Stochastic homogenization of viscous and non-viscous HJ equations with non-convex Hamiltonians

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joint work with E. Kosygina (Baruch College & CUNY)

LMS Durham Symposium, August 20th–24th, 2018

Introduction

Viscous and non–viscous HJ equations

$$\begin{cases} \partial_t u^\varepsilon - \varepsilon \operatorname{tr}\left(A\left(\frac{x}{\varepsilon}\right) D_x^2 u^\varepsilon\right) + H\left(\frac{x}{\varepsilon}, D_x u^\varepsilon\right) = 0 & \text{in } (0,+\infty)\times\mathbb{R}^d \\ u^\varepsilon(0,\cdot) = g \in \mathrm{UC}(\mathbb{R}^d) \end{cases}$$

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Here $A(x) = (\sigma^T \sigma)(x)$ is a positive semi-definite matrix:

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while the Hamiltonian $H(x,p)$ satisfies

(H1) $H \in \mathrm{UC}(\mathbb{R}^d \times B_R)$ for all $R > 0$;

(H2) $\exists\ \alpha,\ \beta : \mathbb{R}_+ \to \mathbb{R}$ coercive such that

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(i) $\alpha_0 |p|^\gamma - 1/\alpha_0 \leqslant H(x,p) \leqslant \beta_0(|p|^\gamma + 1) \quad \forall x, p \in \mathbb{R}^d$;

(ii) $|H(x,p) - H(y,p)| \leqslant \beta_0(|p|^\gamma + 1)|x-y| \quad \forall x, y, p \in \mathbb{R}^d$;

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- A. Davini, *Commun. Contemp. Math.* (2017).

Homogenization of HJ equations

Assume that the following Cauchy problem is well posed:

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We say that $(\mathrm{HJ}_\varepsilon)$ homogenizes if there exists a continuous $\overline{H}:\mathbb{R}^d\to\mathbb{R}$ such that for every $g\in \mathrm{UC}(\mathbb{R}^d)$

$$u^\varepsilon(t,x) \rightrightarrows_{\text{loc}} \overline{u}(t,x) \quad \text{in } [0,+\infty)\times\mathbb{R}^d \qquad \text{as } \varepsilon\to 0^+$$

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Stationary ergodic setting

- Environment: probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{R}^d$ acts on Ω by shifts $\tau_x : \Omega \to \Omega$, $x \in \mathbb{R}^d$, which preserve $\mathbb{P}$. More precisely:

  (i) $(x, \omega) \to \tau_x \omega$ is jointly measurable;

  (ii) $\tau_0 = \mathrm{id}$; $\tau_{x+y} = \tau_x \circ \tau_y$;

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  We assume that the action is ergodic, i.e.

  $$\forall x \in \mathbb{R}^d \quad f(\tau_x \omega) = f(\omega) \quad \text{a.s. in } \Omega \Rightarrow f = \mathit{const}. \text{ a.s. in } \Omega.$$

  for every measurable $f : \Omega \to \mathbb{R}$.

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- Coefficients:

  $$A(x + y, \omega) = A(y, \tau_x \omega), \quad H(x + y, p, \omega) = H(y, p, \tau_x \omega).$$

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  We assume that A and H satisfy (A1)–(A2) and (H1)–(H2) respectively with bounds independent of ω.

Homogenization of HJ equations in random media

Assume that the following Cauchy problem is well posed for every $\omega \in \Omega$:

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Our main homogenization result

We have proved homogenization for viscous/nonviscous HJ equations for $d = 1$ in the stationary ergodic setting for a class of non–convex Hamiltonians.

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Homogenization in random media: literature

H convex

- P.E. Souganidis, *Asymptot. Anal.* (1999),
  F. Rezakhanlou and J.E. Tarver, *ARMA* (2000): $A \equiv 0$.

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- P.-L. Lions, P.E. Souganidis, *Comm. PDE* (2005),
  E. Kosygina, F. Rezakhanlou, and S.R.S. Varadhan, *CPAM* (2006): $A \not\equiv 0$.

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  $d = 1$, quite general H.

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- W.M. Feldman, P.E. Souganidis, *J. Math. Pures Appl.* (2017)

Literature: $A \not\equiv 0$ and non-convex H

- S. Armstrong, P. Cardaliaguet, J. Eur. Math. Soc. 20 (2018)
  $H \in \mathrm{C}^1(\mathbb{R}^d \times \mathbb{R}^d)$ such that

  $$\exists \alpha \geqslant 1 : \; H(x, tp, \omega) = t^{\alpha} H(x, p, \omega) \qquad \forall t \geqslant 0.$$

  Environments with finite range of dependence.

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- A. Davini, E. Kosygina, *Calc. Var. Partial Differential Equations*, 56 (2017)
  $d = 1$, a class of non–convex Hamiltonians including

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- E. Kosygina, A. Yilmaz, O. Zeitouni, ArXiv e-print (2018+)
  $d = 1$ e $H(x, p, \omega) = |p|^2 - b|p| + V(x, \omega)$.

Our results

Motivating example

Let $d = 1$ and $b : \mathbb{R} \times \Omega \to \mathbb{R}$ stationary such that:

- $a \leqslant b(\cdot, \cdot) \leqslant 1/a$ for some $a \in (0, 1)$;
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$$H(x, p, \omega) := \min\{H_+(x, p, \omega), H_-(x, p, \omega)\} = \begin{cases} H_+(x, p, \omega) & \text{if } p \geqslant 0 \\ H_-(x, p, \omega) & \text{if } p \leqslant 0. \end{cases}$$

with $H_\pm(x, p, \omega) := \frac{1}{2}\, p^2 \mp b(x, \omega)p$.

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Motivating example

Let $d = 1$ and $b : \mathbb{R} \times \Omega \to \mathbb{R}$ stationary such that:

- $a \leqslant b(\cdot, \cdot) \leqslant 1/a$ for some $a \in (0, 1)$;
- $b(\cdot, \omega)$ is Lip in $\mathbb{R}$ uniformly in ω.

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We are interested in the limit of $u_\theta^\varepsilon(t, x) = \varepsilon u_\theta(\frac{t}{\varepsilon}, \frac{x}{\varepsilon})$ as $\varepsilon \to 0^+$.

Solution by Hopf-Cole + control representation

Note that $v_\theta^\varepsilon := e^{-u_\theta^\varepsilon}$ solves

$$\partial_t v_\theta^\varepsilon - \frac{\varepsilon}{2}\,\partial_x^2 v_\theta^\varepsilon + b(x,\omega)|\partial_x v_\theta^\varepsilon| = 0, \quad v_\theta^\varepsilon\big|_{t=0} = e^{-\theta x}.$$

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The control representation formula gives

$$v_\theta^\varepsilon(t,x,\omega) = \inf_{\|c\|_\infty \leqslant 1} \mathbb{E}[\mathrm{e}^{-\theta X(t)}],$$

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By 1d comparison, for $\theta > 0$ ($\theta < 0$) the inf is attained for $c \equiv 1$ (resp., $c \equiv -1$). Thus, v_θ^ε also solves

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Thus $\overline{H}(\pm a) < 0$. Since $\overline{H}(0) = 0$, we infer that $\overline{H}$ is not convex.

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If (HJ_ε) homogenizes, then, in particular,

$$-\overline{H}(\theta) = \lim_{\varepsilon\to 0^+} u_\theta^\varepsilon(1,0) = \lim_{\varepsilon\to 0^+} \varepsilon u_\theta(1/\varepsilon, 0).$$

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Remark. If (H3) holds or (L) holds with $\kappa = \kappa(\theta)$ locally bounded in θ, then $\overline{H}$ is continuous.

Previous results in this direction

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  The authors introduce a notion of ergodicity that is shown to be a sufficient condition for homogenization.

Comparison with Alvarez and Bardi, ARMA (2003)

For our class of problems: let

$$F(x,p,X) := -\operatorname{tr}(A(x)X) + H(x,p) \quad \text{be } \mathbb{Z}^d\text{-periodic in } x.$$

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Theorem 2 (Alvarez-Bardi, 2003). If F is ergodic at each $\theta \in \mathbb{R}^d$, then $(\mathrm{HJ}_\varepsilon)$ homogenizes with $\overline{H}(\theta) := -c(\theta)$.

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The ergodicity is equivalent to the statement that, for every fixed $t > 0$,

$$\lim_{\varepsilon \to 0+} u_\theta^\varepsilon(t, x) = \langle \theta, x \rangle - t\,\overline{H}(\theta) \quad \text{uniformly in } x \in \mathbb{R}^d.$$

Stationary ergodic refinement

Lemma 3 (AD, E. Kosygina (2017)). Assume that, for a fixed $\theta \in \mathbb{R}^d$,

$$\lim_{\varepsilon \to 0+} u_\theta^\varepsilon(1, 0, \omega) = -\overline{H}(\theta) \qquad \text{a.s. in } \Omega.$$

Then

$$u_\theta^\varepsilon(t, x, \omega) \rightrightarrows_{\text{loc}} \theta \cdot x - t\overline{H}(\theta) \quad \text{in } [0, +\infty) \times \mathbb{R}^d \qquad \text{a.s. in } \Omega.$$

1d homogenization in random media

Let us put ourself in a stationary ergodic setting with $d = 1$.

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Remark. The effective Hamiltonian $\overline{H}$ is not convex in general.

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The solution u_θ^ε of

$$u_t^\varepsilon - \varepsilon A\left(\frac{x}{\varepsilon}\right) u_{xx}^\varepsilon + H\left(\frac{x}{\varepsilon}, u_x^\varepsilon, \omega\right) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R},$$

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Since this equation homogenizes, we get

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The argument for $\theta \leqslant 0$ is similar. $\square$

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yielding the asserted monotonicity of $u_\theta^\varepsilon(t,\cdot)$. $\square$

A class of 1–dimensional examples

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Then ($\mathrm{HJ}_\varepsilon^\omega$) homogenizes.

Sketch of the proof

The Hamiltonian H can be written in the following form:

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Thank you

for your attention!