Cutoff for SRW on Ramanujan graphs via degree inflation

Jonathan Hermon

Goals

- Lubetzky & Peres (15) - SRW on Ramanujan graphs exhibits cutoff.
- We give a short alternative proof exploiting hit-mix connections.

General Reversible MCs - Notation

- Transition matrix - P.
- Stationary dist. - π.
- State space Ω.
- Reversibility: $\pi(x)P(x,y) = \pi(y)P(y,x)$ for all x, y.
- The hitting-time of $A \subset \Omega$ is $T_A := \inf\{t : X_t \in A\}$.

Expander graphs

- P transition matrix of SRW on a graph G.
- λ_i - ith largest e.v. of P.
- **Expander family** - a seq. of reg. graphs G_n with

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- Alon-Milman (85) - equivalent to lack of sparse cuts (discrete Cheeger's ineq.).
- Best "infinite expander" - the d-ary tree $\mathbb{T}_d$.

$$\rho_d := \text{spectral-radius of SRW on } \mathbb{T}_d = \frac{2\sqrt{d-1}}{d}.$$

- Alon-Boppana (86) - $\lambda_2(G) \geqslant \rho_d - o(1)$.

Motivation - Expander graphs and cutoff

- δ-TV mixing time

$$t_{\text{mix}}(\delta) := \inf\{k : \frac{1}{2}\sum_u |P^k(v,u) - \pi(u)| \leqslant \delta \text{ for all } v\}.$$

- Easy fact - For an n-vertex d-regular expander, mixing in TV of SRW is not "very gradual": i.e. for some $o(1)$ terms

$$t_{\text{mix}}(1 - o(1)) \geqslant (1 - o(1))\frac{d}{d-2}\log_{d-1} n.$$

$$t_{\text{mix}}(1/n) \leqslant 2(1 - \lambda_2)^{-1}\log n.$$

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$$t_{\mathrm{mix}}(1/n) \leqslant 2(1-\lambda_2)^{-1}\log n.$$

- Conjecture (Peres (04)) - SRW on transitive expanders exhibits cutoff:

$$t_{\mathrm{mix}}(1-o(1)) = (1+o(1))t_{\mathrm{mix}}(o(1)).$$

- Until (15) not a single such example was understood!

- **Ramanujan graph** - A connected $3 \leqslant d$-reg graph with all non-unit e.v.'s (of P) in $[-\rho_d, \rho_d = \frac{2\sqrt{d-1}}{d}]$ ("optimal expanders" by AB).
- Used in quantum computing and in constructions of codes with some extremal properties.

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- Constructions: Lubotzky, Phillips, Sarnak (88), Margulis (88) - for $d-1 =$ prime, Morgenstern (94) - for $d-1 =$ prime power.
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- Marcus, Spielman, Srivastava (13) - existence for all d.
- A seq. of connected d-reg. G_n is **almost Ramanujan** if all e.v.'s lie in

$$[-\rho_d^{1+o(1)}, \rho_d^{1+o(1)}] \cup \{\pm 1\}.$$

- Friedman (01) - A seq. of random d-reg. graphs of increasing sizes is w.h.p. almost Ramanujan (conjectured by Alon).

Diameter lower bound on TV mixing

- G d-reg. $\Longrightarrow \text{Diameter}(G) \geqslant \log_{d-1} n$.
- The "average speed" of SRW is $\leqslant \frac{d-2}{d}$ (sharp for $\mathbb{T}_d$).
- $\Longrightarrow$ To "see" at least εn vertices the walk needs

$$\frac{d}{d-2}\log_{d-1}(\varepsilon n) \approx \frac{d}{d-2}\log_{d-1} n.$$

  steps.
- For a d-ary tree of size n, SRW starting from the root exhibit abrupt convergence around time $\frac{d}{d-2}\log_{d-1} n$.

Cutoff for Ramanujan graphs

Lemma (Easy diameter lower bound - Lubetzky & Peres (15))

Let G be an n-vertex $d \geqslant 3$-regular graph. Then SRW on G satisfies:[a]

$$\forall \varepsilon \in (0,1), \quad t_{\mathrm{mix}}(\varepsilon \pm o(1)) \geqslant \frac{d}{d-2} \log_{d-1} n + c_{d,\varepsilon} \sqrt{\log_{d-1} n}. \tag{1}$$

[a] $c_{d,\varepsilon} := \frac{d\rho_d}{(d-2)^{3/2}} \Phi^{-1}(1-\varepsilon)$.

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Theorem (LP (15))

If G is Ramanujan then (1) *holds also with $\leqslant$ instead of $\geqslant$*
(i.e. cutoff at time $\frac{d}{d-2} \log_{d-1} n$).

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Corollary

For an n-vertex d-reg. Ramanujan graph, for all x all but $o(n)$ vertices are within distance $(1+o(1)) \log_{d-1} n$ from x.

Motivation

- In this talk - an easy alt. proof assuming diverging girth[1][2].

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[1] Enough that for some diverging k_n, every ball of radius k_n in G_n has $O(1)$ disjoint cycles.

[2] Always true for transitive Ramanujans of diverging sizes.

Motivation

- In this talk - an easy alt. proof assuming diverging girth[1][2].
- Observation - cutoff for almost Ramanujan is trivial if $d \to \infty$:

Proof: $\rho_d = \frac{2\sqrt{d-1}}{d} \implies \rho_d^{1+o(1)} = d^{-\frac{1}{2}+o(1)}$. Let

$$t := (1+\delta)\log_{d-1} n \sim \frac{1}{2}(1+\delta)\log_{\sqrt{d}} n$$

$$\implies \quad \rho_d^{2t} = n^{-(1+\delta-o(1))},$$

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so by Poincaré's ineq.:

$$\|\mathrm{P}_x^t - \pi\|_{2,\pi}^2 \leqslant (\rho_d^{1+o(1)})^{2t} \|\mathrm{P}_x^0 - \pi\|_{2,\pi}^2 \approx \rho_d^{2t}(n-1) \approx n^{-\delta}.$$

□

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Cutoff for Ramanujan graphs

- Def.: Given $G = (V, E)$, define $G(k) = (V, E(k))$ via
$$E(k) := \{\{u, v\} : \mathrm{dist}(u, v) = k\}.$$

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- Consider SRW on $G(k)$ for some $1 \ll k \ll g$.
- Morally, cutoff for $G(k)$ around time t should imply cutoff for G around time $\frac{d}{d-2}kt$.
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- Want $G(k)$ to be almost Ramanujan and deduce cutoff for $G(k)$.
- We'll show something similar (bypassing the “morally”) exploiting hit-mix machinery...

t_{mix} and hitting times - under reversibility

- Aldous (83) - $t_{\mathrm{mix}} \asymp \max_{a,A} \pi(A)\mathbb{E}_a[T_A]$.
- Peres & Sousi and independently Oliveira (12)[3] $t_{\mathrm{mix}} \asymp \max_{a,A:\pi(A)\geqslant 1/2} \mathbb{E}_a[T_A]$.

[3] + an extension by Griffiths et al. (2012) for size exactly 1/2.

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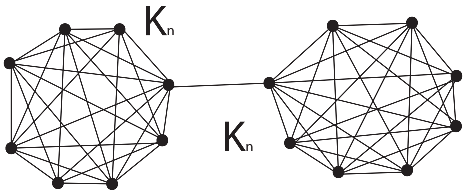

Figure: 2 copies of K_n connected by a single edge.

- $d_{\mathrm{TV}}(t) \sim \frac{1}{2}\mathrm{P}$(the other clique was not hit by time t) $\Longrightarrow$ maybe we should look at tails rather than on expectations!

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t_{mix} and hitting times - under reversibility

- Let $0 < \varepsilon < 1$.

$$\mathrm{hit}_\alpha(\varepsilon) := \min\{t : \mathrm{P}_a[T_{A^c} > t] \leqslant \varepsilon : \text{for all } a \in A \subset \Omega \text{ s.t. } \pi(A) \leqslant \alpha\}$$

$=$ the first time by which every “small” (size $\leqslant \alpha$) set is escaped from w.p. $\geqslant 1-\varepsilon$.

- Basu, H., Peres (13) - $t_{\mathrm{mix}}(\varepsilon) \approx \mathrm{hit}_{\frac{1}{2}}(\varepsilon)$.

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Why is this useful?

Using hit-mix connections we can:

- Can prove theoretical result about MCs.
- Construct surprising counter-examples.
- Analyze mixing when we know what sets are hardest to hit (so far only trees).
- Analyze mixing when we can control hitting times of all large sets uniformly (Ramanujan).

More precisely

For any reversible finite chain, $0 < \varepsilon < 1$ and $0 < \alpha < \min(\varepsilon, 1-\varepsilon)$

$$\mathrm{hit}_\alpha(\varepsilon+\alpha) - \frac{8}{\lambda_{\mathrm{abs}}}|\log\alpha| \leqslant t_{\mathrm{mix}}(\varepsilon) \leqslant \mathrm{hit}_\alpha(\varepsilon-\alpha) + \frac{8}{\lambda_{\mathrm{abs}}}|\log\alpha|$$

- Terms involving $\lambda_{\mathrm{abs}} := 1 - \max\{\lambda_2, |\lambda_{|\Omega|}|\}$ are often negligible (and always $1/\lambda_{\mathrm{abs}} \leqslant t_{\mathrm{mix}}$).

- Let P_A be the restriction of P to $A \subset \Omega$ (killed when escaping A).
- Let $\lambda(A)$ be the largest e.v. of P_A.
- Let π_A be π conditioned on A.

Proof of cutoff for Ramanujan graphs

- Consider SRW on $G(k)$ for some $1 \ll k \ll \sqrt{\mathrm{girth}}$.
- Let
$$\alpha := d^{-3k^2} = o(1).$$
- If $\mathrm{hit}_\alpha(\alpha) \approx \frac{1}{k}\log_{d-1} n =: s$ for $G(k)$, then:
$$\text{for } G\text{: } \mathrm{hit}_\alpha(\alpha + o(1)) \leqslant (\frac{d}{d-2}k)s = \frac{d}{d-2}\log_{d-1} n$$
(up to $o(1)$ terms on r.h.s.).
- $\Longrightarrow$ cutoff for SRW on G.

Application - Cutoff for Ramanujan graphs

- Back to $G(k)$ - Denote the transition matrix corresponding to SRW on it by K (and for G by P)

$$\sum_{b \in A} \pi_A(b)(\mathrm{P}_b[T_{A^c} > t])^2 = \|K_A^t 1_A\|_{2,A}^2 \leqslant [\lambda_K(A)]^{2t}$$

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- $\Longrightarrow (\mathrm{P}_a[T_{A^c} > t])^2 \leqslant n[\lambda_K(A)]^{2t}$. We are done if $\lambda_K(A) \leqslant d^{-\frac{k}{2}(1-o(1))}$.
- Will show this via a simple comparison technique:

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Proposition

Let P and Q be reversible w.r.t. π. Assume $K(x,y) \leqslant CQ(x,y)$ for all x, y.

$$\lambda_K(A) \leqslant C\lambda_Q(A)$$

($\lambda_K(A)$ and $\lambda_Q(A)$ - largest eigenvalues of K_A and Q_A, resp.).

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Proof: Denote $\langle f,g\rangle_{\pi_A}:=\sum_{x\in\Omega}\pi_A(x)g(x)f(x)$. By Perron-Frobenius

$$\lambda_P(A)=\frac{\max_{f\in\mathbb{R}_+^A,f\neq 0}\langle K_A f,f\rangle_{\pi_A}}{\|f\|_{2,A}^2}\leqslant C\frac{\max_{f\in\mathbb{R}_+^A,f\neq 0}\langle Q_A f,f\rangle_{\pi_A}}{\|f\|_{2,A}^2}=C\lambda_Q(A).\quad\square$$

Let $Q := P^{k+2k^2}$ (as before K SRW on $G(k)$). Let $A \subset V$ be s.t. $\pi(A) \leqslant \alpha = o(1)$.

Recall - want $\lambda_K(A) \leqslant (d-1)^{-\frac{k}{2}(1-o(1))}$.

By last Lemma: $\lambda_K(A) \leqslant \lambda_Q(A) / [\min_{x,y} K(x,y)/Q(x,y)]$.

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General fact - $\lambda(A) \leqslant \pi(A) + \lambda_2 \pi(A^c)$

$$\Longrightarrow \lambda_Q(A) \leqslant \lambda_2^{k+2k^2} + \alpha \leqslant 2\rho_d^{k+2k^2}.$$

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Let x, y be adjacent in $G(k)$. Then

$$Q(x,y) = \frac{1}{\text{number of such } y\text{'s}} \binom{k+2k^2}{k^2} (d-1)^{k^2+k-o(k)} d^{-(k+2k^2)}$$

$= K(x,y) \cdot C_k$, where

$$C_k := 2^{k+2k^2}(d-1)^{k^2+k-o(k)} d^{-(k+2k^2)} = \rho_d^{k+2k^2}(d-1)^{\frac{k}{2}(1-o(1))}.$$

$$\Longrightarrow \lambda_K(A) \leqslant \lambda_Q(A)/C_k \leqslant 2\rho^{k+2k^2}/C_k = (d-1)^{-\frac{k}{2}(1-o(1))}.$$

□

Unrelated hitting-time results

- (H., Peres (16)): The L_∞ mixing-time started from $x \asymp$

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- Same for relative-entropy with $\frac{1}{|\log \pi(A)|}$ instead of $\pi(A)$.
- First sharp bounds!
- $c_{\mathrm{Log-Sobolev}} \asymp \inf_{A:\pi(A)\leqslant 1/2} [1-\lambda(A)]/|\log \pi(A)|$

Applications - Positive results

- (Basu, H., Peres (13)): Characterize cutoff using hitting times.
- H. and Peres (15) - one lazy step mixes twice quicker than always being lazy.
- For (weighted RW on) trees:
- Peres and Sousi (12) - t_{mix} is robust[4].
- H. and Peres (16) - τ_∞ is robust (and is $\asymp \max(t_{\mathrm{mix}}, 1/c_{\mathrm{Log-Sobolev}})$).
- BHP - cutoff iff (spectral-gap)$\times t_{\mathrm{mix}} \to \infty$.

---

[4] A parameter is **robust** if changing the edge-weights by a bounded amount can change it only by a constant factor.

Negative results - counterexamples inspired by hit-mix connections

- Ding and Peres (12) - t_{mix} is not robust.
- H. (16) - τ_{∞} and $\tau_{\mathrm{relative-entropy}}$ are not robust (resolves a conjuncture of Kozma; variants asked by various authors Aldous, Diaconis and Saloff-Coste).
- H. and Peres (16) - separation cutoff may depend on the holding prob.!

Thank you!