Balanced truncation model reduction: algorithms and applications
Part II

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Outline

Part I (Tuesday)

- Model order reduction problem
- Balanced truncation model reduction
- Balancing-related model reduction techniques

Part II (today)

- Balanced truncation for differential-algebraic equations
- Balanced truncation for second-order systems

Part III (Thursday)

- Balanced truncation for parametric systems
- Related topics and open problems

Balanced truncation

Idea: Balance the system (A, B, C, D) and truncate the states corresponding to small Hankel singular values

Algorithm:

1. Solve the Lyapunov equations

$$A\,X + X A^T = -BB^T, \qquad A^T Y + Y\,A = -C^T C$$

for $X \approx \widetilde{R}\widetilde{R}^T$ and $Y \approx \widetilde{L}\widetilde{L}^T$.

2. Compute the SVD $\widetilde{L}^T \widetilde{R} = [\,U_1,\; U_2\,]\begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}[\,V_1,\; V_2\,]^T$,

with $\Sigma_1 = \mathrm{diag}(\xi_1, \ldots, \xi_\ell)$, $\Sigma_2 = \mathrm{diag}(\xi_{\ell+1}, \ldots, \xi_n)$.

3. $(\,\widetilde{A},\; \widetilde{B},\; \widetilde{C},\; \widetilde{D}\,) = (\,W^T A T,\; W^T B,\; CT,\; D\,)$ with

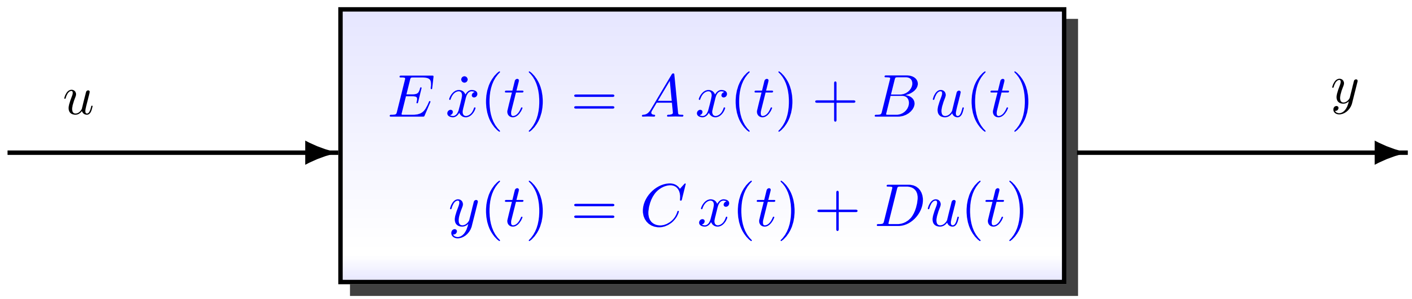
$$W = \widetilde{L} U_1 \Sigma_1^{-1/2} \in \mathbb{R}^{n\times\ell}, \qquad T = \widetilde{R} V_1 \Sigma_1^{-1/2} \in \mathbb{R}^{n\times\ell}.$$

Outline

- Model order reduction problem
- Balanced truncation model reduction
- Balancing-related model reduction techniques
- Balanced truncation for differential-algebraic equations
  - properties of DAEs
  - proper and improper Gramians
  - proper and improper Hankel singular values
  - numerical methods for projected Lyapunov equations
- Balanced truncation for second-order systems
- Balanced truncation for parametric systems
- Related topics and open problems

Linear DAE control systems

Time domain representation



where $E,\, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$,

$\lambda E - A$ is regular ($\det(\lambda E - A) \not\equiv 0$).

Frequency domain representation

Laplace transform: $u(t) \mapsto \boldsymbol{u}(s)$, $y(t) \mapsto \boldsymbol{y}(s)$

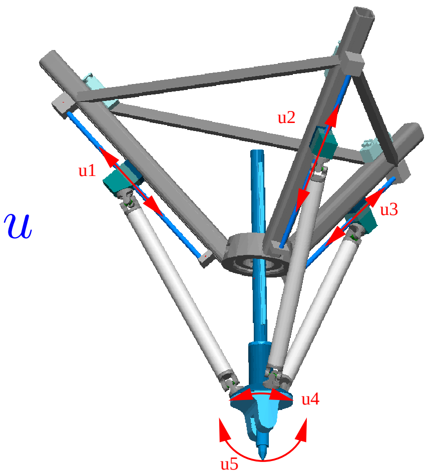
$$\hookrightarrow \quad \boldsymbol{y}(s) = \big(C(sE - A)^{-1}B + D\big)\boldsymbol{u}(s) + C(sE - A)^{-1}Ex(0)$$

with the transfer function $\boldsymbol{G}(s) = C(sE - A)^{-1}B + D$

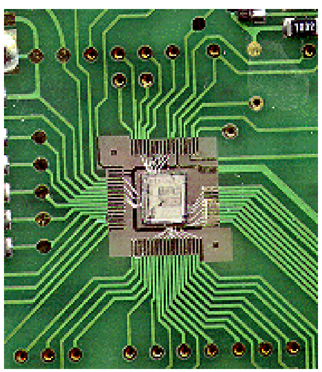
Applications

- Multibody systems with constraints

$$\begin{bmatrix} I & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{v}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ K & D & -G^T \\ G & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \\ B_3 \end{bmatrix} u$$

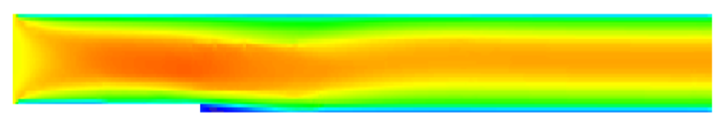


- Electrical circuits

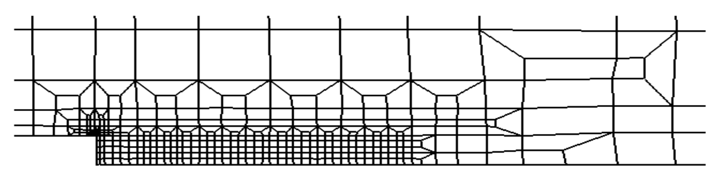


$$\begin{bmatrix} A_C \mathcal{C} A_C^T & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{e}} \\ \dot{\mathbf{j}}_L \\ \dot{\mathbf{j}}_V \end{bmatrix} = \begin{bmatrix} -A_R R^{-1} A_R^T & -A_L^T & -A_V^T \\ A_L^T & 0 & 0 \\ A_V^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{j}_L \\ \mathbf{j}_V \end{bmatrix} - \begin{bmatrix} A_I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} i_V \\ v_I \end{bmatrix}$$

- Semidiscretized Stokes equation



$$\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$



DAEs are not ODEs! [Petzold'82]

- DAEs may have no solutions or solution may be nonunique
- Initial conditions $x(0) = x_0$ should be consistent
  $\rightsquigarrow$ distributional solutions
- Control $u(t)$ should be sufficiently smooth
  $\rightsquigarrow$ distributional solutions
- Drift off effects may occur in the numerical solution
- Index concepts:
  differentiation index, geometric index, perturbation index, strangeness index, structural index, tractability index, unsolvability index, ...

Model reduction problem

Given a large-scale DAE control system

$$u \longrightarrow \boxed{\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}} \longrightarrow y$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$,

find a reduced-order model

$$u \longrightarrow \boxed{\begin{aligned} \widetilde{E}\,\dot{\widetilde{x}}(t) &= \widetilde{A}\,\widetilde{x}(t) + \widetilde{B}\,u(t) \\ \widetilde{y}(t) &= \widetilde{C}\,\widetilde{x}(t) + \widetilde{D}\,u(t) \end{aligned}} \longrightarrow \widetilde{y}$$

where $\widetilde{E}, \widetilde{A} \in \mathbb{R}^{\ell\times\ell}$, $\widetilde{B} \in \mathbb{R}^{\ell\times m}$, $\widetilde{C} \in \mathbb{R}^{p\times\ell}$, $\widetilde{D} \in \mathbb{R}^{p\times m}$, $\ell \ll n$.

Decoupling of DAEs

Weierstraß canonical form:

$$E = T_l \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} T_r, \qquad A = T_l \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} T_r,$$

where J – Jordan block ($\lambda_j(J)$ are finite eigenvalues of $\lambda E - A$),
N – nilpotent ($N^{\nu-1} \neq 0$, $N^{\nu} = 0$ $\rightsquigarrow$ ν is index of $\lambda E - A$).

Slow subsystem

$$\dot{x}_1(t) = J\, x_1(t) + B_1 u(t)$$
$$y_1(t) = C_1 x_1(t)$$
$$\Rightarrow \quad x_1(t) = e^{Jt} x_1(0) + \int_0^t e^{J(t-\tau)} B_1 u(\tau)\, d\tau$$

Fast subsystem

$$N\dot{x}_2(t) = x_2(t) + B_2 u(t)$$
$$y_2(t) = C_2 x_2(t) + D\, u(t)$$
$$\Rightarrow \quad x_2(t) = -\sum_{k=0}^{\nu-1} N^k B_2 u^{(k)}(t)$$

Idea: define the controllability and observability Gramains for each subsystem and reduce the subsystems separately.

Proper and improper Gramians

Consider the projectors

$$P_r = T_r^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_r, \qquad P_l = T_l \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_l^{-1}, \qquad \begin{aligned} Q_r &= I - P_r, \\ Q_l &= I - P_l. \end{aligned}$$

- The proper controllability and observability Gramians solve the projected continuous-time Lyapunov equations

$$E\,\mathcal{G}_{pc}A^T + A\,\mathcal{G}_{pc}E^T = -P_l B B^T P_l^T, \qquad \mathcal{G}_{pc} = P_r \mathcal{G}_{pc} P_r^T,$$
$$E^T \mathcal{G}_{po}\, A + A^T \mathcal{G}_{po} E = -P_r^T C^T C P_r, \qquad \mathcal{G}_{po} = P_l^T \mathcal{G}_{po} P_l.$$

- The improper controllability and observability Gramians solve the projected discrete-time Lyapunov equations

$$A\,\mathcal{G}_{ic}\,A^T - E\,\mathcal{G}_{ic}E^T = \; Q_l B B^T Q_l^T, \qquad \mathcal{G}_{ic} = Q_r \mathcal{G}_{ic} Q_r^T,$$
$$A^T \mathcal{G}_{io}\, A - E^T \mathcal{G}_{io}\, E = \; Q_r^T C^T C Q_r, \qquad \mathcal{G}_{io} = Q_l^T \mathcal{G}_{io} Q_l.$$

Balanced truncation for DAEs

- $\boldsymbol{G} = (\,E,\ A,\ B,\ C,\ D\,)$ is balanced, if the Gramians satisfy

$$\mathcal{G}_{pc} = \mathcal{G}_{po} = \begin{bmatrix} \Sigma & \\ & 0 \end{bmatrix}, \qquad \mathcal{G}_{ic} = \mathcal{G}_{io} = \begin{bmatrix} 0 & \\ & \Theta \end{bmatrix}$$

with $\Sigma = \mathrm{diag}(\xi_1, \ldots, \xi_{n_f})$ and $\Theta = \mathrm{diag}(\theta_1, \ldots, \theta_{n_\infty})$.

- $\xi_j = \sqrt{\lambda_j(\mathcal{G}_{pc}E^T\mathcal{G}_{po}E)}$ are the proper Hankel singular values

  $\theta_j = \sqrt{\lambda_j(\mathcal{G}_{ic}A^T\mathcal{G}_{io}A)}$ are the improper Hankel singular values

Idea: balance the system and truncate the states corresponding to small proper and zero improper Hankel singular values.

Example

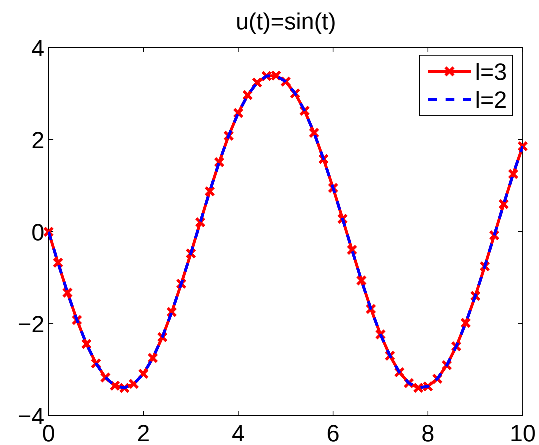
$$N\dot{x}(t) = x(t) + Bu(t)$$
$$y(t) = Cx(t)$$

with $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 0.1 \\ 0 \end{bmatrix}, \quad C^T = \begin{bmatrix} 0.04 \\ 30 \\ 1 \end{bmatrix}$

Improper Hankel singular values $\theta_1 = 3.4,\ \theta_2 = 4.7 \cdot 10^{-6},\ \theta_3 = 0$

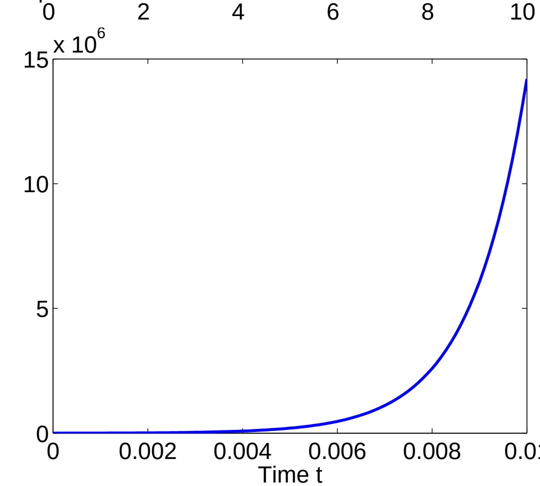
- Reduced-order system: $\ell = 2$

$$\begin{bmatrix} 1.2 & 1.2 \\ -1.2 & -1.2 \end{bmatrix} \dot{\tilde{x}}(t) = \begin{bmatrix} 10^3 & 0 \\ 0 & 10^3 \end{bmatrix} \tilde{x}(t) + \tilde{B}u(t)$$
$$\tilde{y}(t) = \tilde{C}\,\tilde{x}(t)$$



- Reduced-order system: $\ell = 1$

$$\dot{\tilde{x}}(t) = 850\,\tilde{x}(t) + 1567u(t)$$
$$\tilde{y}(t) = 1.9\,\tilde{x}(t)$$



Balanced truncation for DAEs

1. Solve the projected Lyapunov equations for

$$\mathcal{G}_{pc} = R_p R_p^T, \quad \mathcal{G}_{po} = L_p L_p^T, \quad \mathcal{G}_{ic} = R_i R_i^T, \quad \mathcal{G}_{io} = L_i L_i^T;$$

2. Compute the SVD

$$L_p^T E R_p = [U_1,\, U_2] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} [V_1,\, V_2]^T;$$

3. Compute the SVD

$$L_i^T A R_i = [U_3,\, U_4] \begin{bmatrix} \Theta & \\ & 0 \end{bmatrix} [V_3,\, V_4]^T;$$

4. $(\,\tilde{E},\, \tilde{A},\, \tilde{B},\, \tilde{C},\, \tilde{D}\,) = (\,W^T E T,\, W^T A T,\, W^T B,\, CT,\, D\,)$ with

$$W = [\,L_p U_1 \Sigma_1^{-1/2},\, L_i U_3 \Theta^{-1/2}],\quad T = [\,R_p V_1 \Sigma_1^{-1/2},\, R_i V_3 \Theta^{-1/2}\,].$$

Balanced truncation: properties

- Asymptotic stability is preserved
- Error bound:
  - $\boldsymbol{G}(s) = C(sE-A)^{-1}B + D = \boldsymbol{G}_{\rm sp}(s) + \boldsymbol{P}(s)$,

    where $\boldsymbol{G}_{\rm sp}(s) = C_1(sI-J)^{-1}B_1$ is strictly proper,

    $$\boldsymbol{P}(s) = C_2(sN-I)^{-1}B_2 + D = -\sum_{k=0}^{\nu-1} C_2N^kB_2s^k + D$$
  - $\widetilde{\boldsymbol{G}}(s) = \widetilde{C}(s\widetilde{E}-\widetilde{A})^{-1}\widetilde{B} + \widetilde{D} = \widetilde{\boldsymbol{G}}_{\rm sp}(s) + \boldsymbol{P}(s)$

    $$\hookrightarrow \quad \|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathcal{H}_\infty} \le 2(\xi_{\ell_f} + \ldots + \xi_{n_f})$$
- $\mathrm{Index}(\widetilde{E}, \widetilde{A}) \le \mathrm{Index}(E, A)$

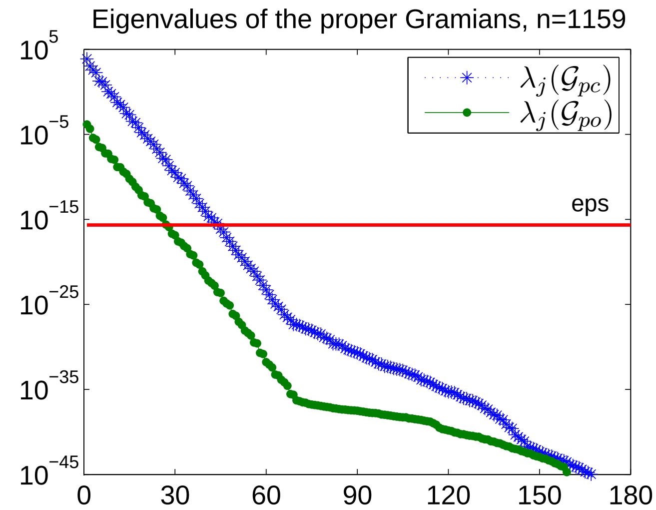
Computing the Gramians

- Instead of the proper Gramians compute their low-rank approximations

$$\mathcal{G}_{pc} \approx \tilde{R}_p \tilde{R}_p^T \quad \text{and} \quad \mathcal{G}_{po} \approx \tilde{L}_p \tilde{L}_p^T$$

with $\tilde{R}_p \in \mathbb{R}^{n \times r_{pc}}$, $\tilde{L}_p \in \mathbb{R}^{n \times r_{po}}$, $r_{pc}, r_{po} \ll n$

$\hookrightarrow$ use the generalized ADI method [St.'08]



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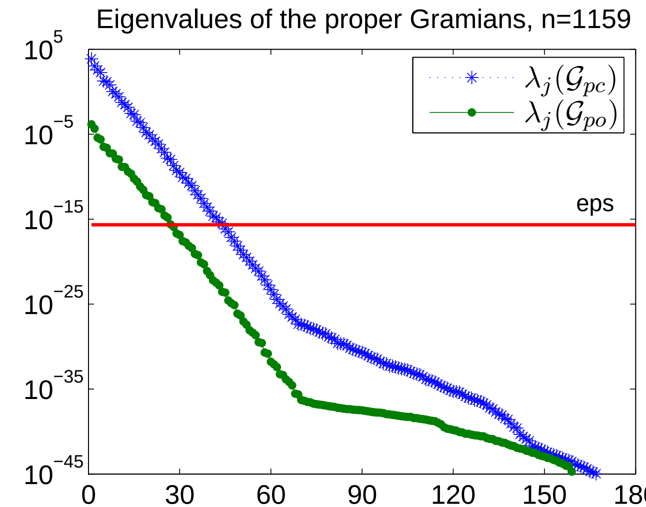


- Since $r_{ic} = \mathrm{rank}\,(\mathcal{G}_{ic}) \le \nu m$ and $r_{io} = \mathrm{rank}\,(\mathcal{G}_{io}) \le \nu p$, compute the full-rank factors of the improper Gramians

$$\mathcal{G}_{ic} = R_i R_i^T, \; R_i \in \mathbb{R}^{n\times r_{ic}} \quad \text{and} \quad \mathcal{G}_{io} = L_i L_i^T, \; L_i \in \mathbb{R}^{n\times r_{io}}$$

$\hookrightarrow$ use the generalized Smith method [St.'08]

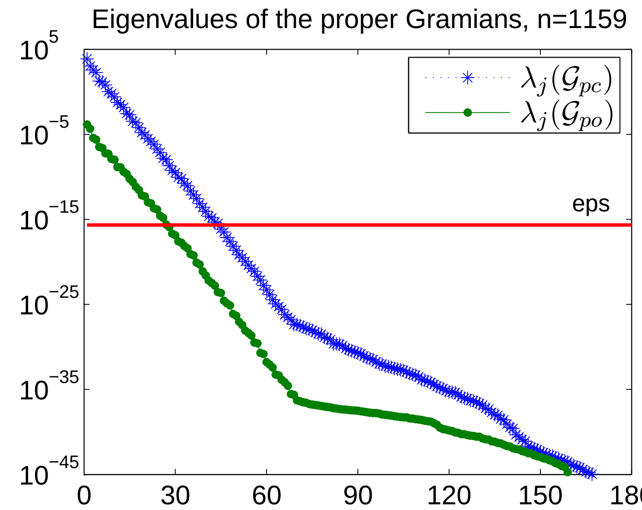
Computing the Gramians

- Instead of the proper Gramians compute their low-rank approximations

$$\mathcal{G}_{pc} \approx \tilde{R}_p \tilde{R}_p^T \quad \text{and} \quad \mathcal{G}_{po} \approx \tilde{L}_p \tilde{L}_p^T$$

with $\tilde{R}_p \in \mathbb{R}^{n \times r_{pc}}$, $\tilde{L}_p \in \mathbb{R}^{n \times r_{po}}$, $r_{pc}, r_{po} \ll n$

$\hookrightarrow$ use the generalized ADI method [St.'08]



- Since $r_{ic} = \operatorname{rank}(\mathcal{G}_{ic}) \le \nu m$ and $r_{io} = \operatorname{rank}(\mathcal{G}_{io}) \le \nu q$, compute the full-rank factors of the improper Gramians

$$\mathcal{G}_{ic} = R_i R_i^T, \; R_i \in \mathbb{R}^{n \times r_{ic}} \quad \text{and} \quad \mathcal{G}_{io} = L_i L_i^T, \; L_i \in \mathbb{R}^{n \times r_{io}}$$

$\hookrightarrow$ use the generalized Smith method [St.'08]

- Projectors P_r and P_l are required

$\hookrightarrow$ exploit the structure of the matrices E and A

Computing the projectors

[✓] semi-explicit systems (index 1) [St.'08]

$$E = \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

[✓] Stokes-like systems (index 2)

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

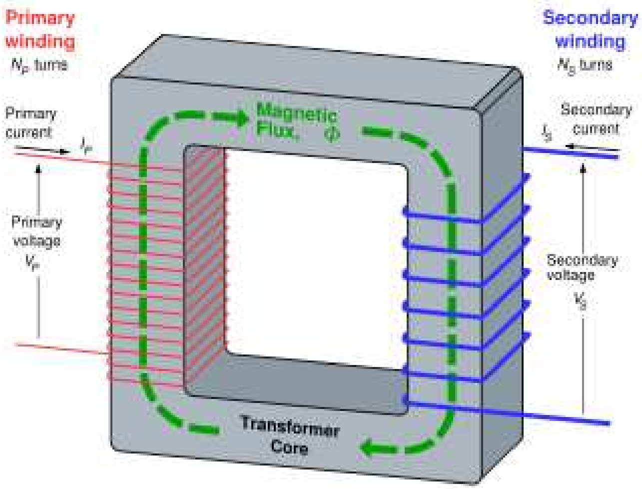
[✓] mechanical systems (index 3)

$$E = \begin{bmatrix} I & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I & 0 \\ D & K & -G^T \\ G & 0 & 0 \end{bmatrix}$$

[✓] electrical circuits (index 1 and 2) [Reis/St.'10,'11]

Remark: For some problems, the explicit computation of the projectors can be avoided [Heinkenschloss/Sorensen/Sun'08, Freitas/Rommes/Martins'08]

Example: one-phase transformer



- Mathematical model

$$
\begin{aligned}
\sigma \frac{\partial A}{\partial t} + \nabla \times (\nu_{\mathrm{ir}} \nabla \times A) &= 0 && \text{in} \quad \Omega_{\mathrm{ir}} \times (0,T) \\
\nabla \times (\nu_{ca} \nabla \times A) &= \omega\, i && \text{in} \quad \Omega_{\mathrm{c}} \cup \Omega_{\mathrm{a}} \times (0,T) \\
\int_\Omega \omega^T \frac{\partial}{\partial t} A\, dz + R\, i &= u && \text{in} \quad (0,T) \\
A \times n &= 0 && \text{on} \quad \partial\Omega \times (0,T) \\
A &= A_0 && \text{in} \quad \Omega_{\mathrm{ir}}
\end{aligned}
$$

- FEM model

$$
\begin{bmatrix} M_{11} & 0 & 0 \\ 0 & 0 & 0 \\ X_1^T & X_2^T & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} a_1 \\ a_2 \\ i \end{bmatrix} = \begin{bmatrix} -K_{11} & -K_{12} & X_1 \\ -K_{12}^T & -K_{22} & X_2 \\ 0 & 0 & -R \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} u
$$

$$
y = i
$$

Example: one-phase transformer

Transform the DAE into the ODE form [Kerler-Back/St.'17]

$$\hat{E}\,\dot{\hat{x}} = \hat{A}\,\hat{x} + \hat{B}\,u$$
$$y = \hat{C}\,\hat{x}$$

with

$$\hat{E} = \begin{bmatrix} M_{11} + X_1 R^{-1} X_1^T & X_1 R^{-1} X_2^T Z \\ Z^T X_2 R^{-1} X_1^T & Z^T X_2 R^{-1} X_2^T Z \end{bmatrix} > 0, \qquad \hat{x} = \begin{bmatrix} a_1 \\ Z^T a_2 \end{bmatrix} \in \mathbb{R}^{n_d},$$

$$\hat{A} = -\begin{bmatrix} K_{11} & K_{12} Z \\ Z^T K_{12}^T & Z^T K_{22} Z \end{bmatrix} + \begin{bmatrix} K_{12} \\ Z^T K_{22} \end{bmatrix} Y \left(Y^T K_{22} Y\right)^{-1} Y^T \left[\, K_{12}^T,\ K_{22} Z \,\right] < 0,$$

$$\hat{B} = \begin{bmatrix} X_1 \\ Z^T X_2 \end{bmatrix} R^{-1}, \qquad \operatorname{im} Y = \ker X_2^T, \qquad Z = X_2 (X_2^T X_2)^{-1/2},$$

$$\hat{C} = (X_2^T X_2)^{-1} X_2^T \left(I - K_{22} Y (Y^T K_{22} Y)^{-1} Y^T\right) \left[\, K_{12}^T,\ K_{22} Z \,\right] = -\hat{B}^T \hat{E}^{-1} \hat{A}.$$

Example: one-phase transformer

Goal: solve $(\hat{A}+\tau\hat{E})z = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ with

$$\hat{E} = \begin{bmatrix} M_{11}+X_1R^{-1}X_1^T & X_1R^{-1}X_2^TZ \\ Z^TX_2R^{-1}X_1^T & Z^TX_2R^{-1}X_2^TZ \end{bmatrix}, \qquad Z = X_2(X_2^TX_2)^{-1/2}$$

$$\hat{A} = -\begin{bmatrix} K_{11} & K_{12}Z \\ Z^TK_{21} & Z^TK_{22}Z \end{bmatrix} + \begin{bmatrix} K_{12} \\ Z^TK_{22} \end{bmatrix} Y\left(Y^TK_{22}Y\right)^{-1}Y^T[\,K_{21},\ K_{22}Z\,]$$

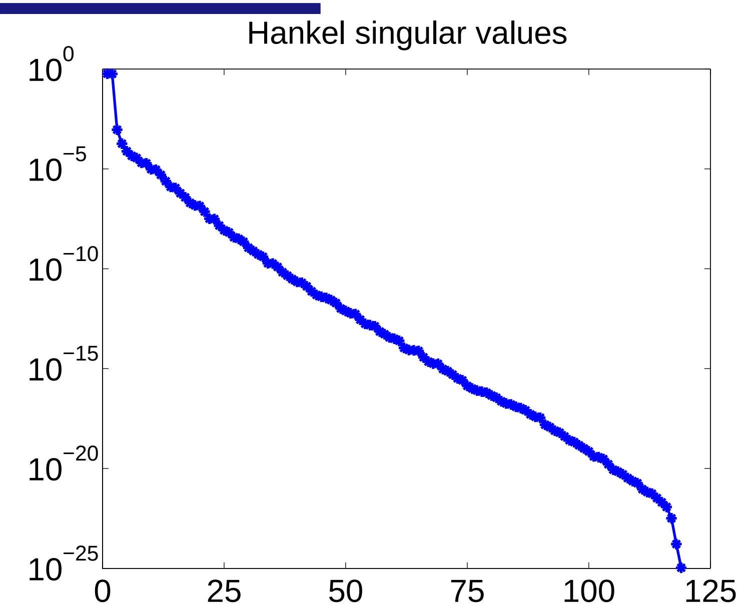
- Solve $\begin{bmatrix} \tau M_{11}-K_{11} & -K_{12} & X_1 \\ -K_{21} & -K_{22} & X_2 \\ \tau X_1^T & \tau X_2^T & -R \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ Zv_2 \\ 0 \end{bmatrix}$.

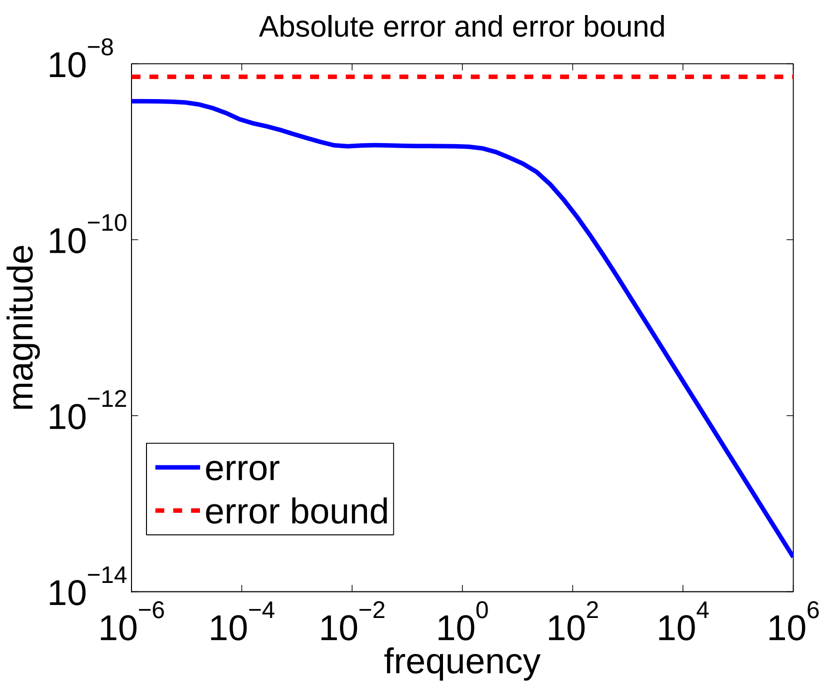
- Compute $z = \begin{bmatrix} z_1 \\ Z^Tz_2 \end{bmatrix}$.

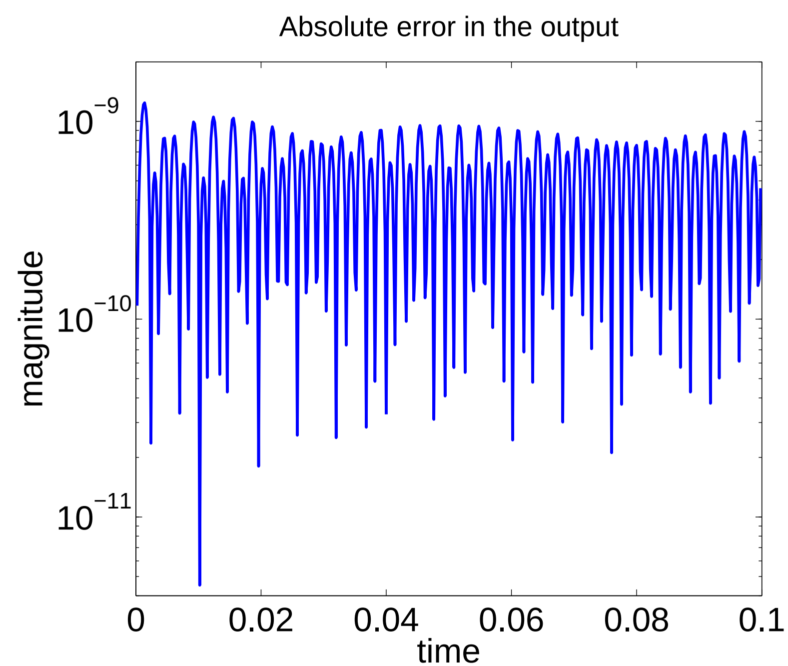
Note: Y is not required!

Example: one-phase transformer

- $n = 17733$, $n_d = 7202$, $n_a = 12531$, $m = 2$
- $X \approx \widetilde{R}\widetilde{R}^T, \quad \widetilde{R} \in \mathbb{R}^{n_d \times 126}$
- Reduced system: $r = 29$
- $t_{orig} = 180.74$ sec, $t_{red} = 0.06$ sec







Outline

- Model order reduction problem
- Balanced truncation model reduction
- Balancing-related model reduction techniques
- Balanced truncation for differential-algebraic equations
- Balanced truncation for second-order systems
  - structure-preserving model reduction
  - position and velocity Gramians
  - position and velocity Hankel singular values
  - second-order balanced truncation
- Balanced truncation for parametric systems
- Related topics and open problems

Second-order control systems

Time domain representation

$u \rightarrow$

$$\begin{aligned} M\ddot{q}(t) + D\dot{q}(t) + Kq(t) &= B_2u(t) \\ C_2\dot{q}(t) + C_1q(t) &= y(t) \end{aligned}$$

$\rightarrow y$

where $M, D, K \in \mathbb{R}^{n\times n}$, $B_2 \in \mathbb{R}^{n\times m}$, $C_1, C_2 \in \mathbb{R}^{p\times n}$,

$u \in \mathbb{R}^m$ – input, $q \in \mathbb{R}^n$ – state, $y \in \mathbb{R}^p$ – output.

Frequency domain representation

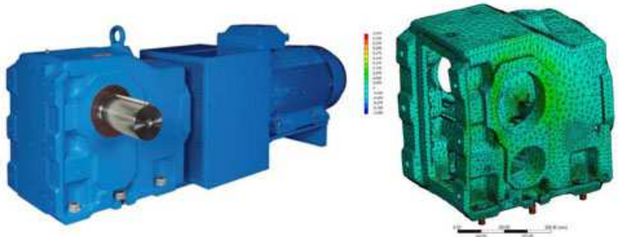
Laplace transform: $u(t) \mapsto \boldsymbol{u}(s)$, $y(t) \mapsto \boldsymbol{y}(s)$ ($q(0) = 0$, $\dot{q}(0) = 0$)

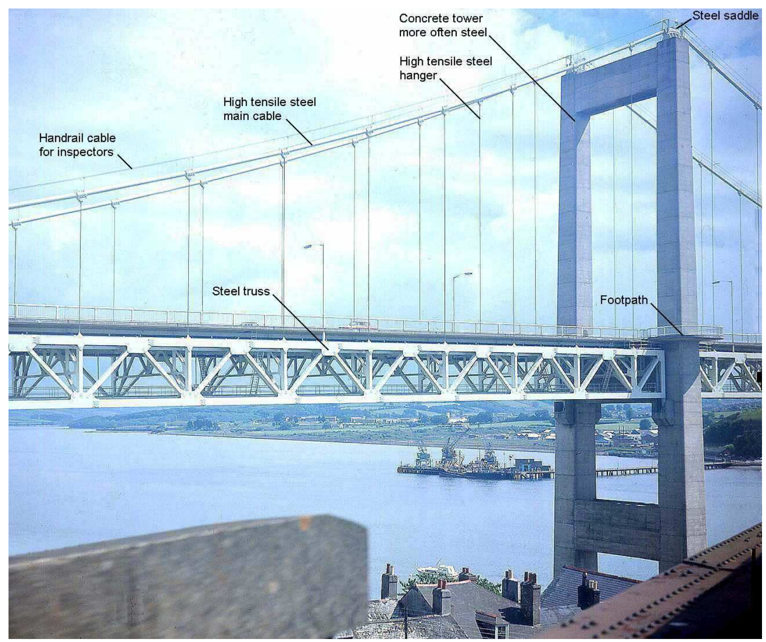
$\hookrightarrow$ $\boldsymbol{y}(s) = (C_1 + sC_2)(s^2M + sD + K)^{-1}B_2\,\boldsymbol{u}(s) = \boldsymbol{G}(s)\boldsymbol{u}(s)$

with $\boldsymbol{G}(s) = (C_1 + sC_2)(s^2M + sD + K)^{-1}B_2$

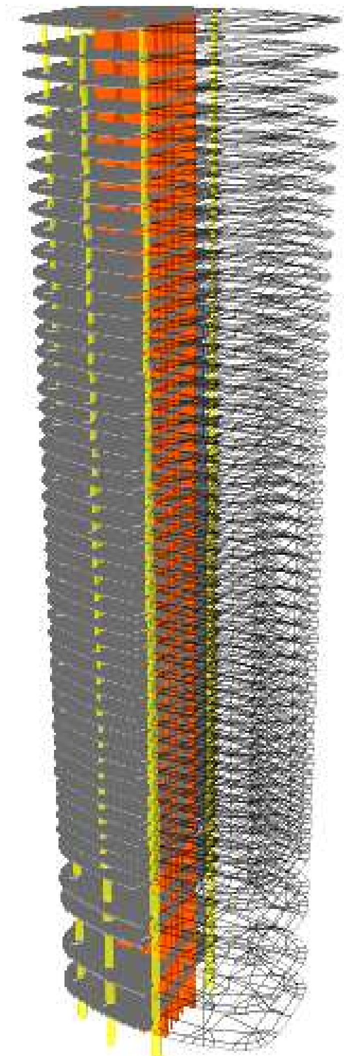
Applications

- Vibration and acoustic systems
  (automotive industry, rotor dynamics, machine tools,
  civil and earthquake engineering, ...)
- Control of large flexible structures
- MEMS devices design





The Tamar Bridge in England



50-Storey Tower in Kuala Lumpur, Malaysia

Model reduction problem

Given a second-order system

$$\begin{aligned} M\ddot q(t) + D\dot q(t) + Kq(t) &= B_2\,u(t) \\ C_2\dot q(t) + C_1 q(t) &= y(t) \end{aligned}$$

with $M, D, K \in \mathbb{R}^{n\times n}$, $B_2 \in \mathbb{R}^{n\times m}$, $C_1, C_2 \in \mathbb{R}^{p\times n}$,

find a reduced-order model

$$\begin{aligned} \tilde M\ddot{\tilde q}(t) + \tilde D\dot{\tilde q}(t) + \tilde K\tilde q(t) &= \tilde B_2\,u(t) \\ \tilde C_2\dot{\tilde q}(t) + \tilde C_1 \tilde q(t) &= \tilde y(t) \end{aligned}$$

with $\tilde M, \tilde D, \tilde K \in \mathbb{R}^{\ell\times \ell}$, $\tilde B_2 \in \mathbb{R}^{\ell\times m}$, $\tilde C_1, \tilde C_2 \in \mathbb{R}^{p\times \ell}$ and $\ell \ll n$.

Structure-preserving model reduction

$$\begin{aligned} M\ddot{q}(t) + D\dot{q}(t) + Kq(t) &= B_2u(t) \\ C_2\dot{q}(t) + C_1q(t) &= y(t) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \tilde{M}\,\ddot{\tilde{q}}(t) + \tilde{D}\,\dot{\tilde{q}}(t) + \tilde{K}\,\tilde{q}(t) &= \tilde{B}_2u(t) \\ \tilde{C}_2\dot{\tilde{q}}(t) + \tilde{C}_1\tilde{q}(t) &= \tilde{y}(t) \end{aligned}$$

Second-order $\Rightarrow$ first-order

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = B_2u(t)$$
$$C_2\dot{q}(t) + C_1q(t) = y(t)$$

$$\tilde{M}\,\ddot{\tilde{q}}(t) + \tilde{D}\,\dot{\tilde{q}}(t) + \tilde{K}\,\tilde{q}(t) = \tilde{B}_2u(t)$$
$$\tilde{C}_2\dot{\tilde{q}}(t) + \tilde{C}_1\tilde{q}(t) = \tilde{y}(t)$$

$$\Downarrow$$

$$\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned}$$

$$E = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \qquad C = [\,C_1,\ C_2\,]$$

or

$$E = \begin{bmatrix} D & M \\ M & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \qquad B = \begin{bmatrix} B_2 \\ 0 \end{bmatrix}, \qquad C = [\,C_1,\ C_2\,]$$

$$\hookrightarrow \quad \boldsymbol{G}(s) = (C_1 + sC_2)(s^2M + sD + K)^{-1}B_2 = C(sE - A)^{-1}B$$

Model reduction of the first-order system

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = B_2u(t)$$
$$C_2\dot{q}(t) + C_1q(t) = y(t)$$

$$\tilde{M}\,\ddot{\tilde{q}}(t) + \tilde{D}\,\dot{\tilde{q}}(t) + \tilde{K}\,\tilde{q}(t) = \tilde{B}_2u(t)$$
$$\tilde{C}_2\dot{\tilde{q}}(t) + \tilde{C}_1\tilde{q}(t) = \tilde{y}(t)$$

$$\Downarrow$$

$$\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \tilde{E}\,\dot{\tilde{x}}(t) &= \tilde{A}\,\tilde{x}(t) + \tilde{B}\,u(t) \\ \tilde{y}(t) &= \tilde{C}\,\tilde{x}(t) \end{aligned}$$

$$E = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \qquad C = [\,C_1,\; C_2\,]$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\tilde{E} = W^TET \qquad \tilde{A} = W^TAT \qquad \tilde{B} = W^TB \qquad \tilde{C} = CT$$

First-order $\Rightarrow$ second-order

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = B_2u(t)$$
$$C_2\dot{q}(t) + C_1q(t) = y(t)$$

$$\tilde{M}\,\ddot{\tilde{q}}(t) + \tilde{D}\,\dot{\tilde{q}}(t) + \tilde{K}\,\tilde{q}(t) = \tilde{B}_2u(t)$$
$$\tilde{C}_2\dot{\tilde{q}}(t) + \tilde{C}_1\tilde{q}(t) = \tilde{y}(t)$$

$$\Downarrow \qquad\qquad \Uparrow\ ?$$

$$\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \tilde{E}\,\dot{\tilde{x}}(t) &= \tilde{A}\,\tilde{x}(t) + \tilde{B}\,u(t) \\ \tilde{y}(t) &= \tilde{C}\,\tilde{x}(t) \end{aligned}$$

$$E = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad C = [\,C_1,\ C_2\,]$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\tilde{E} = W^TET, \quad \tilde{A} = W^TAT, \quad \tilde{B} = W^TB, \quad \tilde{C} = CT$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\tilde{M} = ?, \quad \tilde{D} = ?, \quad \tilde{K} = ?, \quad \tilde{B}_2 = ?, \quad \tilde{C}_1 = ?, \quad \tilde{C}_2 = ?$$

First-order $\Rightarrow$ second-order

Is it always possible to rewrite a first-order control system as a second-order control system ?

Answer: NO!

But ...

for $W = \begin{bmatrix} W_1 & \\ & W_2 \end{bmatrix}$ and $T = \begin{bmatrix} T_1 & \\ & T_1 \end{bmatrix}$, we have

$$\tilde{E} = W^T E T = \begin{bmatrix} W_1^T T_1 & 0 \\ 0 & W_2^T M T_1 \end{bmatrix}, \quad \tilde{A} = W^T A T = \begin{bmatrix} 0 & W_1^T T_1 \\ -W_2^T K T_1 & -W_2^T D T_1 \end{bmatrix},$$

$$\tilde{B} = W^T B = \left[\, 0, \; (W_2^T B_2)^T \,\right]^T, \quad \tilde{C} = C T = \left[\, C_1 T_1, \; C_2 T_1 \,\right]$$

$$\hookrightarrow \quad \tilde{\boldsymbol{G}} = (\, W_2^T M T_1, \; W_2^T D T_1, \; W_2^T K T_1, \; W_2^T B_2, \; C_1 T_1, \; C_2 T_1 \,)$$

Position and velocity Gramians

$$AXE^T + EXA^T = -BB^T \qquad A^TYE + E^TYA = -C^TC$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$X = \begin{bmatrix} X_p & X_{12} \\ X_{12}^T & X_v \end{bmatrix}, \qquad Y = \begin{bmatrix} Y_p & Y_{12} \\ Y_{12}^T & Y_v \end{bmatrix}$$

X_p – position controllability Gramian

X_v – velocity controllability Gramian

Y_p – position observability Gramian

Y_v – velocity observability Gramian

[Meyer/Srinivasan'96]

Hankel singular values

First-order system:

$$\xi_j = \sqrt{\lambda_j(XE^TYE)}$$ – Hankel singular values

Second-order system:

$$\xi_j^p = \sqrt{\lambda_j(X_pY_p)}$$ – position singular values

$$\xi_j^v = \sqrt{\lambda_j(X_vM^TY_vM)}$$ – velocity singular values

$$\xi_j^{pv} = \sqrt{\lambda_j(X_pM^TY_vM)}$$ – position-velocity singular values

$$\xi_j^{vp} = \sqrt{\lambda_j(X_vY_p)}$$ – velocity-position singular values

[Reis/St.'08]

Balancing

First-order system:

$(E,\ A,\ B,\ C)$ is balanced, if $X = Y = \operatorname{diag}(\xi_1, \ldots, \xi_{2n})$.

Second-order system:

$(M,\ K,\ D,\ B_2,\ C_1,\ C_2)$ is position balanced, if

$$X_p = Y_p = \operatorname{diag}(\xi_1^p, \ldots, \xi_n^p).$$

$(M,\ K,\ D,\ B_2,\ C_1,\ C_2)$ is velocity balanced, if

$$X_v = Y_v = \operatorname{diag}(\xi_1^v, \ldots, \xi_n^v).$$

$(M,\ K,\ D,\ B_2,\ C_1,\ C_2)$ is position-velocity balanced, if

$$X_p = Y_v = \operatorname{diag}(\xi_1^{pv}, \ldots, \xi_n^{pv}).$$

$(M,\ K,\ D,\ B_2,\ C_1,\ C_2)$ is velocity-position balanced, if

$$X_v = Y_p = \operatorname{diag}(\xi_1^{vp}, \ldots, \xi_n^{vp}).$$

Second-order balanced truncation (SOBTp)

1. Compute $X = \begin{bmatrix} X_p & X_{12} \\ X_{12}^T & X_v \end{bmatrix}$, $Y = \begin{bmatrix} Y_p & Y_{12} \\ Y_{12}^T & Y_v \end{bmatrix}$ $\Bigg|$ $X_p = R_p R_p^T, \; X_v = R_v R_v^T$ $Y_p = L_p L_p^T, \; Y_v = L_v L_v^T$

2. Compute the SVD $R_p^T L_p = [\, U_{p1}, \; U_{p2} \,] \begin{bmatrix} \Sigma_{p1} & \\ & \Sigma_{p2} \end{bmatrix} [\, V_{p1}, \; V_{p2} \,]^T$,

   where $\Sigma_{p1} = \text{diag}(\xi_1^p, \ldots, \xi_\ell^p)$ and $\Sigma_{p2} = \text{diag}(\xi_{\ell+1}^p, \ldots, \xi_n^p)$;

3. Compute the SVD $R_v^T M^T L_v = [\, U_{v1}, \; U_{v2} \,] \begin{bmatrix} \Sigma_{v1} & \\ & \Sigma_{v2} \end{bmatrix} [\, V_{v1}, \; V_{v2} \,]^T$,

   where $\Sigma_{v1} = \text{diag}(\xi_1^v, \ldots, \xi_\ell^v)$ and $\Sigma_{v2} = \text{diag}(\xi_{\ell+1}^v, \ldots, \xi_n^v)$;

3. Compute $\tilde{M} = \tilde{W}^T M \tilde{T}, \; \tilde{D} = \tilde{W}^T D \tilde{T}, \; \tilde{K} = \tilde{W}^T K \tilde{T}, \; \tilde{B}_2 = \tilde{W}^T B_2,$
   $\tilde{C}_1 = C_1 \tilde{T}, \; \tilde{C}_2 = C_2 \tilde{T}$ with $\tilde{W} = L_v V_{v1} \Sigma_{p1}^{-1/2}, \quad \tilde{T} = R_p U_{p1} \Sigma_{p1}^{-1/2}$.

Properties of the SOBT

- Stability is not necessarily preserved in the reduced model and, in general, no error bounds
- For symmetric second-order systems with

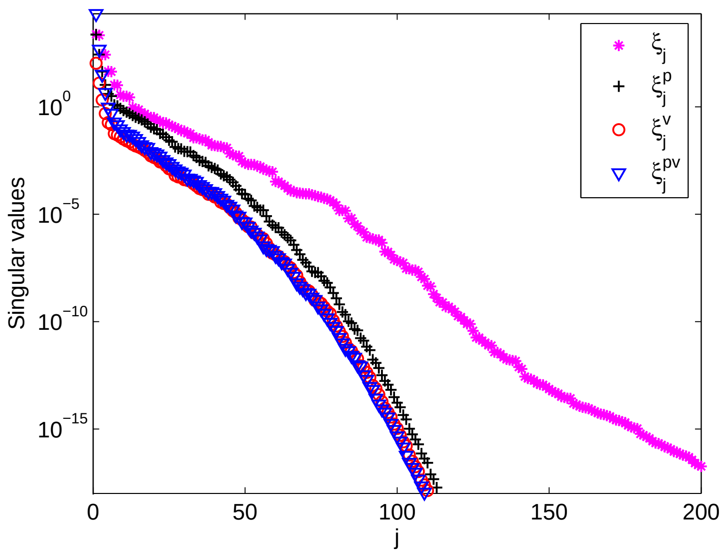
  $M = M^T > 0, \;\; D = D^T > 0, \;\; K = K^T > 0, \;\; B_2 = C_1^T, \;\; C_2 = 0,$

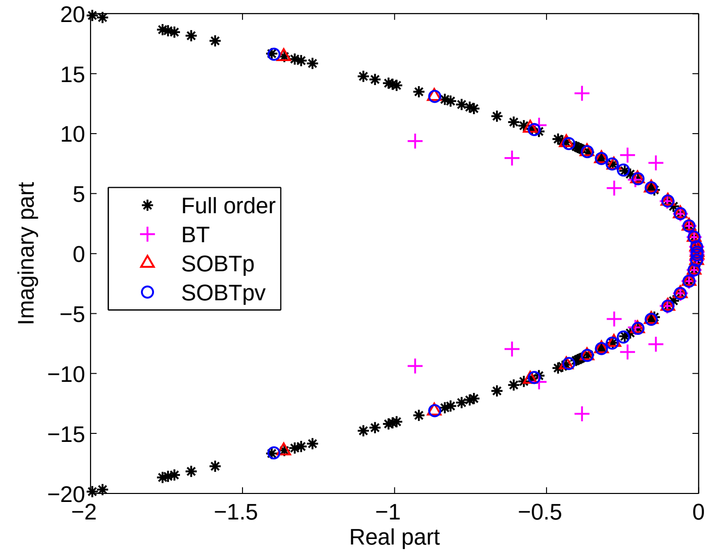
  we have
  - $\boldsymbol{G}(s) = \boldsymbol{G}^T(s)$
  - $\lambda^2 M + \lambda D + K$ is stable
  - $X_p = Y_v$
  - only SOBTpv preserves symmetry and stability
  - no error bound
- Position and velocity Graminas can be computed using the ADI method without explicit forming the double sized matrices

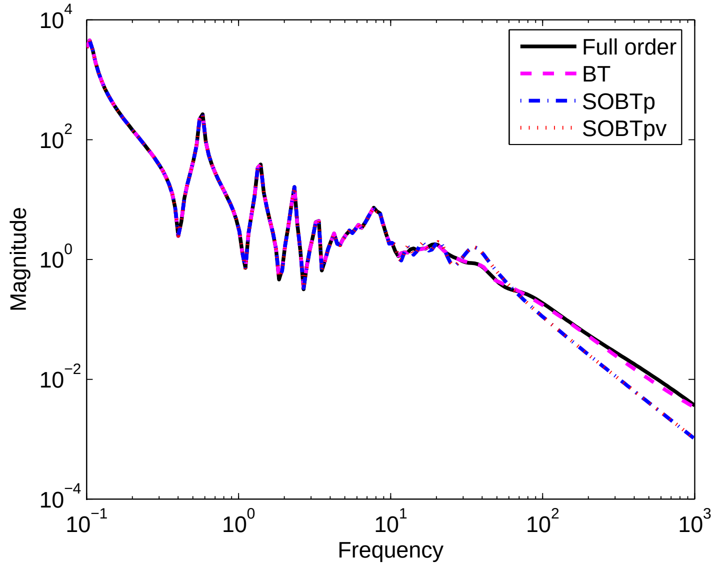
[Benner/Saak'11]

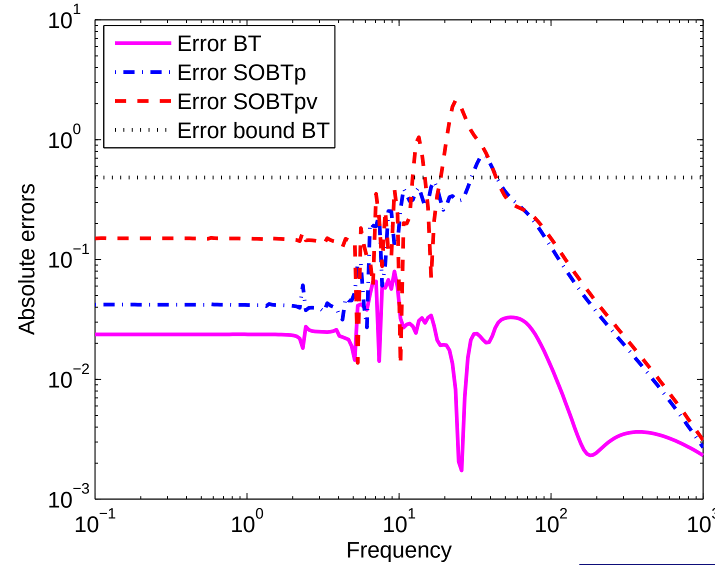
Clamped beam model

$n = 174, \;\; m = p = 1 \;\;\Longrightarrow\;\; \ell = 17$ [Oberwolfach Benchmark Collection]









Conclusions

- Balanced truncation for DAEs
  - proper and improper Gramians
  - algebraic constraints are preserved
  - exploiting the structure of system matrices for computing P_l and P_r and solving the Lyapunov equations
  - other balancing techniques can also be extended to DAEs

[Reis/St.'10,11, Möckel/Reis/St.'11, Benner/St.'17]

- Balanced truncation for second-order systems
  - position and velocity Gramians
  - second-order structure is preserved
  - stability is not always guaranteed
  - no error bound