The generalised singular perturbation approximation for bounded-real and positive-real control systems

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London Mathematical Society — EPSRC Durham Symposium

Model Order Reduction

UNIVERSITY OF BATH

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Overview

- I shall present recent research on model order reduction of bounded real and positive real linear control systems by the generalised singular perturbation approximation

Model order reduction

- Model order reduction refers to approximating an elaborate model with a simpler one which is close to the original

- Simpler means of the same form, but with lower state-space dimension $r < n$

- Close refers to qualitative properties: (stability, minimality, dissipativity etc) of the system and quantitatively: the input-output maps $u \mapsto y$ "close" in some sense

- Model reduction is important for simulation and controller design

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- Model reduction is important for simulation and controller design

Linear control systems

- We shall consider linear control systems

$$\left.\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x^0 \\ y &= Cx + Du, \end{aligned}\right\} \tag{1}$$

where for $n, m, p \in \mathbb{N}$

$$(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}.$$

- We shall assume that A is stable (or Hurwitz), meaning $\alpha(A) < 0$
- Transfer function of (1) is

$$\mathbf{G}(s) = C(sI - A)^{-1}B + D,$$

  - maps $\hat{u} \mapsto \hat{y}$ via $\hat{y}(s) = \mathbf{G}(s)\hat{u}(s)$ and is defined for $s \in \mathbb{C}_\alpha$ for some $\alpha \in \mathbb{R}$
  - is rational and proper

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Linear control systems

- Conversely, given $\mathbf{G} : \mathbb{C}_0 \to \mathbb{C}^{p \times m}$ proper rational, we can find a realisation of the form (1), denoted by (A, B, C, D)
- Realisations are never unique
- Indeed, if (A, B, C, D) is a realisation of $\mathbf{G}$, then so is $(T^{-1}AT, T^{-1}B, CT, D)$ for every invertible $T \in \mathbb{C}^{n \times n}$

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Properties of the transfer function

- There are many!
- For input $u \in L^2$ and output $y \in L^2$, we have

$$\|y\|_{L^2} \leq \|\mathbf{G}\|_{H^\infty} \|u\|_{L^2} \,,$$

  where

$$\|\mathbf{G}\|_{H^\infty} := \sup_{z \in \mathbb{C}_0} \|\mathbf{G}(z)\|_2 = \sup_{\omega \in \mathbb{R}} \|\mathbf{G}(i\omega)\|_2 \,.$$

- In the SISO case, if $u(t) = \sin(\omega t)$ for $\omega \in \mathbb{R}$, then for large t

$$y(t) \approx |\mathbf{G}(i\omega)| \sin(\omega(t + \arg \mathbf{G}(i\omega))) \,.$$

- If $u(t)$ has a limit as $t \to \infty$, then for all $x_0 \in \mathbb{R}^n$

$$\lim_{t \to \infty} y(t) = \mathbf{G}(0) \lim_{t \to \infty} u(t) \,.$$

- Plays a crucial role in stability theory when connecting versions of (1), or when (1) is in feedback connection with a nonlinear term.

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Model reduction

- We approximate **G** by approximating a state-space realisation of **G**
- Given (A, B, C, D) partition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

with $A_{11} \in \mathbb{C}^{r \times r}$, $r < n$ and B_1, C_1 conformly sized

- To connect with prevailing notation of workshop

$$A_{11} = W^T AV = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad B_1 = W^T B, \quad C_1 = CV,$$

(although I tend to think that

$$X = X_1 \oplus X_2 \quad \text{and} \quad A_{11} = P_{X_1} A|_{X_1} : X_1 \hookrightarrow X \xrightarrow{A} X \xrightarrow{P_{X_1}} X \approx X_1).$$

- Many model reduction schemes build a reduced order model from these components, somehow.
- Note that the components may change with realisation

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The generalised singular perturbation approximation

- For $\xi \in \mathbb{C}$, $\mathrm{Re}(\xi) \geq 0$ and $r \in \{1, 2, \ldots, n-1\}$, the generalised singular perturbation approximation (GSPA) is given by

$$A_\xi := A_{11} + A_{12}(\xi I - A_{22})^{-1} A_{21}\,, \quad B_\xi := B_1 + A_{12}(\xi I - A_{22})^{-1} B_2\,,$$
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provided $\xi \notin \sigma(A_{22})$

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- Defining property: for $\xi \notin \sigma(A_\xi)$

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  that is, GSPA interpolates original transfer function at ξ

- Proven by algebraic manipulation

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- Key disadvantage: If $\mathrm{Im}\,(\xi) \neq 0$, then $(A_\xi, B_\xi, C_\xi, D_\xi)$ will have non-real components in general
- However, if $\xi \in \mathbb{R}$ and $\xi \geq 0$, then

$$(A_\xi, B_\xi, C_\xi, D_\xi) \in \mathbb{R}^{r \times r} \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r} \times \mathbb{R}^{p \times m}$$

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- From a state-space perspective we have

$$\left.\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1 u \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2 u \\ y &= C_1 x_1 + C_2 x_2 + Du \end{aligned}\right\}$$

- The GSPA arises by assuming that $\dot{x}_2 = \xi x_2$ above and subsequently eliminating x_2

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$$\left.\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1 u \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2 u \\ y &= C_1 x_1 + C_2 x_2 + Du \end{aligned}\right\}$$

- The GSPA arises by assuming that $\dot{x}_2 = \xi x_2$ above and subsequently eliminating x_2

The generalised singular perturbation approximation

- For $\xi \in \mathbb{C}$, $\mathrm{Re}(\xi) \geq 0$ and $r \in \{1, 2, \dots, n-1\}$, the generalised singular perturbation approximation (GSPA) is given by

$$\begin{aligned} A_\xi &:= A_{11} + A_{12}(\xi I - A_{22})^{-1} A_{21}\,, & B_\xi &:= B_1 + A_{12}(\xi I - A_{22})^{-1} B_2\,, \\ C_\xi &:= C_1 + C_2(\xi I - A_{22})^{-1} A_{21}\,, & D_\xi &:= D + C_2(\xi I - A_{22})^{-1} B_2\,. \end{aligned}$$

  provided $\xi \notin \sigma(A_{22})$

- The case $\xi = 0$ corresponds to x_2 at equilibrium and so is the (usual) singular perturbation approximation (SPA)

$$\begin{aligned} A_0 &:= A_{11} - A_{12} A_{22}^{-1} A_{21}\,, & B_0 &:= B_1 - A_{12} A_{22}^{-1} B_2\,, \\ C_0 &:= C_1 - C_2 A_{22}^{-1} A_{21}\,, & D_0 &:= D - C_2 A_{22}^{-1} B_2\,, \end{aligned}$$

- The SPA has the property that $\mathbf{G}(0) = \mathbf{G}^0(0)$ — interpolation at zero — the steady-state gains coincide

The generalised singular perturbation approximation

- As mentioned, key question is how to choose realisation and decomposition to give

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

- To answer that, first look at the case $|\xi| \to \infty$ in the GSPA which gives reduced order system

$$(A_{11}, B_1, C_1, D)$$

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Balanced realisations

- Recall the controllability $\mathcal{Q}$ and observability $\mathcal{O}$ Gramians,

$$\mathcal{Q} = \int_{\mathbb{R}_+} \mathrm{e}^{At} BB^* \mathrm{e}^{A^* t}\, dt, \quad \mathcal{O} = \int_{\mathbb{R}_+} \mathrm{e}^{A^* t} C^* C \mathrm{e}^{At}\, dt\,.$$

- Note these quantities depend on the realisation
- The Gramians of $\tilde{\mathcal{Q}}$, $\tilde{\mathcal{O}}$ of $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (T^{-1}AT, T^{-1}B, CT, D)$ are given by

$$\tilde{\mathcal{Q}} = T^{-1}\mathcal{Q}T^{-*}, \quad \tilde{\mathcal{O}} = T^*\mathcal{O}T \quad \Rightarrow \quad \tilde{\mathcal{Q}}\tilde{\mathcal{O}} = T^{-1}\mathcal{Q}\mathcal{O}T,$$

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Hankel operators and singular values

- The Hankel operator H of (1) is given by

$$H = \mathfrak{C}\mathfrak{B} : L^2(\mathbb{R}_-;\mathbb{C}^m) \to L^2(\mathbb{R}_+;\mathbb{C}^p)$$

where

$$\mathfrak{B} : L^2(\mathbb{R}_-;\mathbb{C}^m) \to \mathbb{C}^n, \quad \mathfrak{B}u = \int_{-\infty}^0 \mathrm{e}^{-As} Bu(s)\, ds,$$
$$\mathfrak{C} : \mathbb{C}^n \to L^2(\mathbb{R}_+;\mathbb{C}^p), \quad (\mathfrak{C}x_0)(t) = C\mathrm{e}^{At} x_0,$$

- The maps $\mathbb{B}$ and $\mathbb{C}$ satisfy

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Hankel singular values continued

- We see that apart from zero

$$\sigma(H^*H) = \sigma(\mathfrak{B}^*\mathfrak{C}^*\mathfrak{C}\mathfrak{B}) = \sigma(\mathfrak{B}\mathfrak{B}^*\mathfrak{C}^*\mathfrak{C}) = \sigma(\mathcal{Q}\mathcal{O}),$$

  and so there are only finitely many (non-zero) singular values

- Thus, the singular values of H are the squareroots of the eigenvalues of $\mathcal{Q}\mathcal{O}$, denoted by σ_k. They are ordered

$$\sigma_1 > \sigma_2 > \cdots > \sigma_k \geq 0 \,,$$

  (each with a multiplicity possibly bigger than one)

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Energy interpretation

- If (1) is controllable, then

$$\inf_{u \in L^2} \|u\|_{L^2}^2 = \langle x_f, \mathcal{Q}^{-1} x_f \rangle =: C_{x_f},$$

subject to (1) with $x(-\infty) = 0$ and $x(0) = x_f$

- Morally, C_{x_f} captures how "hard" it is to reach the state x_f
- Similarly, the "energy" of the uncontrolled output in forwards time starting at state $x(0) = x_f$ is

$$\|y\|_{L^2}^2 = \langle x_f, \mathcal{O} x_f \rangle =: E_{x_f}.$$

- So E_{x_f} captures how much the state x_f contributes to the energy of the output

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- Suppose the system (1) is Lyapunov balanced with simple singular values, so

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- Further, states corresponding to small singular values require lots of energy to reach and yield little energy from observing

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Lyapunov balanced truncation

- Lyapunov balanced truncation is to truncate states that correspond to small singular values
- Suppose we keep $\sigma_1, \ldots, \sigma_r$. The reduced order system

$$(A_{11}, B_1, C_1, D)$$

(the GSPA with $\xi \to \infty$) is called the Lyapunov balanced truncation
- Balanced truncations inherit stability and minimality from (1)
- Lyapunov balanced truncations may be computed by computing the solutions of Lyapunov equations

$$AQ + QA^* + BB^* = 0 \quad \text{and} \quad A^*O + OA + C^*C = 0 .$$

- An appealing facet of balanced truncation is the *a priori* error bound

$$\|\mathbf{G} - \mathbf{G}_1\|_{\mathcal{H}^\infty} \leq 2 \sum_{i=r+1}^{k} \sigma_i ,$$

proved independently by Enns and Glover in 1984.

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$$\|\mathbf{G} - \mathbf{G}_1\|_{H^\infty} \leq 2 \sum_{i=r+1}^{k} \sigma_i,$$

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$$\sigma_{r+1} \leq \|H - H_r\| \leq \|\mathbf{G} - \mathbf{G}_r\|_{H^\infty}\,,$$

  always holds
- [Glover,'84] showed that the transfer function $\tilde{\mathbf{G}}$ corresponding to the Hankel operator $\tilde{H}$ satisfies

$$\|\mathbf{G} - \tilde{\mathbf{G}} - D_0\|_{H^\infty} \leq \sum_{i=r+1}^{k} \sigma_i\,,$$

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Generalised singular perturbation approximation

Theorem

Given $\xi \in \mathbb{C}$ *with* $\mathrm{Re}(\xi) \geq 0$ *and stable, minimal, balanced quadruple* (A, B, C, D)*, assume that the Hankel singular values are simple.*

Generalised singular perturbation approximation

Theorem

Given $\xi \in \mathbb{C}$ *with* $\mathrm{Re}(\xi) \geq 0$ *and stable, minimal, balanced quadruple* (A, B, C, D)*, assume that the Hankel singular values are simple. Then* $(A_\xi, B_\xi, C_\xi, D_\xi)$*, the generalised singular perturbation approximation of order* $r \in \underline{n-1}$*, is well-defined and:*

(i) A_ξ *is Hurwitz and* (A_ξ, B_ξ, C_ξ) *is minimal.*

(ii) *If* $\xi \in i\mathbb{R}$*, then* (A_ξ, B_ξ, C_ξ) *is balanced.*

Generalised singular perturbation approximation

Theorem

Given $\xi \in \mathbb{C}$ *with* $\mathrm{Re}(\xi) \geq 0$ *and stable, minimal, balanced quadruple* (A, B, C, D)*, assume that the Hankel singular values are simple. Then* $(A_\xi, B_\xi, C_\xi, D_\xi)$*, the generalised singular perturbation approximation of order* $r \in \underline{n-1}$*, is well-defined and:*

(i) A_ξ *is Hurwitz and* (A_ξ, B_ξ, C_ξ) *is minimal.*

(ii) *If* $\xi \in i\mathbb{R}$*, then* (A_ξ, B_ξ, C_ξ) *is balanced.*

Let $\mathbf{G}_r^\xi$ *denote the transfer function of the GSPA. Then* $\mathbf{G}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times m})$ *has McMillan degree* r*,* $\mathbf{G}_r^\xi(\xi) = \mathbf{G}(\xi)$ *and*

$$\|\mathbf{G} - \mathbf{G}_r^\xi\|_{H^\infty} \leq 2 \sum_{j=r+1}^{n} \sigma_j \,. \tag{2}$$

Generalised singular perturbation approximation

GSPA proposed in a control theoretic setting in [1]–[2] and properties studied across [3]–[6].

[1] K. V. Fernando and H. Nicholson. Singular perturbational model reduction of balanced systems, *IEEE Trans. Automat. Control*, **27** (1982), 466–468.

[2] K. V. Fernando and H. Nicholson. Singular perturbational model reduction in the frequency domain, *IEEE Trans. Automat. Control*, **27** (1982), 969–970.

[3] U. M. Al-Saggaf and G. F. Franklin. Model reduction via balanced realizations: an extension and frequency weighting techniques, *IEEE Trans. Automat. Control*, **33** (1988), 687–692.

[4] Y. Liu and B. D. O. Anderson. Singular perturbation approximation of balanced systems, *Internat. J. Control*, **50** (1989), 1379–1405.

[5] P. Heuberger. A family of reduced order models based on open-loop balancing, in *Selected Topics in Identification, Modelling and Control*, Delft University Press, 1990, 1–10.

[6] G. Muscato and G. Nunnari. On the σ-reciprocal system for model order reduction, *Math. Model. Systems*, **1** (1995), 261–271.

Model reduction for dissipative systems

- Bounded-realness and positive-realness are important qualitative properties pertaining to dissipation of energy in control systems
- May well be desirable for these properties to be retained in a reduced order model
- Need not be preserved in Lyapunov balanced GSPA (including balanced truncation and SPA)
- Balanced truncation and SPA have been extended to bounded-real and positive-real systems and make use of bounded-real and positive-real balanced realisations. Shown that:
  - stability inherited
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Bounded-real systems

- $\mathbf{G} \in H^\infty$ is bounded real if $\|\mathbf{G}\|_{H^\infty} \leq 1$ (and so $\|y\|_{L^2} \leq \|u\|_{L^2}$)
- $\mathbf{G} \in H^\infty$ is strictly bounded real if $\|\mathbf{G}\|_{H^\infty} < 1$
- Let (A, B, C, D) denote a minimal realisation of $\mathbf{G}$. The following are equivalent.
  - (i) $\mathbf{G}$ is bounded real
  - (ii) There exists a triple (P, K, W) with $P = P^*$ positive-definite satisfying the bounded-real Lur'e equations

$$\begin{aligned} A^*P + PA + C^*C &= -K^*K, \\ PB + C^*D &= -K^*W, \\ I - D^*D &= W^*W \end{aligned}$$

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- The previous equivalences are often called the bounded-real lemma
- If $I - D^*D$ is invertible and P solves

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then P solves the bounded-real algebraic Riccati equation

$$A^*P + PA + C^*C + (PB + C^*D)(I - D^*D)^{-1}(B^*P + D^*C) = 0\,.$$

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- If (i) or (ii) hold, then (3) has extremal solutions P_m, P_M in the sense that any $P = P^* \geq 0$ solving (3) satisfies

$$0 < P_m \leq P \leq P_M \,.$$

- The extremal operators P_m, P_M are the optimal cost operators of the bounded real optimal control problems:

$$\langle P_M x_0, x_0 \rangle_{\mathscr{X}} = \inf_{u \in L^2(\mathbb{R}_-)} \int_{\mathbb{R}_-} \|u(s)\|^2 - \|y(s)\|^2 \, ds \,,$$

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Bounded-real balanced realisations

Definition

The realisation (A, B, C, D) is bounded-real balanced if $P_m = P_M^{-1} = \Sigma$.

- If P solves (3), then P^{-1} solves the dual equations

$$AQ + QA^* + BB^* = -LL^*,$$
$$QC^* + BD^* = -LX^*,$$
$$I - DD^* = XX^*,$$

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Bounded-real balanced realisations

- It is always possible to construct a bounded-real balanced realisation via a state-space transformation of a given realisation
- The eigenvalues of Σ are called the bounded-real singular values or bounded-real characteristic values
- Truncation takes place according to the size of these singular values
- Given a bounded real balanced realisation (A, B, C, D), $\xi \in \mathbb{C}$, $\mathrm{Re}(\xi) \geq 0$ and $r \in \{1, 2, \dots, n-1\}$, the bounded-real GSPA is given as before

$$A_\xi := A_{11} + A_{12}(\xi I - A_{22})^{-1} A_{21}\,, \quad B_\xi := B_1 + A_{12}(\xi I - A_{22})^{-1} B_2\,,$$
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Given $\xi \in \mathbb{C}$ *with* $\mathrm{Re}(\xi) \geq 0$ *and stable, minimal, and bounded real balanced quadruple* (A, B, C, D), *assume that the bounded real singular values are simple.*

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$$\|\mathbf{G} - \mathbf{G}_r^\xi\|_{H^\infty} \leq 2 \sum_{j=r+1}^{n} \sigma_j \,.$$

If $\|\mathbf{G}\|_{H^\infty} < 1$, then $\|\mathbf{G}_r^\xi\|_{H^\infty} < 1$.

Spectral factors

- Recall the bounded-real Lur'e equations

$$\begin{aligned} A^*P + PA + C^*C &= -K^*K \\ PB + C^*D &= -K^*W \\ I - D^*D &= W^*W \end{aligned}$$

- If $\mathbf{G}$ is realised by (A, B, C, D) and $\mathbf{R}$ realised by (A, B, K, W), then

$$I - (\mathbf{G}(s))^*\mathbf{G}(s) = (\mathbf{R}(s))^*\mathbf{R}(s) \quad \forall\, s \in i\mathbb{R}\,.$$

- $\mathbf{R}$ is a so-called spectral factor of $I - \mathbf{G}^*\mathbf{G}$
- Using dual equations, can also obtain a spectral factor $\mathbf{S}$ such that $I - \mathbf{G}\mathbf{G}^* = \mathbf{S}\mathbf{S}^*$ on imaginary axis

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Bounded-real GSPA also approximates spectral factors

Proposition (G.'17)

Imposing notation and assumptions of previous theorem, assume that $\xi \in i\mathbb{R}$, *there exists* $\mathbf{R}, \mathbf{R}_r^\xi \in H^\infty$ *such that*

(i) $I - (\mathbf{G}(s))^*\mathbf{G}(s) = (\mathbf{R}(s))^*\mathbf{R}(s)$ *for all* $s \in i\mathbb{R}$

(ii) $I - (\mathbf{G}_r^\xi(s))^*\mathbf{G}_r^\xi(s) = (\mathbf{R}_r^\xi(s))^*\mathbf{R}_r^\xi(s)$ *for all* $s \in i\mathbb{R}$

(iii)
$$\left\| \begin{bmatrix} \mathbf{G} - \mathbf{G}_r^\xi \\ \mathbf{R} - \mathbf{R}_r^\xi \end{bmatrix} \right\|_{H^\infty} \leq 2 \sum_{j=r+1}^{n} \sigma_j \,,$$

(iv) $\mathbf{R}_r^\xi$ *may be chosen with the interpolation property* $\mathbf{R}(\xi) = \mathbf{R}_r^\xi(\xi)$

- Similar statements apply to other spectral factor $\mathbf{S}$ and $\mathbf{S}_r^\xi$
- Obtain sub-spectral factors when $\mathrm{Re}\,(\xi) > 0$

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Positive-real systems

- $\mathbf{G}$ is positive real if $\mathbf{G}(s) + (\mathbf{G}(s))^* \geq 0$ for all $s \in \mathbb{C}_0$
- $\mathbf{G}$ is strongly positive real if $\mathbf{G}(s) + (\mathbf{G}(s))^* \geq \delta I$ for all $s \in \mathbb{C}_0$
- Positive-real functions need not be stable $s \mapsto 1/s$ or proper $s \mapsto s$
- Let (A, B, C, D) denote a stable, minimal realisation of $\mathbf{G}$. The following are equivalent
  - (i) $\mathbf{G}$ is positive real
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$$A^*P + PA = -K^*K,$$
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  - (iii) For input $u \in L^2$ and output $y \in L^2$ with initial condition $x_0 = 0$

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Positive-real balanced realisations

- The previous equivalences are often called the positive-real lemma or KYP lemma
- Positive-real balanced realisations are morally the same as the bounded-real versions...
- ...now balance extremal solutions of positive-real Lur'e equations or positive-real algebraic Riccati equation
- Can either work from first principles or use bounded-real case and Cayley transform

$$\mathbf{G} \mapsto (I - \mathbf{G})(I + \mathbf{G})^{-1}$$

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Theorem (G.'17)

Given $\xi \in \mathbb{C}$ *with* $\mathrm{Re}(\xi) \geq 0$ *and stable, minimal, and positive real balanced quadruple* (A, B, C, D)*, assume that the positive real singular values are simple.*

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Given $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) \geq 0$ and stable, minimal, and positive real balanced quadruple (A, B, C, D), assume that the positive real singular values are simple. Then $(A_\xi, B_\xi, C_\xi, D_\xi)$, the positive real generalised singular perturbation approximation of order $r \in \underline{n-1}$, is well-defined and

(i) *$(A_\xi, B_\xi, C_\xi, D_\xi)$ is positive real, and is positive real balanced if $\xi \in i\mathbb{R}$*

(ii) *A_ξ is Hurwitz*

(iii) *If (A, B, C, D) is strictly positive real, then $(A_\xi, B_\xi, C_\xi, D_\xi)$ is minimal and strictly positive real*

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Summary

- Model order reduction for linear control systems by the generalised singular perturbation approximation has been revisited
- Specifically, the GSPA preserves the same properties of bounded-real and positive-real systems as SPA and balanced truncation when defined in terms of dissipative balannced realisations
- The defining property of the GSPA is that the reduced order transfer function interpolates the original at ξ with $\mathrm{Re}\,(\xi) \geq 0$ — Lyapunov balanced truncation and the SPA are special cases of this
- The usual error bounds hold
- Possible application is to choose ξ to trade off interpolating at zero, or at infinity