Homogenization for the Stochastic heat equation $d \geq 3$

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Durham University, August 2018

Stochastic heat equation

$$\partial_t u = \frac{1}{2}\Delta u + \lambda V(t,x)u, \quad x \in \mathbb{R}^d, d \geq 3.$$

Here, $V(t,x)$ is random field, molification of space-time white noise:

$$V(t,x) = \int_{\mathbb{R}^{d+1}} \phi(t-s)\psi(x-y)dW(s,y),$$

For simplicity, we always take ϕ, ψ compactly supported with ψ isotropic.
Rescale: $u_\varepsilon(t,x) := u(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$ satisfies

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \frac{\lambda}{\varepsilon^2} V(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})u_\varepsilon. \quad u_\varepsilon(0,x) = u_0(x) \in \mathcal{C}_b(\mathbb{R}^d).$$

The noise $\varepsilon^{-2}V(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$ does not converge to white noise $\dot{W}$ - rather to $\varepsilon^{d/2-1}\dot{W}$.

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The Feynmann-Kac representation

$$\partial_t u = \frac{1}{2}\Delta u + \lambda V(t,x)u$$

$$u(t,x) = E_B^x\left(u_0(X_t)\exp(\lambda\int_0^t V(t-\tau,B_\tau)d\tau)\right)$$

In particular, if V is white in time, can be made into a martingale (in t) using time reversal and substraction of the (deterministic) quadratic variation.

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Advantage: Martingale!

In non-white in time case, the correction term is itself not deterministic.

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Special case: V - white in time, $u_0 = 1$.

Theorem (Mukherjee, Shamov, Z. '16)

There exits $\lambda_* \in (0, \infty)$ so that:

- *(Weak disorder) For $\lambda < \lambda^*$, solutions converge weakly in distribution to a deterministic limit, and $u_\varepsilon(x)$ converges to a random variable $Z_\infty > 0$.*
- *(Strong disorder) For $\lambda > \lambda^*$, $u_\epsilon(0) \to 0$ in probability.*

In this talk, we focus on the weak disorder phase, and try to understand better the convergence.

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Back to non-white in time. $\lambda < \lambda_0 < \lambda_*$

Theorem (Ryzhik, Gu, Z. '17)

There exist c_1, c_2 depending on λ such that for any $t > 0$ and $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} u_\varepsilon(t,x) \exp\Big\{-\frac{c_1 t}{\varepsilon^2} - c_2\Big\} g(x)dx \to_{\varepsilon\to 0} \int_{\mathbb{R}^d} \bar{u}(t,x) g(x) dx,$$

$$\frac{1}{\varepsilon^{d/2-1}} \int_{\mathbb{R}^d} (u_\varepsilon(t,x) - \mathbb{E}[u_\varepsilon(t,x)]) \exp\Big\{-\frac{c_1 t}{\varepsilon^2} - c_2\Big\} g(x)dx \Rightarrow_{\varepsilon\to 0} \int_{\mathbb{R}^d} \mathbf{U}(t,x) g(x) dx$$

in distribution. $\bar{u}$ - solution of effective heat equation

$$\partial_t \bar{u} = \frac{1}{2}\nabla\cdot \boldsymbol{a}_{\text{eff}} \nabla \bar{u}, \quad \bar{u}(0,x) = u_0(x), \boldsymbol{a}_{\text{eff}} \in \mathbb{R}^{d\times d}_{\text{sym}} \text{ effective diffusion,}$$

$\mathbf{U}$ solves the additive stochastic heat equation

$$\partial_t \mathbf{U} = \frac{1}{2}\nabla\cdot \boldsymbol{a}_{\text{eff}} \nabla \mathbf{U} + \lambda \nu_{\text{eff}} \bar{u} \dot{W}, \quad \mathbf{U}(0,x) = 0, \nu^2_{\text{eff}} > 0 \text{ effective variance}$$

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Edwards-Wilkinson equation (additive noise).

Mostly probabilistic methods, more below.

Related results: Magnen-Unterberger '17, applies to Hopf-Cole transform (KPZ) and gives same EW limit. Different methods. Mukherjee '17 averaged CLT; Comets, Cosco, Mukherjee '18 rates of convergence to limit Z_∞, fluctuations from limit. (White in time noise).

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Some homogenization...

$$\partial_t u^\varepsilon = \frac{1}{2}\Delta u^\varepsilon + \frac{1}{\varepsilon^2}\Big(\beta V(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon}) - \lambda\Big)u^\varepsilon, u^\varepsilon(0,x) = u_0(x).$$

Rename variables:

$$\partial_t u = \frac{1}{2}\Delta u + (\beta V - \lambda)u, \quad (1)$$

$u(0,x) = u_0(\epsilon x)$. Denote by Ψ solution of (1) with $u_0 = 1$. Recall that $\bar{u}$ is (weak) limit of homogenized equation $\bar{u}_t = (a_{\mathrm{eff}}/2)\Delta\bar{u}$.

Theorem (Dunlap,Gu,Ryzhik,Z. '18)

For $\beta < \beta_0 < \beta_*$ there exist $\lambda = \lambda(\beta)$ and a stationary solution $\tilde{\Psi}(t,x)$ so that

$$\lim_{t\to\infty} E|\Psi(t,x) - \tilde{\Psi}(t,x)|^2 = 0.$$

Further,

$$E|u^\varepsilon(t,x) - \bar{u}(t,x)\Psi^\varepsilon(t,x)|^2 \to_{\varepsilon\to 0} 0$$

where $\Psi^\varepsilon(t,x) = \Psi(t/\varepsilon^2, x/\varepsilon)$.

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$$\partial_t \overline{u} = \frac{1}{2} a_{\mathrm{eff}} \Delta \overline{u}, \overline{u}(0,x) = u_0(x).$$

Introduce the corrector

$$\begin{aligned}``\partial_s u_1(t,x,s,y) &= \frac{1}{2}\Delta_y u_1(t,x,s,y) + (\beta V(s,y)-\lambda)u_1(t,x,s,y)\\ &\quad + \nabla_y \Psi(s,y)\cdot\nabla_x \overline{u}(t,x)''\end{aligned}$$

not quite..

Theorem (Dunlap,Gu,Ryzhik,Z. '18 - second order convergence)

$0 \le \beta < \beta_0$, $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $\gamma \in (1,2)$. *For any* $\zeta < (1-\gamma/2)\wedge(\gamma-1)$, *there exists* $C > 0$ *so that*

$$\mathrm{Var}\left(\varepsilon^{-d/2+1}\int g(x)\left[u^\varepsilon(t,x) - \Psi^\varepsilon(t,x)\overline{u}(t,x) - \varepsilon u_1^\varepsilon(t,x)\right]\mathrm{d}x\right) \le C\varepsilon^{2\zeta}.$$

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Recall the Edwards-Wilkinson limit:

$$\partial_t \mathbf{U} = \frac{1}{2}\nabla \cdot \boldsymbol{a}_{\mathrm{eff}} \nabla \mathbf{U} + \lambda \nu_{\mathrm{eff}} \bar{u} \dot{W}, \quad \mathbf{U}(0,x) = 0,$$

Theorem (Dunlap,Gu,Ryzhik,Z. '18 - effective noise strength)

$$\nu_{\mathrm{eff}}^2 = \frac{a_{\mathrm{eff}} \lim_{\varepsilon\to 0} \int\int g(x)g(\tilde{x}) \left(\frac{1}{\varepsilon^{d-2}} Cov\left(\widetilde{\Psi}\left(0,\frac{x}{\varepsilon}\right), \widetilde{\Psi}\left(0,\frac{\tilde{x}}{\varepsilon}\right)\right)\right) \mathrm{d}x\, \mathrm{d}\tilde{x}}{\bar{c}\beta^2 e^{2\alpha_\infty} \int\int g(x)g(\tilde{x})|x-\tilde{x}|^{2-d}\, \mathrm{d}x\, \mathrm{d}\tilde{x}}$$

where α_∞ has an explicit representation.

Weak version of

$$Cov(\widetilde{\Psi}(0,0), \widetilde{\Psi}(0,y)) \sim \frac{\bar{c}\beta^2\nu^2 e^{2\alpha_\infty}}{a_{\mathrm{eff}}|y|^{d-2}}, y \gg 1$$

An expression for a_{eff} in terms of a solvability condition for a second order corrector is also available.

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$$\partial_t u^\varepsilon = \frac{1}{2}\Delta u^\varepsilon + \frac{1}{\varepsilon^2}\Big(\beta V(\tfrac{t}{\varepsilon^2},\tfrac{x}{\varepsilon}) - \lambda\Big)u^\varepsilon, u^\varepsilon(0,x) = u_0(x).$$

Recall the Edwards-Wilkinson limit:

$$\partial_t \mathbf{U} = \frac{1}{2}\nabla\cdot \boldsymbol{a}_{\text{eff}}\nabla\mathbf{U} + \lambda\nu_{\text{eff}}\bar{u}\dot{W}, \quad \mathbf{U}(0,x) = 0,$$

Theorem (Dunlap,Gu,Ryzhik,Z. '18 - effective noise strength)

$$\nu_{\text{eff}}^2 = \frac{a_{\text{eff}} \lim_{\varepsilon\to 0}\int\int g(x)g(\tilde{x})\left(\frac{1}{\varepsilon^{d-2}} Cov\left(\widetilde{\Psi}\left(0,\frac{x}{\varepsilon}\right),\widetilde{\Psi}\left(0,\frac{\tilde{x}}{\varepsilon}\right)\right)\right)\mathrm{d}x\,\mathrm{d}\tilde{x}}{\bar{c}\beta^2 e^{2\alpha_\infty}\int\int g(x)g(\tilde{x})|x-\tilde{x}|^{2-d}\,\mathrm{d}x\,\mathrm{d}\tilde{x}}$$

where α_∞ has an explicit representation.

Weak version of

$$Cov(\widetilde{\Psi}(0,0),\widetilde{\Psi}(0,y)) \sim \frac{\bar{c}\beta^2\nu^2 e^{2\alpha_\infty}}{a_{\text{eff}}|y|^{d-2}}, y \gg 1$$

An expression for a_{eff} in terms of a solvability condition for a second order corrector is also available.

The white in time case I

$$\partial_t u = \frac{1}{2}\Delta u + \lambda V(t,x) u$$

$$u(t,x) = E_B^x\left(u_0(X_t)\exp\left(\int_0^t V(t-\tau, B_\tau)d\tau - \frac{\lambda^2 t}{2} R_V(0)\right)\right)$$

In law, after rescalling and reversing time, and recalling that $V(t,x) = \int_{\mathbb{R}^d}\phi(x-y)\dot{W}(t,dy)$, need to compute

$$\hat{u}_\varepsilon(0) = E_B\left(\exp\left(\lambda\int_0^{t/\varepsilon^2}\int_{\mathbb{R}^d}\phi(y-B_s)\dot{W}(s,dy) - \frac{\lambda^2}{2\varepsilon^2}V(0)\right)\right),$$

where $V(x) = \int\phi(x-y)\phi(y)dy$
Martingale, link with polymer measures.

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The white in time case II

For λ small, can do L^2 computations: for example,

$$E(\hat{u}_\varepsilon(0)^2) = E_{B,B'}\Big(\exp(\lambda \int_0^{t/\varepsilon^2} \int_{\mathbb{R}^d} \phi(y - B_s)\dot{W}(s, dy) + \int_0^{t/\varepsilon^2} \int_{\mathbb{R}^d} \phi(y' - B'_{s'})\dot{W}(s', dy') - \frac{\lambda^2}{\varepsilon^2} V(0))\Big)$$

Because ϕ is compactly supported, this involves the total time that two independent BM's spend at bouded distance from each other. In dimension $d \geq 3$, this has exponential moments.

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Λ_ε is a map from a $X \times Y$ to $\mathbb{R}$ where $X = \mathcal{C}(\mathbb{R}_+; \mathbb{R}^d)$ supports the Wiener measure P_0 and Y is a Gaussian space with measure G. Define the measures $dQ_\varepsilon/(dP_0 \times dG) = \Lambda_\varepsilon(x, y)$. Note that $\hat{u}_\varepsilon = E_0\Lambda_\varepsilon$, i.e. random "total mass" of Q_ε. Example of a Gaussian Multiplicative Chaos.

From general theory, convergence will occur if (and only if) $\hat{u}_\varepsilon$ is uniformly integrable; If not, it will converge to 0!

Similar arguments go back to random polymer measures: Comets-Yosida '05, ..

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The limits, I

We only work in the L^2 phase.
Define the measure $\widehat{P}$ and normalization constant ζ:

$$\zeta_t := \log E_B\Big[\exp\Big\{\frac{\lambda^2}{2}\int_{[0,t]^2} R(s-u, B_s - B_u)dsdu\Big\}\Big],$$

$$\widehat{E}_{B,t}[f(B)] := E_B\Big[f(B)\exp\Big\{\frac{\lambda^2}{2}\int_{[0,t]^2} R(s-u, B_s - B_u)dsdu - \zeta_t\Big\}\Big].$$

We will see that $\zeta_t \sim c_1 t + c_2$. On the other hand, rescaling,

$$\mathbb{E}[u_\varepsilon(t,x)]e^{-\zeta_{t/\varepsilon^2}} = \widehat{\mathbb{E}}_{B,t/\varepsilon^2}[u_0(x+\varepsilon B_{t/\varepsilon^2})]$$

Thus, we need to show that $\varepsilon B_{t/\varepsilon^2}$ converges to a Brownian motion.
The main computational tool is a general Markov device, coming from the theory of Doeblin chains, described next.

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Independence decomposition

Suppose that $\{X_n\}$ is a Markov chain, on an abstract space $\mathcal{X}$, with transition probabilities $\pi(x, dy)$ satisfying

$$\pi(x, dy) \geq p\mu(dy)$$

for some probability measure μ and $p > 0$.

Then $\sum_{i=1}^{n}[f(X_i) - E_{\text{stat}} f(X)]/\sigma_f\sqrt{n}$ satisfies the invariance principle.

$\{X_n\}$ can be constructed as follows: let $\{B_n\}$ be a collection of iid Bernolli(p). Then write

$$X_i = B_i Y_i + (1 - B_i) Z_i$$

where Y_i are iid, $\sim \mu$, and Z_i, conditioned on history, is distributed as $\sim (\pi(X_{i-1}, dy) - p\mu(dy))/(1-p)$.

Apply it here to pieces of paths of length 1: $X_i = \{B_{i+t}\}_{t\in(0,1)}$.

Asymptotics of ζ_t from Krein-Rutman.

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The Edwards-Wilkinson limit I

Define $\Phi_{t,x,B}(s,y) := \int_0^t \phi(t-r-s)\psi(x+B_r-y)dr$, and martingale

$$M_{t,x,B}(r) := \int_{-\infty}^{r}\int_{\mathbb{R}^d} \Phi_{t,x,B}(s,y)dW(s,y), \langle M_{t,x,B}\rangle_r = \int_{-\infty}^{r}\int_{\mathbb{R}^d} |\Phi_{t,x,B}(s,y)|^2 dsdy$$

Then, by the Clark-Ocone formula,

$$\begin{aligned}(u(t,x)-\mathbb{E}[u(t,x)])e^{-\zeta_t} &= \lambda \int_{-1}^{t}\int_{\mathbb{R}^d} \widehat{\mathbb{E}}_{B,t}\Big[u(0,x+B_t)\Phi_{t,x,B}(r,y) \\ &\quad \exp\Big\{\lambda M_{t,x,B}(r) - \frac{\lambda^2}{2}\langle M_{t,x,B}\rangle_r\Big\}\Big]dW(r,y).\end{aligned}$$

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$$A_\varepsilon := \int_{\mathbb{R}^d} g(x)(u(t,x) - \mathbb{E}[u(t,x)])e^{-\zeta_t} = \lambda \int_{-1}^{t/\varepsilon^2} \int_{\mathbb{R}^d} Z_t^\varepsilon(r,y) dW(r,y)$$

Writing the time integral as sum over intervals (of length $\varepsilon^{-\beta}$ with $\beta < 2$) with short deletions (of order $\varepsilon^{-\alpha}$, $\alpha < \beta$) essentially represents A_ε as sum of iids, hence need only to understand variances.

Computing variances involves expectation with respect to pairs of Brownian motions B, B', under the measure $\widehat{\mathbb{P}}$. Note that the interaction involves only compact (in time) intervals: B_t interacts only with B'_{t+s}, $|s| \le 1$.

More explicitly:

$$(u(t,x) - \mathbb{E}[u(t,x)])e^{-\zeta_t} = \lambda \int_{-1}^{t} \int_{\mathbb{R}^d} \widehat{\mathbb{E}}_{B,t}\Big[u(0, x + B_t)\Phi_{t,x,B}(r,y) \exp\Big\{\lambda M_{t,x,B}(r) - \frac{\lambda^2}{2}\langle M_{t,x,B}\rangle_r\Big\}\Big] dW(r,y).$$

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More explicitly:

The Edwards-Wilkinson limit II

$$\mathcal{J}_\varepsilon(M_1, M_2) = \lambda^2 \int_{-1}^{M_1}\int_{-1}^{M_2}$$

$$R_\phi(u_1, u_2) R_\psi(x_1 - x_2 + \Delta B^1_{\frac{t-r}{\varepsilon^2}-s_1, \frac{t-r}{\varepsilon^2}+u_1} - \Delta B^2_{\frac{t-r}{\varepsilon^2}-s_2, \frac{t-r}{\varepsilon^2}+u_2}) du_1 du_2.$$

$$\mathcal{I}_\varepsilon = \prod_{i=1}^{2} g(\varepsilon x_i + y - \varepsilon B^i_{\frac{t-r}{\varepsilon^2}-s_i}) u_0(\varepsilon x_i + y + \varepsilon \Delta B^i_{\frac{t-r}{\varepsilon^2}-s_i, \frac{t}{\varepsilon^2}})$$

Variance involves computing

$$\frac{1}{\varepsilon^{d-2}}\mathbb{E}\left[\int_{t_1/\varepsilon^2}^{t_2/\varepsilon^2}\int_{\mathbb{R}^d}|Z_t^\varepsilon(r,y)|^2 dydr\right]$$

$$= \int_{t_1}^{t_2}\int_{\mathbb{R}^{3d}}\int_{[0,1]^2}\widehat{\mathbb{E}}_{B^1,B^2,t/\varepsilon^2}\left[\mathcal{I}_\varepsilon e^{\mathcal{J}_\varepsilon(\frac{r}{\varepsilon^2},\frac{r}{\varepsilon^2})}\right]\prod_{i=1}^{2}\phi(s_i)\psi(x_i)\, d\bar{s}d\bar{x}dydr.$$

Proceed now with the Doeblin trick to compute the expectation.

$$\mathcal{J}_\varepsilon(M_1, M_2) = \lambda^2 \int_{-1}^{M_1} \int_{-1}^{M_2}$$

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Variance involves computing

$$\frac{1}{\varepsilon^{d-2}} \mathbb{E} \left[\int_{t_1/\varepsilon^2}^{t_2/\varepsilon^2} \int_{\mathbb{R}^d} |Z^\varepsilon_t(r, y)|^2 dy dr \right]$$

$$= \int_{t_1}^{t_2} \int_{\mathbb{R}^{3d}} \int_{[0,1]^2} \widehat{\mathbb{E}}_{B^1, B^2, t/\varepsilon^2} \left[\mathcal{I}_\varepsilon e^{\mathcal{J}_\varepsilon(\frac{r}{\varepsilon^2}, \frac{r}{\varepsilon^2})} \right] \prod_{i=1}^{2} \phi(s_i) \psi(x_i)\, d\bar{s} d\bar{x} dy dr.$$

Proceed now with the Doeblin trick to compute the expectation.

The Edwards-Wilkinson limit II

$$\mathcal{J}_\varepsilon(M_1, M_2) = \lambda^2 \int_{-1}^{M_1} \int_{-1}^{M_2}$$

$$R_\phi(u_1, u_2) R_\psi(x_1 - x_2 + \Delta B^1_{\frac{t-r}{\varepsilon^2} - s_1, \frac{t-r}{\varepsilon^2} + u_1} - \Delta B^2_{\frac{t-r}{\varepsilon^2} - s_2, \frac{t-r}{\varepsilon^2} + u_2}) du_1 du_2.$$

$$\mathcal{I}_\varepsilon = \prod_{i=1}^{2} g(\varepsilon x_i + y - \varepsilon B^i_{\frac{t-r}{\varepsilon^2} - s_i}) u_0(\varepsilon x_i + y + \varepsilon \Delta B^i_{\frac{t-r}{\varepsilon^2} - s_i, \frac{t}{\varepsilon^2}})$$

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Proceed now with the Doeblin trick to compute the expectation.

Stationary limits and correctors

$$\partial_t \Psi = \frac{1}{2}\Delta\Psi + (\beta V - \lambda)\Psi, \Psi(0,x) = 1.$$

To show convergence to stationary solution, start at $-S$ and show convergence by computing L^2 norm of difference starting from different (large) S - again, use F-K and decoupling of chains.
Corrector construction: formally, write

$$u^\varepsilon(t,x) = u^{(0)}(t,x,t/\varepsilon^2,x/\varepsilon) + \varepsilon u^{(1)}(t,x,t/\varepsilon^2,x/\varepsilon) + \varepsilon^2 u^{(2)}(t,x,t/\varepsilon^2,x/\varepsilon) + \cdots$$

From variance computation, $u^{(0)} = \bar{u}(t,x)\Psi(t/\varepsilon^2, x/\varepsilon)$.
Formally,

$$\begin{aligned}\partial_s u_1(t,x,s,y) &= \frac{1}{2}\Delta_y u_1(t,x,s,y) + (\beta V(s,y) - \lambda)u_1(t,x,s,y) \\ &\quad + \nabla_y \Psi(s,y)\cdot\nabla_x \bar{u}(t,x), \text{ which solves}\end{aligned}$$

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Slightly modify the above by taking the forcing only at discrete times along a sequence $j\varepsilon^{-\gamma}$, $\gamma\in(1,2)$.
Now evaluate variances, using the Doeblin approach for decomposition.

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