Model Order Reduction of Hyperbolic Systems

H. Bansal[1]**, L. Iapichino**[1]**, W.H.A. Schilders**[1]**, N. van de Wouw**[1]

[1]Eindhoven University of Technology

1. Introduction

This work focusses towards model order reduction (MOR) of hyperbolic systems. The main contribution of this work is the application of Proper Orthogonal Decomposition - (Discrete) Empirical Interpolation Method, known as **POD-(D)EIM** to model-reduce the multi-phase flow model. This work also sheds light towards hybrid and alternative approaches for MOR.

2. Governing Models - Mathematical framework

Equation (1) represents a multiphase flow model, popularly called Drift Flux Model **(DFM)** and Equation (2) represents the acoustic propagation in homogeneous/ heterogeneous media.

$$\begin{pmatrix} \alpha_l\rho_l \\ \alpha_g\rho_g \\ \alpha_l\rho_l v_l + \alpha_g\rho_g v_g \end{pmatrix}_t + \begin{pmatrix} \alpha_l\rho_l v_l \\ \alpha_g\rho_g v_g \\ \alpha_l\rho_l {v_l}^2 + \alpha_g\rho_g {v_g}^2 + p \end{pmatrix}_x = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \text{ and } \begin{cases} \alpha_g + \alpha_l = 1 \\ \rho_g = p/{a_g}^2 \\ \rho_l = \rho_0 + (p - p_0)/{a_l}^2 \\ v_g = \frac{(K v_l \alpha_l + S)}{(1 - K\alpha_g)} \end{cases} \quad (1)$$

$$\begin{pmatrix} p \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & B \\ 1/\rho & 0 \end{pmatrix} \begin{pmatrix} p \\ v \end{pmatrix}_x = 0. \quad (2)$$

Parameters α, ρ, v, p, a and B are void fraction, density, velocity, pressure, sound speed in the medium and bulk modulus, respectively. Subscripts l and g refer to liquid and gaseous phase respectively. ρ_0 is the reference density and p_0 is the reference pressure. K and S are flow dependent parameters. s_1, s_2, s_3 are the source terms.

Both models belong to the class of **Hyperbolic Partial Differential Equations** (PDEs).
A dynamical system, of the form $\frac{dy}{dt} = L(y) + N(y)$ is obtained after the finite volume method is used to perform spatial discretization. Here, y refers to the states of the system, $L(y)$ refers to the linear term and $N(y)$ refers to the non-linear term in the obtained system of **Ordinary Differential Equations** (ODEs).
PDEs can also be written in the form of **Delay Differential Equations** (DDEs), whose general form is: $\frac{d}{dt}y(t) = f(t, y(t), y_t)$ where $y(t) \in \mathbb{R}^n$ and $y_t = \{y(\tau) : \tau \leq t\}$ represents the trajectory of the solution in the past.

3. Challenges for Model Order Reduction

- Presence of **travelling and standing waves**;
- Dynamical behavior of transport dominated phenomenon requires quite a **large** number of spatial modes for sufficient accurate approximation;
- Real world applications have **multiple transported quantities**.

4. Numerical Results

POD-(D)EIM is a model order reduction technique [1, 2] and it is applied to DFM, governed by Equation (1), in this work. Source terms have been neglected in this particular shock-tube test case.

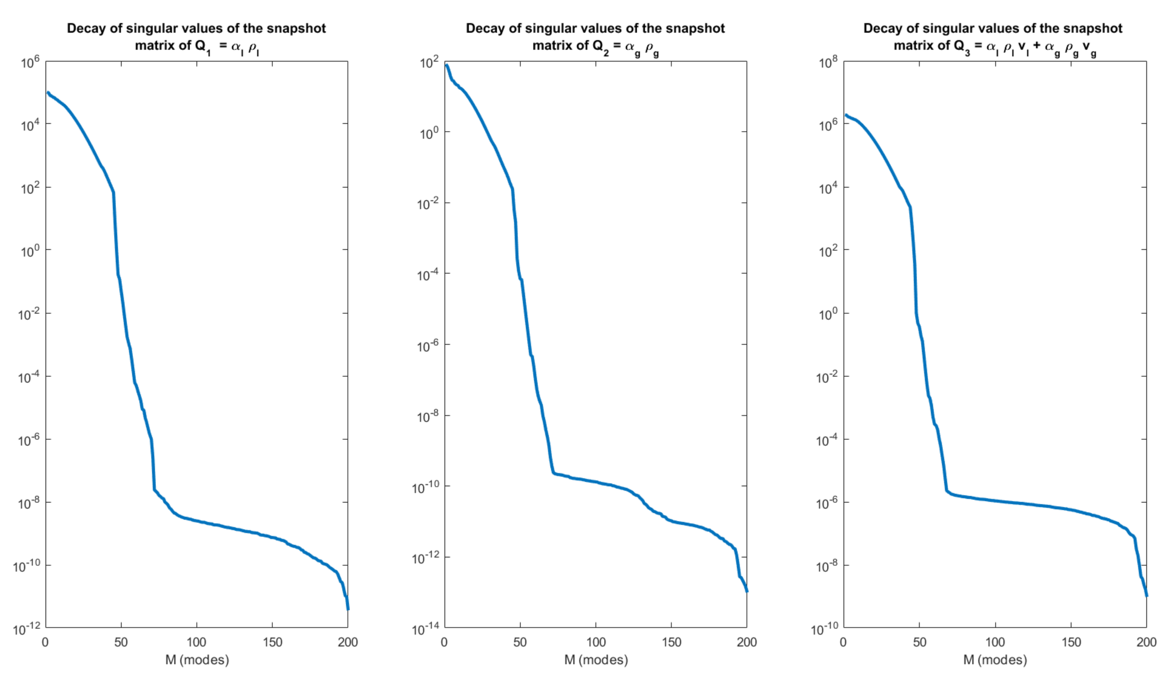

Figure 1: *Behavior of singular values for transported quantities.*

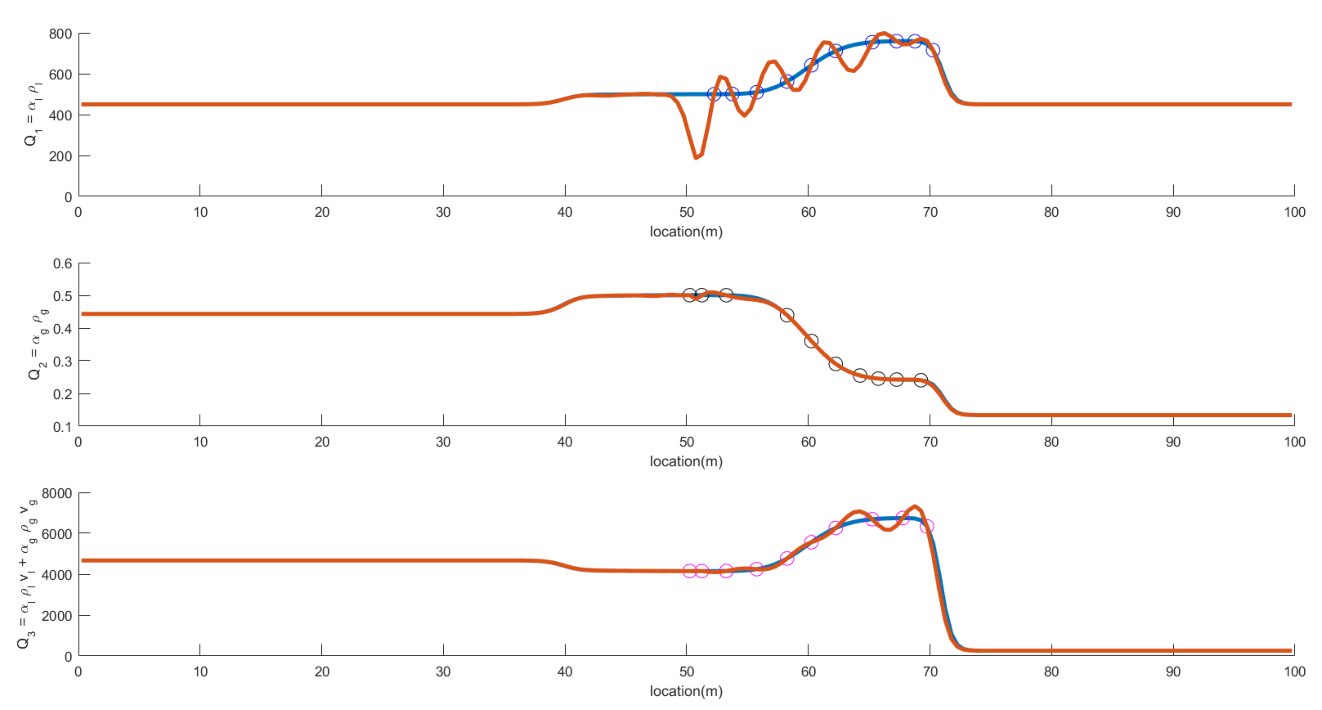

Figure 2: *POD-(D)EIM results for Drift Flux Model at t = 1s. Blue curve represents the high fidelity simulation, Red Curve represents the approximated solution and 'circle' markers represents value at (D)EIM interpolation indices. 10 basis were used to construct separate reconstructions for three different transport quantities. Major reason of using 10 basis was to demonstrate that POD-(D)EIM yields exact results at the (D)EIM interpolation indices.*

5. Envisaged Approach for Model Order Reduction of Hyperbolic Systems

POD-(D)EIM gives good numerical results. The question now is: **Can we do better to obtain lowest possible dimensional representation?**
The two key observations are as follows:

1. Rows 1 to 2 reveal that 60 (D)EIM basis are quite large to represent the physical feature. Row 4 represents that just sufficient basis have been used for accurate representation. Row 5 fails to reveal small gradient around $40m$ as per representation at selected (D)EIM interpolation indices;

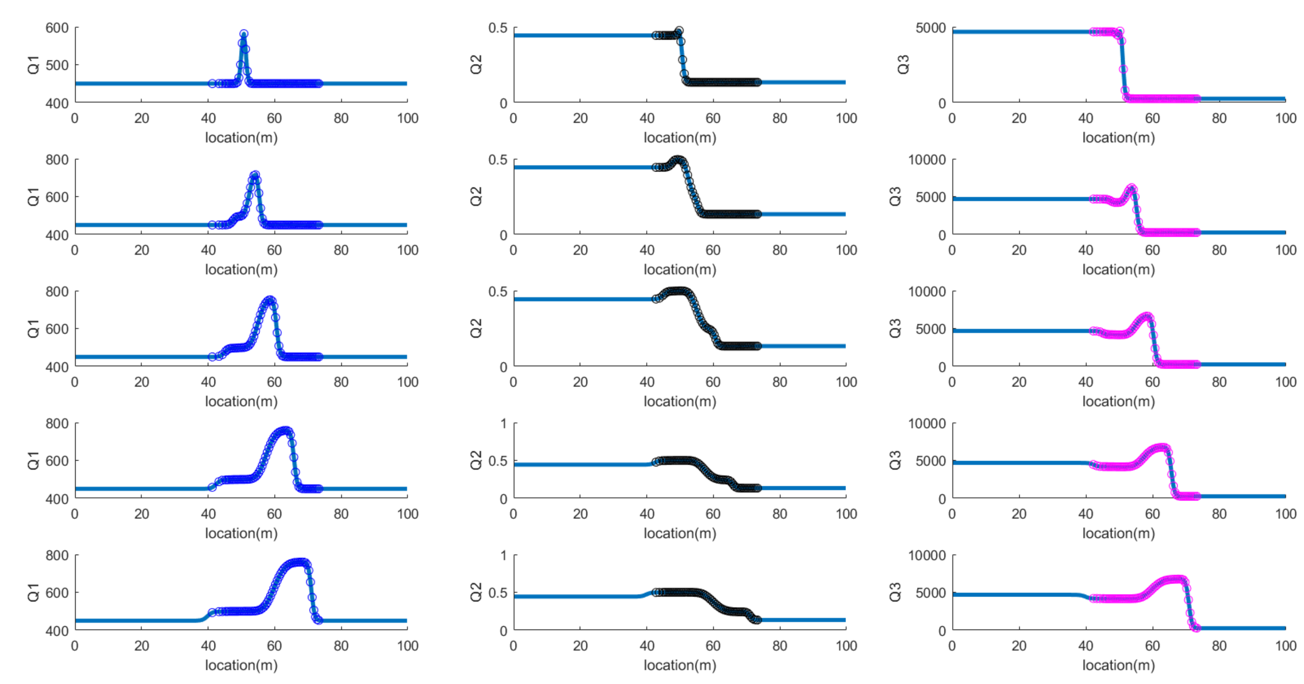

Figure 3: *Evolution of the transport quantities when 60 (D)EIM basis are used. Rows 1 to 5 represent the behavior at Δt, $500\Delta t$, $1000\Delta t$, $1500\Delta t$ and $2000\Delta t$ respectively. Columns 1 to 3 represent the conservative quantities Q_1, Q_2 and Q_3 respectively.*

2. Transport quantities have underlying **diagonal structure** in space-time diagram. Figure 4 (Right) can be represented appropriately by 2 basis if snapshot matrix is pre-processed such that the shifted matrix corresponds to the solution in **co-moving reference frame** [3].

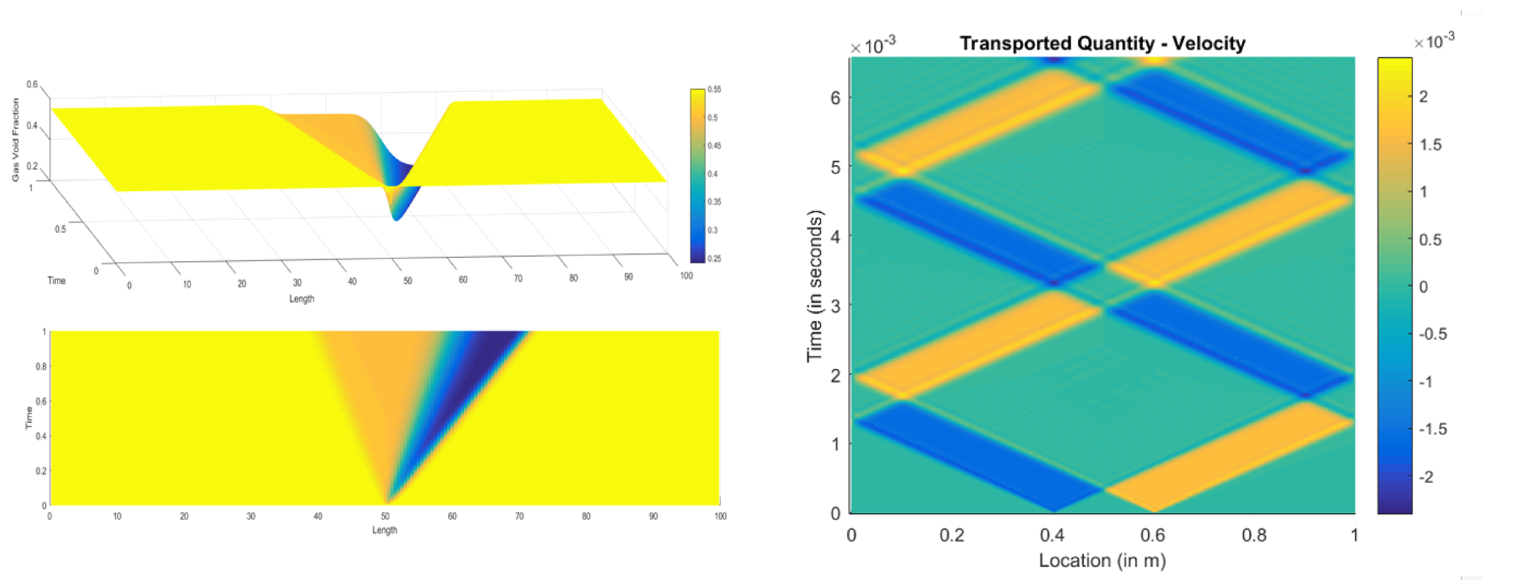

Figure 4: *(Left): Space-time behavior of gas-void fraction obtained from DFM and (Right): Space-time behavior of velocity obtained from acoustic equations.*

It can be concluded that an **adaptive basis construction**, for example, by exploiting **localization across space or time** [4], is required for appropriate feature representation and efficient resource management. Keeping above insights in mind, the envisaged procedure is:

- **Pre-process the snapshot matrix** (of the solution of the PDE or the quantity of interest) to obtain the **stronger decay of the singular values**, as in [5] and references there-in;
- Obtain adaptive basis using the knowledge of underlying dynamics/ transport structure;
- Perform model order **reduction on the transformed matrix and reconstruct** the solution.

Another envisaged approach for MOR is pursued in the form of DDEs to accurately approximate **wave effects** [6].

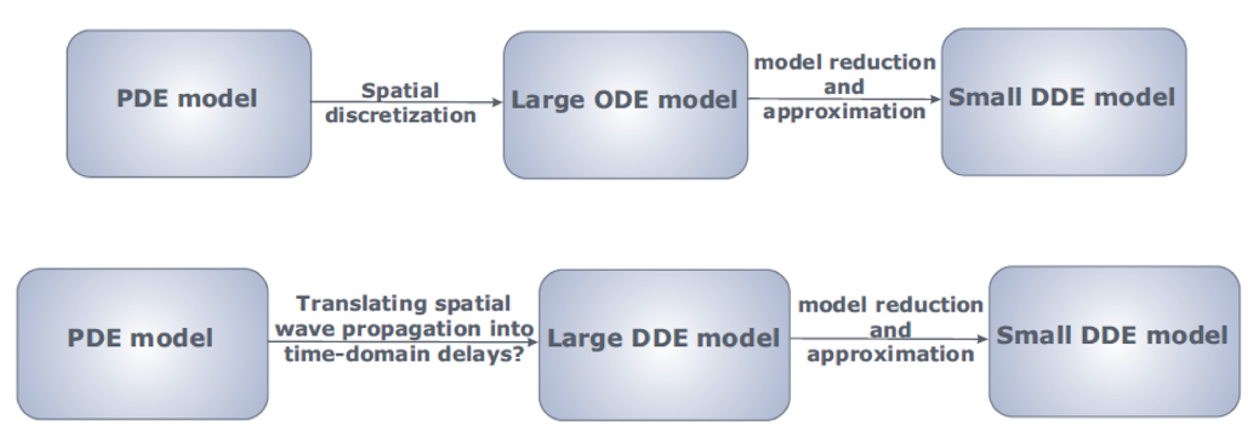

Figure 5: *Two different angles of attack for complexity reduction via DDEs.*

6. Future Work

The aim is to apply model order reduction techniques to study:

- **Fast transients** - Propagation of pressure pulses;
- **Slow transients** - Transport of a gas volume fraction contact discontinuity.

in the context of simulation and control of multi-phase model, governed by Equation (1).

7. References

1. Chaturantabut, S., and Sorensen, D. C. "Nonlinear Model Reduction via Discrete Empirical Interpolation", SIAM Journal on Scientific Computing 32, 5 (Jan. 2010), 2737-2764.
2. Grepl, M. A., Maday, Y., Nguyen, N. C., and Patera, A. T. "Efficient reduced-basis treatment of nonaffine and nonlinear partial differential equations", ESAIM: Mathematical Modelling and Numerical Analysis 41, 3 (May 2007), 575-605.
3. Reiss, J., Schulze, P., and Sesterhenn, J. "The shifted proper orthogonal decomposition: A mode decomposition for multiple transport phenomena", arXiv preprint arXiv:1512.01985 (2015).
4. Peherstorfer, B., Butnaru, D., Willcox, K., and Bungartz, H.-J. "Localized discrete empirical interpolation method", SIAM Journal on Scientific Computing 36, 1 (2014), A168-A192.
5. Rim, D., Moe, S., and LeVeque, R. J. "Transport reversal for model reduction of hyperbolic partial differential equations", arXiv preprint arXiv:1701.07529 (2017).
6. van de Wouw, N., Michiels, W., Besselink, B. "Model Reduction for Delay Differential Equations with Guaranteed Stability and Error Bound ", Automatica, Vol. 35, pp. 132-139, 2015