Cutoff for the random-to-random shuffle

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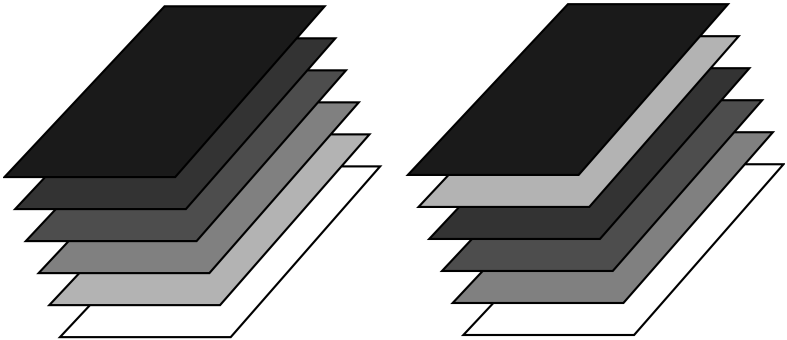
Random-to-random shuffle

Pick a card and position, uniformly at random, and move the card to that position:



More formally...

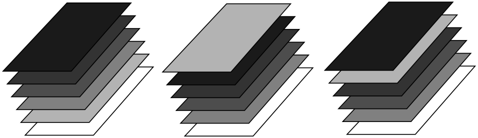
The walk on S_n is given by the matrix $K(g,h) = P(hg^{-1})$ for:

$$P(g) = \begin{cases} \frac{1}{n} & g = e \\ \frac{2}{n^2} & g = (i, i+1) \text{ for some i} \\ \frac{1}{n^2} & g = (i, i+1, \ldots, i+j), (i, i+j, \ldots, i+1) \text{ for some } j > 1 \\ 0 & \text{otherwise} \end{cases}$$

The distribution of the t^{th} step is: $K^t(e, \cdot) = P^{*t}(\cdot)$.

Related shuffles and walks

This walk is the symmetrization of the random-to-top shuffle (Tsetlin library) with its inverse, top-to-random.



Random-to-random should intuitively mix faster, but this could not be shown.

Walk first published in a comparison result by Diaconis and Saloff-Coste in 1993, but known and studied (unsuccessfully) before this.

Top-to-random falls into a broader class: Bidigare-Handlon-Rockmore hyperplane rearrangment random walks

Mixing time

After many steps, the deck should look “close” to random.

We use total variation distance (a scaled l^1 norm) to study how close to uniform the walk is after t steps:

$$||P^{*t} - \pi||_{TV} = \frac{1}{2} \sum_{g \in S_n} |P^{*t}(g) - \pi(g)|$$

Interested in t_{mix}, the minimum number of steps to make the TV distance small:

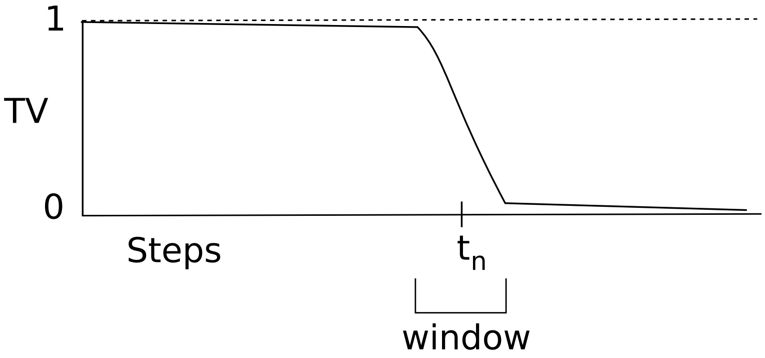
$$t_{\text{mix}} = \min_t \left(||P^{*t} - \pi||_{TV} \leq \frac{1}{4} \right)$$

Cutoff

A shuffle P on S_n mixes with total variation cutoff if there exists a sequence (t_n) s.t. for all $\epsilon > 0$:

$$\lim_{n\to\infty} ||P^{*t_n(1-\epsilon)} - \pi||_{TV} = 1$$

$$\lim_{n\to\infty} ||P^{*t_n(1+\epsilon)} - \pi||_{TV} = 0$$



Mixing bounds on random to random

- (Diaconis, Saloff-Coste 1993) t_{mix} is $O(n \log n)$
- (Uyemura-Reyes 2002) $\frac{1}{2} n \log n \leq t_{\text{mix}} \leq 4n \log n$
- (Diaconis 2005) Conjecture :

$$\left(\frac{3}{4} - o(1)\right) n \log n \leq t_{\text{mix}} \leq \left(\frac{3}{4} + o(1)\right) n \log n$$

- (Saloff-Coste and Zúñiga 2008) $t_{\text{mix}} \leq 2n \log n$
- (Subag 2013) $\frac{3}{4} n \log n - \frac{1}{4} n \log \log(n) - cn \leq t_{\text{mix}}$
- (Morris-Qin 2014) $t_{\text{mix}} \leq 1.5324 n \log n$

Theorem (B.-Nestoridi, 2017+)

$$t_{\text{mix}} \leq \frac{3}{4} n \log n + cn$$

Eigenvalues and the l^2 norm

Let K be a reversible, transitive transition matrix of a random walk on a finite state space Ω with eigenvalues $1 = \lambda_1 > \lambda_2 \geq ... \geq \lambda_{|\Omega|} \geq -1$, then:

$$4||K^t(x,\cdot) - \pi||^2_{T.V.} \leq \left|\left|\frac{K^t(x,\cdot)}{\pi(\cdot)} - 1\right|\right|^2_2 = \sum_{j=2}^{|\Omega|} \lambda_j^{2t}$$

for every starting point $x \in \Omega$.

Diagonalization

- (Uyemura-Reyes 2002) Partial diagonalization (below)
- (Dieker-Saliola 2014+) Diagonalization

Largest eigenvalues include, each with multiplity $n-1$:

$$1 - \frac{n+k^2+k}{n^2}, 1 \leq k \leq n-1$$

- Spectral gap is $\frac{n+2}{n^2}$
- Gap to next is $\frac{4}{n^2}$
- $k = n-1$ eigenvalue is 0

Spectral gap

Largest terms to bound:

$$\sum_{k=1}^{n-1}(n-1)\left(1-\frac{n+k^2+k}{n^2}\right)^{2t}$$

- First term small after $\frac{1}{2}n\log(n)$ steps
- But first $\sqrt{n}$ terms together not small till $\frac{3}{4}n\log(n)-\frac{1}{4}n\log\log(n)$
- If used first eigenvalue bound for all would need $n\log(n)$

The spectral gap with multiplicity does not determine the mixing time... yet still have cutoff!

Spectral methods - the transposition walk

(Diaconis-Shahshahani 1981) Diagonalization of the transposition walk and tight upper bound for t_{mix}

A conjugacy class walk, so by Schur's Lemma:

- A single eigenvalue for each irreduscible representation of S_n
- Formula for eigenvalues from Frobenius

Irreduscible rep's of S_n indexed by partitions λ:

$$\lambda_1 + ... + \lambda_r = n, \lambda_1 \geq .... \geq \lambda_r > 0$$

Non-conjugacy class walks

But the random-to-random walk is not a conjugacy class walk!

- For each irred. rep. get a family of eigenvalues
- No formulaic way to construct eigenvalues

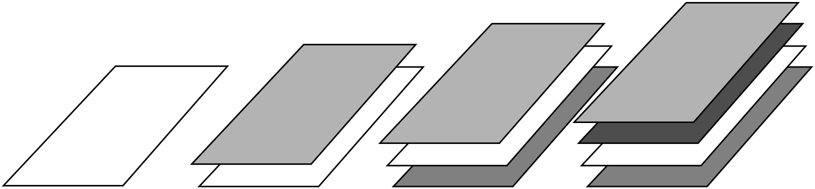
- Specific examples (star transpositions, top-to-random) had been diagonalized

Insight into representation theory

(Dieker-Saliola, 2014+) Direct construction of random-to-random eigenspaces

Observation 1: The 0-eigenspace is the same for random-to-random and random-to-top; intersects each irred. rep.

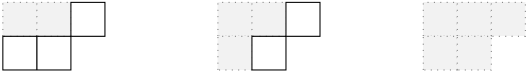
Observation 2: Using random insertion of new cards, can build a uniformly random deck



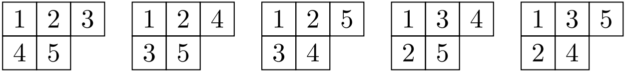
From 0-eigenvectors for walk with fewer cards, recursively construct eigenvectors of random-to-random using random insertion

Spectrum of random-to-random

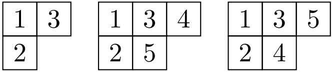
- For each irreducible representation λ of S_n, get a family of eigenvalues indexed by a second partition μ: $\mathrm{eig}(\lambda, \mu)$.



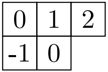
- Each irreducible representation appears d_λ times in the regular representation



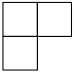
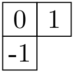
- Each $\mathrm{eig}(\lambda, \mu)$ occurs d^μ times in each copy.



Spectrum of random-to-random

For $\lambda =$  $=$ 

, then $\text{diag}(\lambda) = 0 + 1 + 2 - 1 + 0 = 2$.

For $\mu =$  $=$ 

, $\text{diag}(\mu) = 0 + 1 + -1 = 0$

Theorem (Diecker and Saliola)

The eigenvalue for (λ, μ) *is:*

$$\text{eig}(\lambda, \mu) = \frac{1}{n^2}\left(\binom{n+1}{2} - \binom{|\mu|+1}{2} + \text{diag}(\lambda) - \text{diag}(\mu)\right)$$

E.g. the eigenvalue for $[3, 2]/[2, 1]$ is

$$\frac{1}{5^2}\left(\binom{6}{2} - \binom{4}{2} + 2 - 0\right) = \frac{11}{25}$$

Strategy for upper bound

We need to show for $t = \frac{3}{4}n\log(n) + cn$ that

$$\sum_{(\lambda,\mu)} d_\lambda d^\mu \left(\mathrm{eig}(\lambda/\mu)\right)^{2t} \leq Ce^{-2c}$$

- Cluster λ by λ_1, length of first row
- Get two bounds on eig in terms of λ_1: one for smallest μ, one if μ has $> n - \lambda_1$ more boxes
- Bound for d_λ (from Diaconis-Shahshahani)
- Get bound for d^μ utilizing bijection of Reiner-Saliola-Welker

Thank you!