{div}\left((a^{per} + \tilde{a})\left(p + \nabla w_p\right)\right) = 0$ admits a solution w_p which reads $w_p = w_{p,per} + \tilde{w}_p$, where $w_{p,per}$ is the unperturbed periodic solution and $\nabla \tilde{w}_p \in L^2(\mathbb{R}^d)$. Such a solution is unique up to the addition of a constant.

Remarks:Hölder regularity of a^{per} and $\tilde{a}$ is not needed.

Proof: Write $-\operatorname{div}\left(a\,\nabla \tilde{w}_p\right) = \operatorname{div}\left(\tilde{a}\left(p + \nabla w_{p,per}\right)\right)$. Regularize (adding $+\eta \tilde{w}_p$), solve this equation by Lax-Milgram, bounds, let $\eta \to 0$. This shows existence. For uniqueness, show that

$$-\operatorname{div}\left(a\,\nabla \tilde{v}\right) = 0$$

implies $\tilde{v} = 0$, multiplying by $(\tilde{v} - < \tilde{v} >_{R,2R})\chi_R^2$ and integrating.

Less easy: [Blanc/LB/Lions, Comm. PDE, 2015.]

Theorem (L^r ($r < d$)-perturbation): Assume $\tilde{a} \in L^r(\mathbb{R}^d)$, for some $r < d$. Then, the corrector problem $-\operatorname{div}\left(\left(a^{per} + \tilde{a}\right)(p + \nabla w_p)\right) = 0$ admits a solution w_p which reads $w_p = w_{p,per} + \tilde{w}_p$, where $w_{p,per}$ is the unperturbed solution and $\tilde{w}_p \in L^\infty(\mathbb{R}^d)$ vanishes at infinity. Such a solution is unique up to the addition of a constant.

Remarks: Periodicity of a^{per} is not needed. Hölder regularity of a^{per} and $\tilde{a}$ is not needed.

Proof: Write again $-\operatorname{div}(a\,\nabla \tilde{w}_p) = \operatorname{div}(\tilde{a}\,(p + \nabla w_{p,per}))$. Use

$$\tilde{w}_p(x) = \int \nabla_y G(x,y)\,(\tilde{a}\,(p + \nabla w_{p,per}))\,(y)\,dy,$$

with $\|G(.,y)\|_{L^{d/(d-2),\infty}} + \|\nabla_x G(.,y)\|_{L^{d/(d-1),\infty}} \leq C$.

Defect $\tilde{a} \in L^r(\mathbb{R}^d)$

Theorem (L^r-perturbation of periodic): Assume periodicity of the background and Hölder regularity. Then, the corrector problem has a unique solution w_p, up to the addition of a constant. Moreover, $w_p = w_{p,per} + \tilde{w}_p$, where $w_{p,per}$ is the periodic corrector and

- if $1 \leq r < d$, then, $\lim_{|x| \to +\infty} \tilde{w}_p(x) = 0$;
- if $2 \leq r$, then $\nabla \tilde{w}_p \in L^r$.

Remarks: Periodicity of a^{per} and Hölder regularity of a^{per} and $\tilde{a}$ are needed. The case $r = d$ is clearly critical.

Proof: Estimate the Green function on dyadic rings... Prove $\nabla \tilde{w}_p \in L^\infty$. Next write

$$-\operatorname{div}\left(a^{per} \nabla \tilde{w}_p\right) = \operatorname{div}\left(\tilde{a} \nabla \tilde{w}_p\right) + \operatorname{div}\left(\tilde{a}\left(p + \nabla w_{p,per}\right)\right)$$

and use [Avellaneda-Lin].

Once the corrector equation is written under the form

$$-\,\text{div}\,(a\,\nabla \tilde{w}_p) = \text{div}\,(\tilde{a}\,(p + \nabla w_{p,per})) \quad ,$$

it is immediate to see that proving the existence/uniqueness of the corrector amounts to establishing the estimate

$$-\,\text{div}\,(a\,\nabla u) = \text{div}\,(f) \quad \Rightarrow \|\nabla u\|_{L^q(\mathbb{R}^d)} \leq C_q \,\|f\|_{L^q(\mathbb{R}^d)}$$

for the coefficient $a = a^{per} + \tilde{a}$ and $\tilde{a} \in L^r(\mathbb{R}^d)$. (Apply to $q = r$).

We indeed show that such an estimate holds true. And we do so in a variety of settings ...

Theorem [Blanc/LB/Lions, 2018, Comm. PDE]:

Assume $a = a^{per} + \tilde{a}$, with: a and a^{per} both elliptic, $C^{0,\alpha}(\mathbb{R}^d)$, $L^\infty(\mathbb{R}^d)$ and $\tilde{a} \in L^r(\mathbb{R}^d)$, for some $1 \leq r < +\infty$.
Fix $1 < q < +\infty$ and $f \in L^q(\mathbb{R}^d)$.
Then, there exists $u \in L^1_{loc}(\mathbb{R}^d)$, unique up to the addition of a constant, such that

$$-\operatorname{div}(a \nabla u) = \operatorname{div}(f) \quad \text{in } \mathbb{R}^d,$$

and there exists a constant C_q, independent of f and u such that

$$\|\nabla u\|_{L^q(\mathbb{R}^d)} \leq C_q \, \|f\|_{L^q(\mathbb{R}^d)} \, .$$

Remark: Same estimate for $a = a^{per}$: [Avellaneda-Lin, 1991].

Proof: Assume, by contradiction, that

$$-\,\text{div}\,(a\,\nabla u_n) = \text{div}\,(f_n)\,,$$

with $\|\nabla u_n\|_{L^q(\mathbb{R}^d)} \equiv 1$ and $\|f_n\|_{L^q(\mathbb{R}^d)} \longrightarrow 0$ as $n \longrightarrow +\infty$.
Assume vanishing (Concentration-compactness principle [Lions, 1980s]), that is $\int_{B_R(x)} |\nabla u_n|^q \longrightarrow 0$, for all $R > 0$ and $x \in \mathbb{R}$. Then, since $\tilde{a}$ is small at infinity (because L^r and regular), the term $\tilde{a}\nabla u_n$ is everywhere small in L^q norm, and we have

$$-\,\text{div}\,(a^{per}\,\nabla u_n) = \text{div}\,(f_n + \tilde{a}\,\nabla u_n)$$

where $f_n + \tilde{a}\,\nabla u_n \longrightarrow 0$ in $L^q(\mathbb{R}^d)$ and $\|\nabla u_n\|_{L^q(\mathbb{R}^d)} \equiv 1$. This contradicts [Avellaneda-Lin, 1991]. Therefore "there is some mass somewhere":

$$\exists\,\eta > 0, \quad \exists\,0 < R < +\infty, \quad \forall\,n \in \mathbb{N}, \quad \|\nabla u^n\|_{(L^q(B_R)} \geq \eta > 0$$

Proof (equation in divergence form) - 2

Pass to the weak limit locally $\nabla u_n \rightharpoonup \nabla u$:

$$-\operatorname{div}\left(a\,\nabla u\right) = 0.$$

The limit is strong, by elliptic regularity using the equation.
So $u \neq 0$.

Finally, $\nabla u \in L^2(\mathbb{R}^d)$: bootstrap from $-\operatorname{div}\left(a^{per}\,\nabla u\right) = \operatorname{div}\left(\tilde{a}\,\nabla u\right)$ using $L^r \times L^q$ repeatedly in the R.H.S. to decrease the exponent q until $q = 2$ (if $q < 2$, duality). Conclude by coerciveness.

Ingredients: (i) locally compact problem (the estimate is easy on a bounded domain), (ii) $\tilde{a}$ vanishes at infinity, so, at infinity, the estimate reduces to the estimate for a periodic coefficient, which is true.

Estimate (equation **NOT** *in divergence form)*

Theorem [Blanc/LB/Lions, 2018, Comm. PDE]:

Same assumptions regarding $a = a^{per} + \tilde{a}$.
Fix $1 < q < +\infty$ and $f \in L^q(\mathbb{R}^d)$.
Then, there exists $u \in L^1_{loc}(\mathbb{R}^d)$, unique up to the addition of an affine function, such that

$$- a_{ij}\, \partial_{ij} u = f \quad \text{in}\, \mathbb{R}^d,$$

and there exists a constant C_q, independent of f and u such that

$$\left\| D^2 u \right\|_{L^q(\mathbb{R}^d)} \leq C_q \, \| f \|_{L^q(\mathbb{R}^d)} \, .$$

By duality, we have the existence and uniqueness of a unique ≥ 0 invariant measure m solution to

$$\partial_{ij}\,(a_{ij}\,m) = 0.$$

It reads as $m = m^{per} + \tilde{m}$ with $\tilde{m} \in L^r(\mathbb{R}^d)$. We have $m \geq c > 0$, Hölder continuous. We then transform

$$-\,a_{ij}(\frac{x}{\varepsilon})\,\partial_{ij}u^{\varepsilon}(x) = g(x)$$

into

$$-\,\text{div}\,\Big(\mathcal{A}(\frac{x}{\varepsilon})\,\nabla u^{\varepsilon}(x)\Big) = m(\frac{x}{\varepsilon})\,g(x)$$

where $\mathcal{A} = \mathcal{A}_{per} + \tilde{\mathcal{A}}$ has the "usual" properties, and apply our theory of homogenization for the equation in divergence form.

Our arguments carry over to

$$-a_{ij}\partial_{ij}u + b_j\,\partial_j u = f \quad \text{in} \quad \mathbb{R}^{\mathrm{d}},$$

$a = a^{per} + \tilde{a}$, $b = b^{per} + \tilde{b}$, for which we show

$$\left\| D^2\, u \right\|_{L^{q^*}(\mathbb{R}^d)} + \|\nabla u\|_{L^{q^*}(\mathbb{R}^d)} \leq C_q\, \|f\|_{L^{q^*}\cap L^q(\mathbb{R}^d)}\,,$$

for $\dfrac{1}{q^*} = \dfrac{1}{q} - \dfrac{1}{d}$, $q < d$.

This allows to address the homogenization theory for the advection-diffusion equation.

[Blanc/LB/Lions, 2018, J. Maths. P. & Appl.]

Not all equations behave like elliptic linear. An elliptic equation is indeed very forgiving...

Example (in 1D): u^ε solution to

$$u^\varepsilon + |(u^\varepsilon)'| = \tilde{a}(x/\varepsilon)$$

for $\tilde{a}(0) = \inf_{\mathbb{R}} \tilde{a} < 0$, $\tilde{a} \in \mathcal{D}(\mathbb{R})$, converges uniformly to $\bar{u}$, solution to

$$\bar{u}(0) = \tilde{a}(0) \quad \text{and} \quad \bar{u}(x) + |(\bar{u})'(x)| = 0, \quad \forall x \neq 0$$

that is

$$\bar{u}(x) = \tilde{a}(0)e^{-|x|}$$

which is different from $u = 0$, the solution when $\tilde{a} \equiv 0$. The microscopic defect $\tilde{a}$ affects the equation macroscopically.

Related works with P. Cardaliaguet and P. Souganidis.

Approximation theory for nonperiodic homogenization problems

Joint works with X. Blanc and M. Josien

Based on
X. Blanc/M. Josien/CLB, C.R. Acad. Sc., 2018, subm.,
X. Blanc/M. Josien/CLB, preprint (2018).

Theorem: Assume $r \neq d$ and $\tilde{a} \in L^r(\mathbb{R}^d)$. Take $a = a^{per} + \tilde{a}$ with the usual properties of ellipticity and Hölder regularity. Consider

$$-\text{div}\left(a(\frac{x}{\varepsilon})\,\nabla u^\varepsilon\right) = f,$$

with $f \in L^p(\Omega)$ and the residual

$$R_\varepsilon = u^\varepsilon - u^* - \varepsilon \sum_{i=1}^{d} \partial_i u^*(\cdot)\, w_i(\cdot/\varepsilon).$$

Then

$$\|\nabla R_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^{\min(1,d/r)} \left(\|f\|_{L^2(\Omega)} + \|\nabla u^*\|_{L^\infty(\partial\Omega)}\right).$$

Quantification of the rate of convergence of the remainder R_ε.

When in addition $f \in L^p(\Omega)$, we have

$$\|\nabla R_\varepsilon\|_{L^p(\Omega)} \leq C\varepsilon^{\min(1,d/r)} \left(\|f\|_{L^p(\Omega)} + \|\nabla u^*\|_{L^\infty(\partial\Omega)}\right),$$

and

$$\frac{1}{B(0,\varepsilon)} \int_{B(0,\varepsilon)} |\nabla R_\varepsilon|^2 \leq C\varepsilon^{\min(1,d/r)-d/p} \left(\|f\|_{L^p(\Omega)} + \|\nabla u^*\|_{L^\infty(\partial\Omega)}\right).$$

If $d \geq 3$ and f is Hölder continuous, then

$$\|\nabla R_\varepsilon\|_{L^\infty(\Omega)} \leq C\varepsilon^{\min(1,d/r)} \left(1 + |\ln \varepsilon^{-1}|\right) \|f\|_{C^{0,\beta}(\Omega)}.$$

Strategy of proof

The proof follows the same pattern as those by Avellaneda/Lin and Kenig/Lin/Shen in the periodic case. It concatenates

1. L_ε converges to L^* thus "what is true for the latter is true for the former when ε is small"
2. estimate of the Green function $\mathcal{G}_\varepsilon(x, y)$ by Grüter/Widman (only ellipticity)
3. estimate of its derivatives $\partial_x \mathcal{G}_\varepsilon(x, y)$ and $\partial_x \partial_y \mathcal{G}_\varepsilon(x, y)$ (structure is needed)
4. estimate of the rate of convergence of R_ε for a regular right-hand side
5. argument by duality for the convergence of the Green function $\mathcal{G}_\varepsilon(x, y) - \mathcal{G}^*(x, y)$

The proof first necessitates uniform boundedness and ellipticity, and Hölder continuity of the coefficient.
The essential properties of the corrector for the proof to hold are that it is "strongly" sublinear at infinity, that is

$$|w_p(x) - w_p(y)| \leq C\, |x-y|^\lambda, \quad \text{with } \lambda < 1,$$

and similarly for its potential ($a(p + \nabla w_p) = \text{curl}\, B$).

In fact, the real key point is that all properties of homogenization (bounds on the gradient of correctors, convergences of averages, etc) are uniform in

$$\frac{x}{\varepsilon_n} + y_n \quad \text{for } \varepsilon_n \longrightarrow 0 \quad |y_n| \longrightarrow +\infty.$$

Works in progress and related works

1. In collaboration with P. Cardaliaguet (Paris Dauphine) and P. Souganidis (Chicago), random perturbations of periodic HJB equations,
   (a) non viscous case: J. Maths Pures & Appl, in press, and `https://arxiv.org/abs/1701.05440`
   (b) viscous case: manuscript in preparation.
2. In collaboration with S. Wolf (ENS Paris), problems in non periodic perforated media
   (a) Poisson problem: extension of the results of the periodic case by J-L. Lions [Rocky Mountains J. of M., 1980]: manuscript in preparation.
   (b) Stokes problem: work in progress.

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Support from ONR and EOARD is gratefully acknowledged.