

The generalised singular perturbation approximation for bounded-real and positive-real control systems

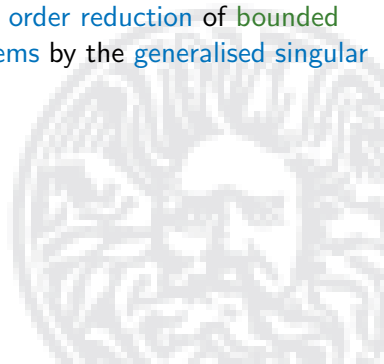
Chris Guiver

London Mathematical Society — EPSRC Durham Symposium
Model Order Reduction



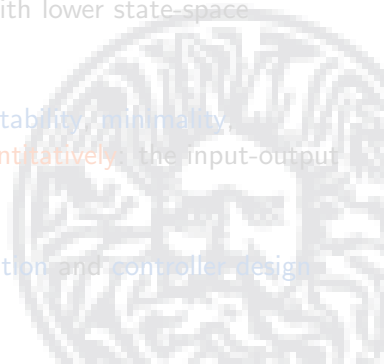
14th August 2017

- I shall present recent research on **model order reduction** of **bounded real** and **positive real** linear control systems by the **generalised singular perturbation approximation**



Model order reduction

- **Model order reduction** refers to **approximating** an elaborate model with a **simpler** one which is **close** to the original
- Simpler means of the same form, but with lower state-space dimension $r < n$
- Close refers to **qualitative properties**: (**stability**, **minimality**, **dissipativity** etc) of the system and **quantitatively**: the input-output maps $u \mapsto y$ “close” in some sense
- Model reduction is important for **simulation** and **controller design**



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- Model reduction is important for **simulation** and **controller design**

Linear control systems

- We shall consider linear control systems

$$\left. \begin{aligned} \dot{x} &= Ax + Bu, & x(0) &= x^0 \\ y &= Cx + Du, \end{aligned} \right\} \quad (1)$$

where for $n, m, p \in \mathbb{N}$

$$(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}.$$

- We shall assume that A is stable (or Hurwitz), meaning $\alpha(A) < 0$
- Transfer function of (1) is

$$\mathbf{G}(s) = C(sI - A)^{-1}B + D$$

- ▶ maps $\hat{u} \mapsto \hat{y}$ via $\hat{y}(s) = \mathbf{G}(s)\hat{u}(s)$ and is defined for $s \in \mathbb{C}$ for some $\alpha \in \mathbb{R}$
- ▶ is rational and proper

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- ▶ is **rational** and **proper**

Linear control systems

- Conversely, given $\mathbf{G} : \mathbb{C}_0 \rightarrow \mathbb{C}^{p \times m}$ proper rational, we can find a realisation of the form (1), denoted by (A, B, C, D)
- Realisations are never unique
- Indeed, if (A, B, C, D) is a realisation of \mathbf{G} , then so is $(T^{-1}AT, T^{-1}B, CT, D)$ for every invertible $T \in \mathbb{C}^{n \times n}$

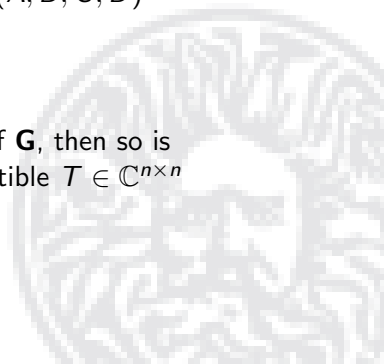


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Properties of the transfer function

- There are many!
- For input $u \in L^2$ and output $y \in L^2$, we have

$$\|y\|_{L^2} \leq \|\mathbf{G}\|_{H^\infty} \|u\|_{L^2},$$

where

$$\|\mathbf{G}\|_{H^\infty} := \sup_{z \in \mathbb{C}_0} \|\mathbf{G}(z)\|_2 = \sup_{\omega \in \mathbb{R}} \|\mathbf{G}(i\omega)\|_2.$$

- In the SISO case, if $u(t) = \sin(\omega t)$ for $\omega \in \mathbb{R}$, then for large t

$$y(t) \approx |\mathbf{G}(i\omega)| \sin(\omega(t + \arg \mathbf{G}(i\omega))).$$

- If $u(t)$ has a limit as $t \rightarrow \infty$, then for all $x_0 \in \mathbb{R}^n$

$$\lim_{t \rightarrow \infty} y(t) = \mathbf{G}(0) \lim_{t \rightarrow \infty} u(t).$$

- Plays a crucial role in stability theory when connecting versions of (1), or when (1) is in feedback connection with a nonlinear term.

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Model reduction

- We approximate \mathbf{G} by approximating a state-space realisation of \mathbf{G}
- Given (A, B, C, D) partition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \ C_2]$$

with $A_{11} \in \mathbb{C}^{r \times r}$, $r < n$ and B_1, C_1 conformly sized

- To connect with prevailing notation of workshop

$$A_{11} = W^T A V = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad B_1 = W^T B, \quad C_1 = C V.$$

(although I tend to think that

$$X = X_1 \oplus X_2 \quad \text{and} \quad A_{11} = P_{X_1} A|_{X_1} : X_1 \hookrightarrow X \xrightarrow{A} X \xrightarrow{P_{X_1}} X \approx X_1.$$

- Many model reduction schemes build a reduced order model from these components, somehow.
- Note that the components may change with realisation

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The generalised singular perturbation approximation

- For $\xi \in \mathbb{C}$, $\operatorname{Re}(\xi) \geq 0$ and $r \in \{1, 2, \dots, n-1\}$, the **generalised singular perturbation approximation** (GSPA) is given by

$$\begin{aligned} A_\xi &:= A_{11} + A_{12}(\xi I - A_{22})^{-1}A_{21}, & B_\xi &:= B_1 + A_{12}(\xi I - A_{22})^{-1}B_2, \\ C_\xi &:= C_1 + C_2(\xi I - A_{22})^{-1}A_{21}, & D_\xi &:= D + C_2(\xi I - A_{22})^{-1}B_2. \end{aligned}$$

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 - ▶ why definition? role of ξ ?
 - ▶ how to choose decomposition of A, B, C ?
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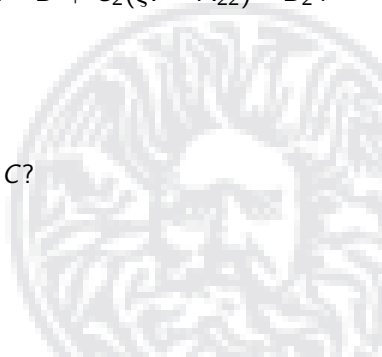
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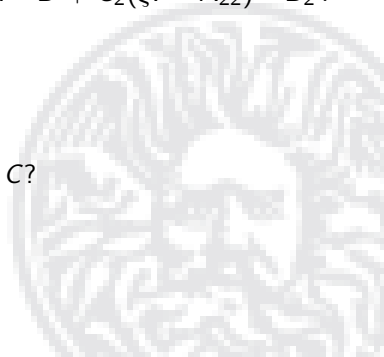
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- **Defining property:** for $\xi \notin \sigma(A_\xi)$

$$\mathbf{G}(\xi) = \mathbf{G}^\xi(\xi),$$

that is, GSPA interpolates original transfer function at ξ

- Proven by algebraic manipulation

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- **Key disadvantage:** If $\text{Im}(\xi) \neq 0$, then $(A_\xi, B_\xi, C_\xi, D_\xi)$ will have non-real components in general
- However, if $\xi \in \mathbb{R}$ and $\xi \geq 0$, then

$$(A_\xi, B_\xi, C_\xi, D_\xi) \in \mathbb{R}^{r \times r} \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r} \times \mathbb{R}^{p \times m}$$

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- For $\xi \in \mathbb{C}$, $\text{Re}(\xi) \geq 0$ and $r \in \{1, 2, \dots, n-1\}$, the **generalised singular perturbation approximation** (GSPA) is given by

$$A_\xi := A_{11} + A_{12}(\xi I - A_{22})^{-1}A_{21}, \quad B_\xi := B_1 + A_{12}(\xi I - A_{22})^{-1}B_2, \\ C_\xi := C_1 + C_2(\xi I - A_{22})^{-1}A_{21}, \quad D_\xi := D + C_2(\xi I - A_{22})^{-1}B_2.$$

provided $\xi \notin \sigma(A_{22})$

- From a state-space perspective we have

$$\left. \begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1u \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2u \\ y &= C_1x_1 + C_2x_2 + Du \end{aligned} \right\}$$

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- The case $\xi = 0$ corresponds to x_2 at equilibrium and so is the (usual) **singular perturbation approximation** (SPA)

$$\begin{aligned}A_0 &:= A_{11} - A_{12}A_{22}^{-1}A_{21}, & B_0 &:= B_1 - A_{12}A_{22}^{-1}B_2, \\C_0 &:= C_1 - C_2A_{22}^{-1}A_{21}, & D_0 &:= D - C_2A_{22}^{-1}B_2,\end{aligned}$$

- The SPA has the property that $\mathbf{G}(0) = \mathbf{G}^0(0)$ — interpolation at zero — the steady-state gains coincide

The generalised singular perturbation approximation

- As mentioned, key question is how to choose realisation and decomposition to give

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \ C_2]$$

- To answer that, first look at the case $|\xi| \rightarrow \infty$ in the GSPA which gives reduced order system

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Balanced realisations

- Recall the **controllability** \mathcal{Q} and **observability** \mathcal{O} Gramians,

$$\mathcal{Q} = \int_{\mathbb{R}_+} e^{At} B B^* e^{A^* t} dt, \quad \mathcal{O} = \int_{\mathbb{R}_+} e^{A^* t} C^* C e^{At} dt.$$

- Note these quantities depend on the realisation
- The Gramians of $\tilde{\mathcal{Q}}, \tilde{\mathcal{O}}$ of $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (T^{-1}AT, T^{-1}B, CT, D)$ are given by

$$\tilde{\mathcal{Q}} = T^{-1} \mathcal{Q} T^{-*}, \quad \tilde{\mathcal{O}} = T^* \mathcal{O} T \quad \Rightarrow \quad \tilde{\mathcal{Q}} \tilde{\mathcal{O}} = T^{-1} \mathcal{Q} \mathcal{O} T,$$

and so the eigenvalues of $\mathcal{Q}\mathcal{O}$ are similarity invariants

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Hankel operators and singular values

- The Hankel operator H of (1) is given by

$$H = \mathfrak{C}\mathfrak{B} : L^2(\mathbb{R}_-; \mathbb{C}^m) \rightarrow L^2(\mathbb{R}_+; \mathbb{C}^p)$$

where

$$\mathfrak{B} : L^2(\mathbb{R}_-; \mathbb{C}^m) \rightarrow \mathbb{C}^n, \quad \mathfrak{B}u = \int_{-\infty}^0 e^{-As} Bu(s) ds,$$
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Hankel singular values continued

- We see that apart from zero

$$\sigma(H^*H) = \sigma(\mathfrak{B}^* \mathfrak{C}^* \mathfrak{C} \mathfrak{B}) = \sigma(\mathfrak{B} \mathfrak{B}^* \mathfrak{C}^* \mathfrak{C}) = \sigma(\mathcal{Q}\mathcal{O}),$$

and so there are only finitely many (non-zero) singular values

- Thus, the singular values of H are the squareroots of the eigenvalues of $\mathcal{Q}\mathcal{O}$, denoted by σ_k . They are ordered

$$\sigma_1 > \sigma_2 > \cdots > \sigma_k \geq 0,$$

(each with a multiplicity possibly bigger than one)

- The Hankel singular values are invariant under state-space similarity transforms

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Energy interpretation

- If (1) is controllable, then

$$\inf_{u \in L^2} \|u\|_{L^2}^2 = \langle x_f, \mathcal{Q}^{-1}x_f \rangle =: C_{x_f},$$

subject to (1) with $x(-\infty) = 0$ and $x(0) = x_f$

- Morally, C_{x_f} captures how “hard” it is to reach the state x_f
- Similarly, the “energy” of the uncontrolled output in forwards time starting at state $x(0) = x_f$ is

$$\|y\|_{L^2}^2 = \langle x_f, \mathcal{O}x_f \rangle =: E_{x_f}$$

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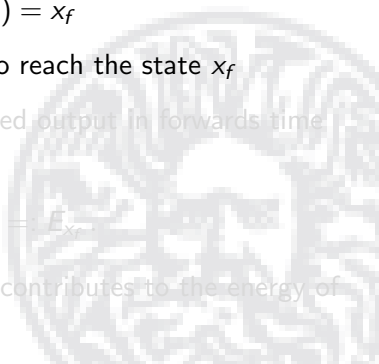
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- Suppose the system (1) is Lyapunov balanced with simple singular values, so

$$Q = O = \Sigma = \text{diag} \{ \sigma_1, \dots, \sigma_n \},$$

with respect to the orthonormal basis $\{v_i\}_{1 \leq i \leq n}$

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Lyapunov balanced truncation

- Lyapunov balanced truncation is to truncate states that correspond to small singular values
- Suppose we keep $\sigma_1, \dots, \sigma_r$. The reduced order system

$$(A_{11}, B_1, C_1, D)$$

(the GSPA with $\xi \rightarrow \infty$) is called the **Lyapunov balanced truncation**

- Balanced truncations inherit **stability** and **minimality** from (1)
- Lyapunov balanced truncations may be computed by computing the solutions of Lyapunov equations

$$AQ + QA^* + BB^* = 0 \quad \text{and} \quad A^*O + OA + C^*C = 0.$$

- An appealing facet of balanced truncation is the *a priori* error bound

$$\|\mathbf{G} - \mathbf{G}_1\|_{\mathcal{H}^\infty} \leq 2 \sum_{i=r+1}^k \sigma_i$$

proved independently by Enns and Glover in 1984.

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was derived in [Opmeer, Reis 2015] for MIMO ($m > 1$) symmetric systems, where m_i is the multiplicity of σ_i as a singular value.



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$$\sigma_{r+1} = \|H - \tilde{H}\|.$$

Generalised singular perturbation approximation

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Given $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) \geq 0$ and stable, minimal, balanced quadruple (A, B, C, D) , assume that the Hankel singular values are simple.

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Let \mathbf{G}_r^ξ denote the transfer function of the GSPA. Then

$\mathbf{G}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p \times m})$ has McMillan degree r , $\mathbf{G}_r^\xi(\xi) = \mathbf{G}(\xi)$ and

$$\|\mathbf{G} - \mathbf{G}_r^\xi\|_{H^\infty} \leq 2 \sum_{j=r+1}^n \sigma_j. \quad (2)$$

Generalised singular perturbation approximation

GSPA proposed in a control theoretic setting in [1]–[2] and properties studied across [3]–[6].

- [1] K. V. Fernando and H. Nicholson. Singular perturbational model reduction of balanced systems, *IEEE Trans. Automat. Control*, **27** (1982), 466–468.
- [2] K. V. Fernando and H. Nicholson. Singular perturbational model reduction in the frequency domain, *IEEE Trans. Automat. Control*, **27** (1982), 969–970.
- [3] U. M. Al-Saggaf and G. F. Franklin. Model reduction via balanced realizations: an extension and frequency weighting techniques, *IEEE Trans. Automat. Control*, **33** (1988), 687–692.
- [4] Y. Liu and B. D. O. Anderson. Singular perturbation approximation of balanced systems, *Internat. J. Control*, **50** (1989), 1379–1405.
- [5] P. Heuberger. A family of reduced order models based on open-loop balancing, in *Selected Topics in Identification, Modelling and Control*, Delft University Press, 1990, 1–10.
- [6] G. Muscato and G. Nunnari. On the σ -reciprocal system for model order reduction, *Math. Model. Systems*, **1** (1995), 261–271.

Model reduction for dissipative systems

- **Bounded-realness** and **positive-realness** are important **qualitative properties** pertaining to dissipation of energy in control systems
- May well be desirable for these properties to be retained in a reduced order model
- **Need not** be preserved in Lyapunov balanced GSPA (including balanced truncation and SPA)
- Balanced truncation and SPA have been extended to **bounded-real** and **positive-real** systems and make use of **bounded-real** and **positive-real balanced realisations**. Shown that:
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Bounded-real systems

- $\mathbf{G} \in H^\infty$ is **bounded real** if $\|\mathbf{G}\|_{H^\infty} \leq 1$ (and so $\|y\|_{L^2} \leq \|u\|_{L^2}$)
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- Let (A, B, C, D) denote a minimal realisation of \mathbf{G} . The following are equivalent.
 - \mathbf{G} is bounded real
 - There exists a triple (P, K, W) with $P = P^*$ positive-definite satisfying the bounded-real Lur'e equations

$$\begin{aligned}A^*P + PA + C^*C &= -K^*K \\ PB + C^*D &= -K^*W \\ I - D^*D &= W^*W\end{aligned}$$



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- If (i) or (ii) hold, then (3) has extremal solutions P_m, P_M in the sense that any $P = P^* \geq 0$ solving (3) satisfies

$$0 < P_m \leq P \leq P_M.$$

- The extremal operators P_m, P_M are the optimal cost operators of the bounded real optimal control problems:

$$\begin{aligned} \langle P_M x_0, x_0 \rangle_{\mathcal{X}} &= \inf_{u \in L^2(\mathbb{R}_-)} \int_{\mathbb{R}_-} \|u(s)\|^2 - \|y(s)\|^2 ds, \\ -\langle P_m x_0, x_0 \rangle_{\mathcal{X}} &= \inf_{u \in L^2(\mathbb{R}_+)} \int_{\mathbb{R}_+} \|u(s)\|^2 - \|y(s)\|^2 ds, \end{aligned}$$

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Definition

The realisation (A, B, C, D) is **bounded-real balanced** if $P_m = P_M^{-1} = \Sigma$.

- If P solves (3), then P^{-1} solves the dual equations

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- It is always possible to construct a **bounded-real balanced realisation** via a state-space transformation of a given realisation
- The eigenvalues of Σ are called the **bounded-real singular values** or **bounded-real characteristic values**
- Truncation takes place according to the size of these singular values
- Given a bounded real balanced realisation (A, B, C, D) , $\xi \in \mathbb{C}$, $\text{Re}(\xi) \geq 0$ and $r \in \{1, 2, \dots, n-1\}$, the **bounded-real GSPA** is given as before

$$\begin{aligned}A_\xi &:= A_{11} + A_{12}(\xi I - A_{22})^{-1}A_{21}, & B_\xi &:= B_1 + A_{12}(\xi I - A_{22})^{-1}B_2, \\C_\xi &:= C_1 + C_2(\xi I - A_{22})^{-1}A_{21}, & D_\xi &:= D + C_2(\xi I - A_{22})^{-1}B_2.\end{aligned}$$

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- \mathbf{R} is a so-called **spectral factor** of $I - \mathbf{G}^*\mathbf{G}$
- Using dual equations, can also obtain a spectral factor \mathbf{S} such that $I - \mathbf{G}\mathbf{G}^* = \mathbf{S}\mathbf{S}^*$ on imaginary axis

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- Similar statements apply to other spectral factor \mathbf{S} and \mathbf{S}_r^ξ
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Positive-real systems

- \mathbf{G} is **positive real** if $\mathbf{G}(s) + (\mathbf{G}(s))^* \geq 0$ for all $s \in \mathbb{C}_0$
- \mathbf{G} is **strongly positive real** if $\mathbf{G}(s) + (\mathbf{G}(s))^* \geq \delta I$ for all $s \in \mathbb{C}_0$
- Positive-real functions **need not** be stable $s \mapsto 1/s$ or proper $s \mapsto s$
- Let (A, B, C, D) denote a stable, minimal realisation of \mathbf{G} . The following are equivalent

- \mathbf{G} is positive real
- There exists a triple (P, K, W) with $P = P^*$ positive-definite satisfying the positive-real Lur'e equations

$$A^*P + PA = -K^*K$$

$$PB - C^* = -K^*W$$

$$D + D^* = W^*W$$

- For input $u \in L^2$ and output $y \in L^2$ with initial condition $x_0 = 0$

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- The previous equivalences are often called the **positive-real lemma** or **KYP lemma**
- Positive-real balanced realisations are morally the same as the bounded-real versions...
- ...now balance extremal solutions of positive-real Lur'e equations or positive-real algebraic Riccati equation
- Can either work from first principles or use bounded-real case and Cayley transform

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- (i) $(A_\xi, B_\xi, C_\xi, D_\xi)$ is positive real, and is positive real balanced if $\xi \in i\mathbb{R}$
- (ii) A_ξ is Hurwitz
- (iii) If (A, B, C, D) is strictly positive real, then $(A_\xi, B_\xi, C_\xi, D_\xi)$ is minimal and strictly positive real

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If \mathbf{G} is strongly positive real, then so is \mathbf{G}_r^ξ

Summary

- Model order reduction for linear control systems by the generalised singular perturbation approximation has been revisited
- Specifically, the GSPA preserves the same properties of bounded-real and positive-real systems as SPA and balanced truncation when defined in terms of dissipative balanced realisations
- The defining property of the GSPA is that the reduced order transfer function interpolates the original at ξ with $\operatorname{Re}(\xi) \geq 0$ — Lyapunov balanced truncation and the SPA are special cases of this
- The usual error bounds hold
- Possible application is to choose ξ to trade off interpolating at zero, or at infinity