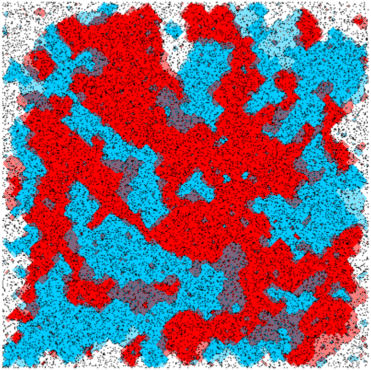
# Percolation games



James Martin

Department of Statistics, University of Oxford

Joint work with Alexander Holroyd and Irène Marcovici

LMS - EPSRC Durham Symposium
Markov chains, mixing times and cut-off
Tuesday 1 August 2017

Nasser Hussain
Durham University 1986-1989
(Natural Sciences)
England captain 1999-2003


Andrew Strauss
Durham University 1995-1998
(Economics)
England captain 2009-2012

Two-player combinatorial games (deterministic, perfect information)

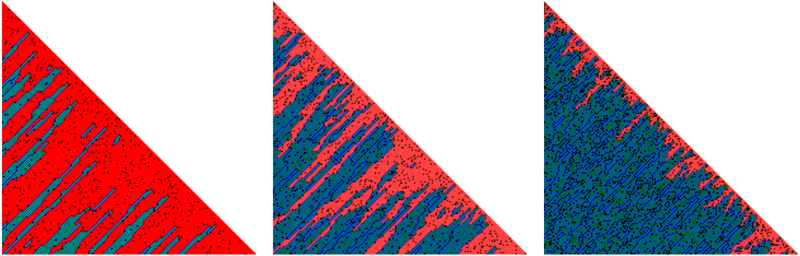
## Directed game on $\mathbb{Z}^d$

Lattice $\mathbb{Z}^d$ with directed edges.

A token starts at some site $x$. The players take turns; a turn consists of moving the token along a directed edge.

Let $p, q \geq 0$ with $p + q < 1$. Each site is forbidden with probability $p$, a target with probability $q$, and open with probability $1 - p - q$ (independently for different sites).

A player moving to a forbidden site loses immediately. A player moving to a target site wins immediately.



$d = 2$. Outcomes of the game on the region $\{x \in \mathbb{Z}_+^2 : x_1 + x_2 \leq 200\}$, declaring a draw if the token reaches the diagonal $x_1 + x_2 = 200$, with $q = 0$ and $p = 0.05, 0.1, 0.2$. Colours indicate the outcome when the game is started from that site: first player win (blue); first player loss (green); draw (red). Forbidden sites are black.

We could also consider boundary conditions corresponding to "next player win" or "next player lose". A site in the interior is a "draw" under "draw boundary conditions" (as above) if its value is different under the two extreme boundary conditions.

Classify open sites as "win", "loss" or "draw" according to the outcome of the game started from the site (from the perspective of the starting player).

In addition count a forbidden site as a win, and a target site as a loss.

Encoding of sites:

$$\eta(x) = \begin{cases} \text{"0"} & \text{if } x \text{ is a win (or forbidden);} \\ \text{"1"} & \text{if } x \text{ is a loss (or a target);} \\ \text{"?"} & \text{if } x \text{ is a draw.} \end{cases}$$

Recurrences:

$$\eta(x) = \begin{cases} 0 & \text{if } x \text{ forbidden, or if } x \text{ open and } \eta(y) = 1 \text{ for some } y \in \mathrm{Out}(x); \\ 1 & \text{if } x \text{ is a target, or if } x \text{ open and } \eta(y) = 0 \text{ for all } y \in \mathrm{Out}(x); \\ ? & \text{otherwise.} \end{cases}$$

Let $d = 2$. Let $S_k$ be the set $\{x = (x_1, x_2) \in \mathbb{Z}^2 : x_1 + x_2 = k\}$ (a NW-SE diagonal of $\mathbb{Z}^2$). The recursion gives the values $\{\eta(x), x \in S_k\}$ in terms of the values $\{\eta(x), x \in S_{k+1}\}$ together with the information about the types of the vertices in $S_k$.

We can regard the configurations on successive diagonals $S_k$, as $k$ decreases, as states of a 1D probabilistic cellular automaton (PCA) at successive times. Specifically, set

$$\eta_t(n) = \eta\big((-t - n, n)\big).$$

Then $\eta_t, t \in \mathbb{Z}$ evolves as a PCA.

## PCA $A_{p,q}$ with alphabet $\{0,1\}$:

$$\begin{array}{lll} 1 & \text{with probability} & 1-p \\ 0 & \text{with probability} & p \end{array}$$

$$\begin{array}{cc} 0 & 0 \end{array}$$

$$\begin{array}{lll} 1 & \text{with probability} & q \\ 0 & \text{with probability} & 1-q \end{array}$$

$$\begin{array}{cc} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{array}$$

## PCA $F_{p,q}$ with alphabet $\{0,?,1\}$ (with $*$ denoting an arbitrary symbol):

$$\begin{array}{lll} 1 & \text{with probability} & 1-p \\ 0 & \text{with probability} & p \end{array}$$

$$\begin{array}{cc} 0 & 0 \end{array}$$

$$\begin{array}{lll} 1 & \text{with probability} & q \\ 0 & \text{with probability} & 1-q \end{array}$$

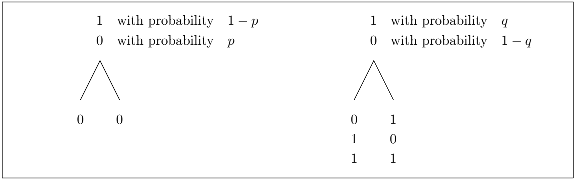
$$\begin{array}{cc} 1 & * \\ * & 1 \end{array}$$

$$\begin{array}{lll} 1 & \text{with probability} & q \\ ? & \text{with probability} & 1-p-q \\ 0 & \text{with probability} & p \end{array}$$

$$\begin{array}{cc} ? & ? \\ 0 & ? \\ ? & 0 \end{array}$$

$F_{p,q}$ is the envelope of the PCA $A_{p,q}$.

PCA $A_{p,q}$ with alphabet $\{0,1\}$:




Formally, let $A_{p,q}$ be the operator on the set of distributions on $\{0,1\}^{\mathbb{Z}}$ representing the action of the PCA; if $\mu$ is the distribution of a configuration in $\{0,1\}^{\mathbb{Z}}$, then $A_{p,q}\mu$ is the distribution of the configuration obtained by performing one update step of the PCA. A stationary distribution of a PCA $G$ is a distribution $\mu$ such that $G\mu = \mu$.

A PCA is ergodic if it has a unique stationary distribution and if from any initial distribution, the iterates of the PCA converge to that stationary distribution.

Question (e.g. Toom, Vasilyev Stavskaya, Mityushin, Kurdyumov and Pirogov, 1990): Is $A_{p,q}$ ergodic for all $p, q$? (Cf. "positive rates").

## Proposition

*For all* $p, q \in (0, 1)$ *with* $0 < p + q < 1$, *the following are equivalent:*

(i) $A_{p,q}$ *is ergodic;*

(ii) $F_{p,q}$ *is ergodic;*

(iii) *The percolation game has probability 0 of a draw.*

Proof is quite straightforward using the fact that $F_{p,q}$ is monotonic decreasing with respect to the ordering $0 <? < 1$, and monotonic increasing with respect to the partial order given by $0 <?$ and $1 <?$.

## Theorem

*For all* $p, q \in (0, 1)$ *with* $0 < p + q < 1$, *(i), (ii) and (iii) above hold.*

Approach: show that $F_{p,q}$ has no translation invariant stationary distribution with positive probability of a ? symbol. (Then (iii) follows.) To do this we introduce a local weighting on instances of ?; we show that if the average weight per site in a stationary distribution is positive, then it should be strictly decreasing under the action of $F_{p,q}$.

# Hard-core model

The case $q = 0$ is closely related to the hard-core model.

Take a finite undirected graph with vertex set $W$ and some $\lambda > 0$. The hard-core model on $W$ with activity $\lambda$ is given by a probability measure on $\{0, 1\}^W$ defined by

$$\nu_\lambda\big((\eta(w), w \in W)\big) \propto \begin{cases} \lambda^{\sum \eta(w)} & \text{if } \eta_v \eta_w = 0 \text{ for all } v \sim w \\ 0 & \text{if } \eta_v = \eta_w = 1 \text{ for some } v \sim w. \end{cases} \qquad (\dagger)$$

State 1 represents an "occupied site" and state 0 represents an "empty site". The form of the RHS ensures that no two neighbours are both occupied (the set of occupied sites forms an independent set).

The measure $\nu_\lambda$ satisfies, for all $v \in W$,

$$\nu_\lambda\big(\eta(v) = 1 \big| (\eta(w)_{w \neq v})\big) = \begin{cases} \frac{\lambda}{1+\lambda} & \text{if } \eta(w) = 0 \text{ for all } w \sim v \\ 0 & \text{if } \eta(w) = 1 \text{ for some } w \sim v. \end{cases} \qquad (*)$$

For $W$ infinite, $(\dagger)$ no longer works, but we can consider $(*)$. A measure satisfying $(*)$ is called a Gibbs measure for the hard-core model on $W$.

## Hard-core model cont.

Take for example $\mathbb{Z}^d$, $d \geq 2$. Then:

- for $\lambda$ sufficiently small, there is a unique Gibbs measure;
- for $\lambda$ sufficiently large, there are multiple Gibbs measures.

## Glauber dynamics

Hard-core measure condition: for $v \in W$,

$$\nu_\lambda\big(\eta(v) = 1 \big| (\eta(w)_{w \neq v})\big) = \begin{cases} \frac{\lambda}{1+\lambda} & \text{if } \eta(w) = 0 \text{ for all } w \sim v \\ 0 & \text{if } \eta(w) = 1 \text{ for some } w \sim v. \end{cases} \qquad (*)$$

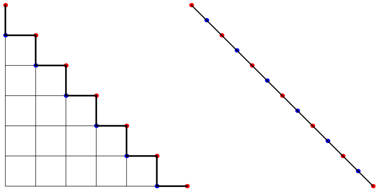
We can resample the value at $v \in W$ according to (*).

If a measure is invariant under all such resamplings, it is a hard-core Gibbs measure.

We can also extend to resample the values at several different sites simultaneously, as long as no two of them are neighbours. For example, if the graph is bipartite, we can resample at all the vertices in one class of the bipartition.

# Doubling graph

The state of the PCA $A_{p,0}$ at a single time corresponds to the values $\{\eta(x), x \in S_k\}$ where $S_k = \{x : x_1 + x_2 = k\}$.



We can consider a doubling graph $D$ corresponding to two successive lines $S_{k+1}$ and $S_k$; then $D$ is isomorphic to $\mathbb{Z}$ and is bipartite. Suppose we set $p = 1/(1+\lambda)$. Then the procedure of obtaining the values on $S_k$ given those on $S_{k+1}$ is identical to the procedure of carrying out a hard-core Glauber update on the sites of $S_k \subset D$, while keeping the values on $S_{k+1} \subset D$ fixed.

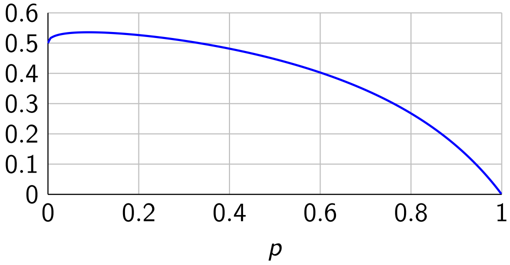
Next we can obtain the values on $S_{k-1}$ from those on $S_k$; this then corresponds to carrying out the Glauber update on the other half of $D$. In this way the PCA evolution corresponds to alternating updates on the two parts of $D$.

The hard-core model on $D = \mathbb{Z}$ is very well-behaved! It has a unique Gibbs measure for all $\lambda$. This can be used to show ergodicity of $A_{p,0}$.

Moreover the Gibbs measure is Markovian (in space), with transition matrix

$$P = \begin{pmatrix} p_{0,0} & p_{0,1} \\ p_{1,0} & p_{1,1} \end{pmatrix} = \begin{pmatrix} \frac{2-p-\sqrt{p(4-3p)}}{2(1-p)^2} & \frac{2p^2-3p+\sqrt{p(4-3p)}}{2(1-p)^2} \\ \frac{-p+\sqrt{p(4-3p)}}{2(1-p)} & \frac{2-p-\sqrt{p(4-3p)}}{2(1-p)} \end{pmatrix}.$$

From this we can derive, for example, the exact probability of a first-player win in the game: $\alpha(p) = \frac{\frac{1}{2}\left(1+\sqrt{p/(4-3p)}-2p\right)}{1-p}$:



Notice that if we had multiple Gibbs measures for the hard-core model on $D = \mathbb{Z}$, then we could conclude that there were multiple two-periodic distributions for $A_{p,0}$.

This would contradict the ergodicity of $A_{p,0}$. However, this idea is useful for some higher-dimensional models, using an analogous "dimension reduction" idea which relates the game on a $d$-dimensional graph to the hard-core model on an appropriate $(d-1)$-dimensional graph.

This yields positive probability of a draw for some games in dimension $d \geq 3$. For this we extend the idea of a doubling graph.

Let $G$ be directed graph. Suppose there is a partition $(S_k : k \in \mathbb{Z})$ of the vertex set $V$ of $G$, and an integer $m \geq 2$, such that the following conditions hold:

(A1) For all $x \in S_k$, we have $\mathrm{Out}(x) \subset S_{k+1} \cup \cdots \cup S_{k+m-1}$.

(A2) There is a graph automorphism $\phi$ of $G$ that maps $S_k$ to $S_{k+m}$ for every $k$, and such that $\mathrm{Out}(x) = \mathrm{In}(\phi(x))$ for all $x$.

Then let $D_k$ be the graph with vertex set $S_k \cup \cdots \cup S_{k+m-1}$, with an undirected edge $(x, y)$ whenever $(x, y)$ is a (directed) edge of $V$. Under conditions (A1) and (A2), the graphs $D_k, k \in \mathbb{Z}$ are isomorphic to each other; write $D$ for a graph isomorphic to any of them.

## Theorem

*Suppose that the directed graph $G$ satisfies (A1) and (A2). If there exist multiple Gibbs measures for the hard-core model on $D$ with activity $\lambda$, then draws occur in the percolation game on $G$ with $p = 1/(1 + \lambda)$ with positive probability.*

(A1) For all $x \in S_k$, we have $\mathrm{Out}(x) \subset S_{k+1} \cup \cdots \cup S_{k+m-1}$.

(A2) There is a graph automorphism $\phi$ of $G$ that maps $S_k$ to $S_{k+m}$ for every $k$, and such that $\mathrm{Out}(x) = \mathrm{In}(\phi(x))$ for all $x$.

Example: Consider $\mathbb{Z}^d$ with $\mathrm{Out}(x) = \{x \pm e_i + e_d : 1 \leq i \leq d-1\}$. Here $|\,\mathrm{Out}(x)| = 2(d-1)$. Since any step preserves parity, it is natural to restrict to the set of even sites $\mathbb{Z}^d_{\text{even}} := \{x \in \mathbb{Z}^d : \sum x_i \text{ is even}\}$.

In two dimensions, the game is isomorphic to the original game on $\mathbb{Z}^2$. For general $d$, conditions (A1) and (A2) hold with $m = 2$ if we set $S_k = \{x \in \mathbb{Z}^d_{\text{even}} : x_d = k\}$ and $\phi(x) = x + 2e_d$.

To obtain the doubling graph, consider $D_k = S_k \cup S_{k+1}$ with an edge between $x \in S_k$ and $y \in S_{k+1}$ whenever $y \in \mathrm{Out}(x)$. This gives a graph isomorphic to the standard cubic lattice $\mathbb{Z}^{d-1}$.

If there exist multiple Gibbs distributions for the hard-core model on $\mathbb{Z}^{d-1}$ with activity $\lambda$, then the percolation game on $G$ with $p = 1/(1+\lambda)$ has positive probability of a draw from any vertex.

Example: $\mathbb{Z}^d$ with $\mathrm{Out}(x) = \{x \pm e_i + e_d : 1 \le i \le d-1\}$. Here $|\,\mathrm{Out}(x)| = 2(d-1)$. We can take $m = 2$, $\phi(x) = x + 2e_d$, and obtain a doubling graph $D$ isomorphic to the standard cubic lattice $\mathbb{Z}^{d-1}$.

If there exist multiple Gibbs distributions for the hard-core model on $\mathbb{Z}^{d-1}$ with activity $\lambda$, then the percolation game on $G$ with $p = 1/(1+\lambda)$ has positive probability of a draw from any vertex.

Example: $\mathrm{Out}(x) = \{x \pm e_1 \pm e_2 \cdots \pm e_{d-1} + e_d\}$, so $|\,\mathrm{Out}(x)| = 2^{d-1}$. Taking $m = 2$ and $\phi(x) = x + 2e_d$, then $D$ is the body-centred cubic lattice in $d-1$ dimensions.

Example: $\mathrm{Out}(x) = \left\{x + \sum_{i \in S} e_i : \emptyset \subset S \subset \{1, \ldots, d\}\right\}$. (A move increments at least one, and not all, coordinates by 1.) Then $|\,\mathrm{Out}(x)| = 2^d - 2$. With $m = d$ and $\phi(x) = x + e_1 + \cdots + e_d$, the graph $D$ is $(d-1)$-dimensional and $d$-partite. For $d = 3$ it is the triangular lattice.

Other examples lead to the hexagonal lattice and the diamond cubic graph.

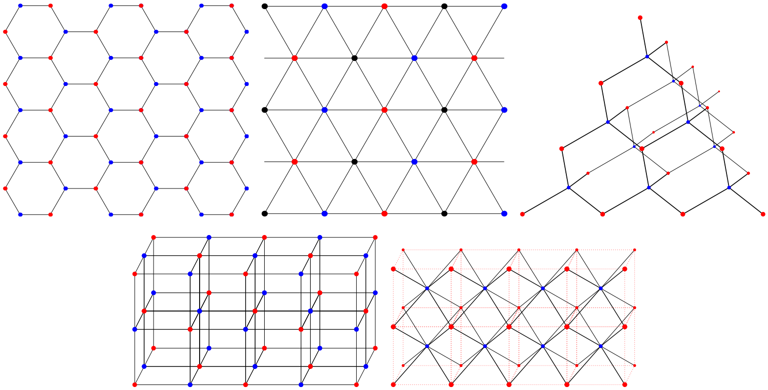
Example: $\mathbb{Z}^d$ with $\mathrm{Out}(x) = \{x \pm e_i + e_d : 1 \leq i \leq d-1\}$. Here $|\mathrm{Out}(x)| = 2(d-1)$. We can take $m = 2$, $\phi(x) = x + 2e_d$, and obtain a doubling graph $D$ isomorphic to the standard cubic lattice $\mathbb{Z}^{d-1}$.

Example: $\mathrm{Out}(x) = \{x \pm e_1 \pm e_2 \cdots \pm e_{d-1} + e_d\}$, so $|\mathrm{Out}(x)| = 2^{d-1}$. Taking $m = 2$ and $\phi(x) = x + 2e_d$, then $D$ is the body-centred cubic lattice in $d-1$ dimensions.

Example: $\mathrm{Out}(x) = \left\{x + \sum_{i \in S} e_i : \emptyset \subset S \subset \{1, \ldots, d\}\right\}$. (A move increments at least one, and not all, coordinates by 1.) Then $|\mathrm{Out}(x)| = 2^d - 2$. With $m = d$ and $\phi(x) = x + e_1 + \cdots + e_d$, the graph $D$ is $(d-1)$-dimensional and $d$-partite. For $d = 3$ it is the triangular lattice.

---

Non-example!!: standard oriented $\mathbb{Z}^d$ for $d \geq 3$, that is, $\mathrm{Out}(x) = \{x + e_1, x + e_2, \ldots, x + e_d\}$.



When $D$ is any of these graphs (and in all dimensions for the lower two) a contour argument can be used to show that there are multiple hard-core Gibbs measures. Dobrushin ('65), Heilmann ('74), Runnels ('75), Galvin + Kahn ('04).

## Open questions

- 
- How to prove existence of draws without link to hard-core model? e.g. standard oriented $\mathbb{Z}^d$, $d \geq 3$? or $q > 0$?
- How to prove non-existence of draws in two dimensions when there is no similar two-type cellular automaton structure (e.g. certain *misère* games).
- When draws are possible, is there monotonicity in $p$ and $q$? (cf. hard-core model)
- Extending the local weighting method to give ergodicity for a more general class of two-dimensional PCA.

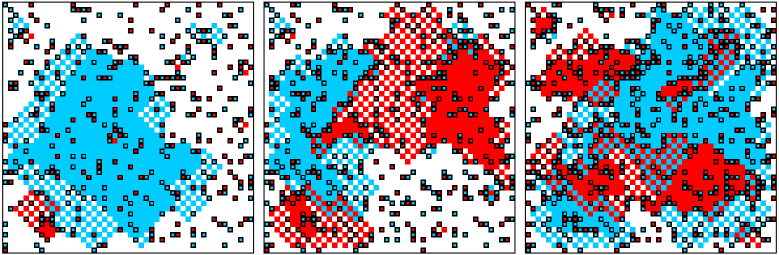
## Games on undirected lattices

Two-player game on the undirected graph $\mathbb{Z}^d$ with nearest-neighbour edges.

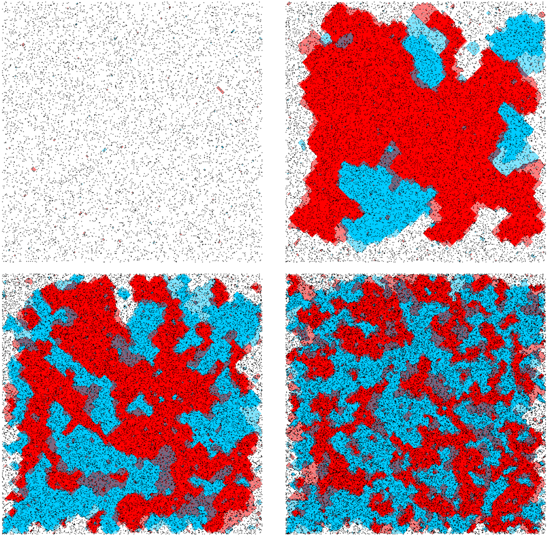
Remove each vertex independently with some probability $p$.

A token starts from a given vertex $v$. A turn consists of moving the token along an edge of the graph to a new vertex which has not been visited before. A player that cannot move loses the game.

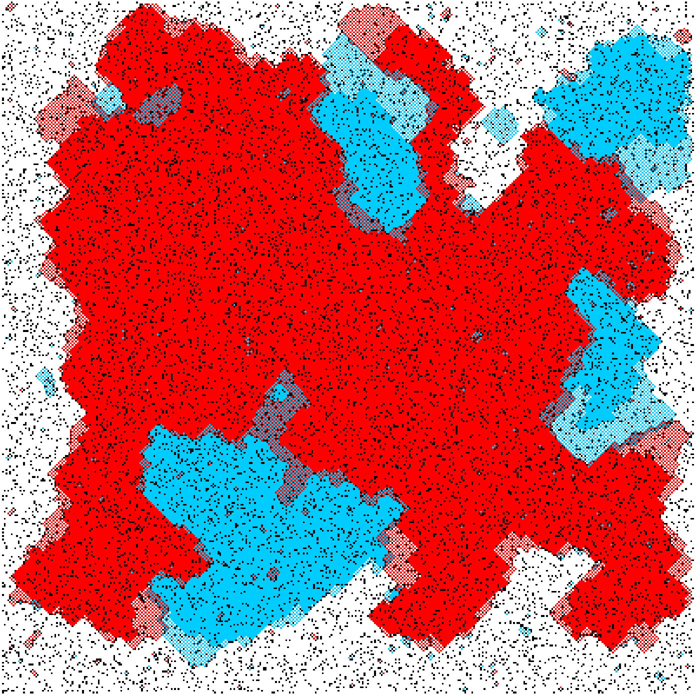
When does the game terminate with optimal play w.p.1, and when is there positive probability of a draw?



Outcomes of the game on a square of side-length $n = 50$, with $p = 0.15$, 0.2 and 0.25 – the game is declared a draw if the token leaves the square. Closed vertices are outlined in black. Wins for "even player" are blue and wins for "odd player" are red. Draws are white.



The same for $n = 400$, with $p = q = 0.05$, $0.1$, $0.15$ and $0.2$.



Some results in Basu, Holroyd, M. and Wästlund (2016), but mostly for asymmetric cases where the closure probability is different for odd and even sites.

Interesting connections to bootstrap percolation processes, and to combinatorial questions involving maximum-size matchings.

Conjecture (for the symmetric case): perhaps draws occur for $d \geq 3$ but not for $d = 2$, as in directed case?

## Games on Galton-Watson trees

- Probabilities of win/loss/draw can be expressed via generating function recursions
- Interesting examples of continuous and discontinuous phase transitions (compare cases of survival of a branching processes, and existence of complete binary tree including the root within the process).
- Relation to leaf-stripping processses (e.g. Karp-Sipser algorithm for finding large matchings or independent sets in a graph).
- Relation to endogeny of recursive distributional equations, and to local / non-local behaviour in optimisation problems. Games on other locally tree-like "boards", e.g. sparse random graphs. (Erdős-Renyi, configuration model, ...).