# Solving high-dimensional nonlinear partial differential equations and high-dimensional nonlinear backward stochastic differential equations using deep learning 

What can data science contribute to stochastic analysis?

$$
\begin{aligned}
& \text { Arnulf Jentzen (ETH Zurich, Switzerland) } \\
& \text { Joint works with } \\
& \text { Weinan E (Beijing Institute of Big Data Research, China, } \\
& \text { Princeton University, USA, \& Beijing University, China), } \\
& \text { Jiequn Han (Princeton University, USA) }
\end{aligned}
$$

London Mathematical Society - EPSRC Durham Symposium on Stochastic Analysis, Durham, UK Organized by Tom Cass (Imperial), Dan Crisan (Imperial), \& David Applebaum (Sheffield)

$$
\text { July 14, } 2017
$$

Introduction

Consider $T>0, d \in \mathbb{N}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}, u \in C\left([0, T] \times \mathbb{R}^{d}, \mathbb{R}\right)$
such that $u(T, x)=g(x),\left.u\right|_{[0, T) \times \mathbb{R}^{d}} \in C^{1,2}\left([0, T) \times \mathbb{R}^{d}, \mathbb{R}\right)$, and

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\begin{equation*}
\frac{\partial u}{\partial t}(t, x)+\frac{1}{2}\left(\Delta_{x} u\right)(t, x)+f\left(u(t, x),\left(\nabla_{x} u\right)(t, x)\right)=0 . \tag{PDE}
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for $t \in[0, T), x \in \mathbb{R}^{d}$. Goal: Solve (PDE) approximatively.

## Applications: Pricing of financial derivatives,

portfolio optimization, operations research
Approximations methods such as finite element methods, finite differences, sparse grids, regression methods suffer under the curse of dimensionality.

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## Deep BSDE solver

Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$, normal filtration $\mathbb{T}$ generated by $W$, continuous and adapted $\gamma:[0, T] \times \Omega \rightarrow \mathbb{R}$ and $Z:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ such that $\forall t \in[0, T] \mathbb{P}$-a.s. :

$$
\begin{equation*}
Y_{t}=g\left(\xi+W_{T}\right)+\int_{t}^{T} f\left(Y_{s}, Z_{s}\right) d s-\int_{t}^{T}\left\langle Z_{s}, d W_{s}\right\rangle_{\mathbb{R}^{d}} \tag{BSDE}
\end{equation*}
$$

Under suitable assumptions (Pardoux \& Peng $1990 \ldots$ ) it holds $\forall t \in[0, T] \mathbb{P}$-a.s. :

$$
Y_{t}=u\left(t, \xi+W_{t}\right) \in \mathbb{R} \quad \text { and } \quad Z_{t}=\left(\nabla_{x} u\right)\left(t_{t} \xi+W_{t}\right) \in \mathbb{R}^{d} .
$$

Hence, $\forall t \in[0, T] \mathbb{P}$-a.s. :
$Y_{t}=g\left(\xi+W_{T}\right)+\int_{t}^{T} f\left(Y_{s},\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right)\right) d s-\int_{t}^{T}\left\langle\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right), d W_{s}\right\rangle_{\mathbb{R}^{d}}$. In particular, $\forall t_{1}, t_{2} \in[0, T]$ with $t_{1} \leq t_{2}$ it holds $\mathbb{P}$-a.s. that

$$
Y_{t_{2}}=Y_{t_{1}}-\int_{t_{1}}^{t_{2}} f\left(Y_{s},\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right)\right) d s+\int_{t_{1}}^{t_{2}}\left\langle\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right), d W_{s}\right\rangle_{\mathbb{R}^{d}}
$$

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that
$Y_{t_{t+1}} \approx$
$Y_{t_{n}}-f\left(Y_{t_{n}},\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}}$.

Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$,
normal filtration $\mathbb{F}$ generated by $W$, continuous and adapted $Y:[0, T] \times \Omega \rightarrow \mathbb{R}$ and $Z:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ such that $\forall t \in[0, T] \mathbb{P}$-a.s. :

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Hence, $\forall t \in[0, T] \mathbb{P}$-a.s.
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In particular, $\forall t_{1}, t_{2} \in[0, T]$ with $t_{1} \leq t_{2}$ it holds $\mathbb{P}$-a.s. that
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$Y_{t_{t+1}} \approx$
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Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$, normal filtration $\mathbb{F}$ generated by $W$, continuous and adapted $Y:[0, T] \times \Omega \rightarrow \mathbb{R}$ and $Z:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ such that $\forall t \in[0, T] \mathbb{P}$-a.s.

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$$

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that
$Y_{t_{n+1}} \approx$
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## Consider

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\Lambda=\left(\Lambda_{1}, \ldots, \Lambda_{\rho}\right)=\operatorname{argmin}_{\theta \in \mathbb{R}^{\rho}} \mathbb{E}\left[\left|\mathcal{Y}_{N}^{\theta}-g\left(\xi+W_{T}\right)\right|^{2}\right]
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(Optimization problem)
We suggest that

$$
\Lambda_{1} \approx u(0, \xi)
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Consider stochastic gradient descent-type approximations
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Consider

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\Lambda=\left(\Lambda_{1}, \ldots, \Lambda_{\rho}\right)=\operatorname{argmin}_{\theta \in \mathbb{R}^{p}} \mathbb{E}\left[\left|\mathcal{Y}_{N}^{\theta}-g\left(\xi+W_{T}\right)\right|^{2}\right]
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(Optimization problem)

## We suggest that

Consider stochastic gradient descent-type approximations
$\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}$ associated to (Optimization problem). We
suggest for sufficiently large $N, \rho, m \in \mathbb{N}$ that

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that
$Y_{t_{n+1}} \approx$
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$\forall n=0,1, \ldots, N-1$ :
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We suggest that

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\Lambda_{1} \approx u(0, \xi)
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## Consider

 $0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$,$0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$, functions $\nu_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}$, $0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function $\mathcal{Y}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k}$ satisfying $\mathcal{Y}_{0}^{\theta, m}=\theta_{1}$ and $\forall n=0,1, \ldots, N-1$. $\mathcal{V}_{n+1}^{\theta, m}=\mathcal{Y}_{n}^{\theta, m}-f\left(\mathcal{Y}_{n}^{\theta, m}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right), W_{t_{n+1}}^{m}-W_{t_{n}}^{m}\right\rangle_{\mathbb{R}}$ for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

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and $\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying


Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$, $0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$, functions $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}$, $0 \leq n \leq N$, a function
for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

$$
\forall \theta \in \mathbb{R}^{n}: \quad \phi^{m}(\theta)=\left|\gamma_{N}^{\theta \cdot m}-g^{\prime}\left(\xi+W_{T}^{m}\right)\right|^{2}
$$

for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying and $\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying


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for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying
and $\Theta=\left(\Theta^{(1)}\right.$


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for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}: \quad \phi^{m}(\theta)=\left|\nu_{N}^{\theta \cdot m}-g^{\prime}\left(\xi+W_{T}^{m}\right)\right|^{2}
$$

for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying


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$$
\begin{aligned}
& \text { Consider } T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}, \\
& 0=t_{0}<t_{1}<\ldots<t_{N}=T, \text { probability space }(\Omega, \mathcal{F}, \mathbb{P}) \text {, independent Brownian } \\
& \text { motions } W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0} \text {, functions } \mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}, \\
& 0 \leq n \leq N, \text { for every } m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho} \text { a function } \\
& \mathcal{Y}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k} \text { satisfying } \mathcal{Y}_{0}^{\theta, m}=\theta_{1} \text { and } \forall n=0,1, \ldots, N-1: \\
& \mathcal{Y}_{n+1}^{\theta, m}=\mathcal{Y}_{n}^{\theta, m}-f\left(\mathcal{Y}_{n}^{\theta, m}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right), W_{t_{n+1}}^{m}-W_{t_{n}}^{m}\right\rangle_{\mathbb{R}^{d}},
\end{aligned}
$$

for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying
for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying


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& \text { Consider } T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}, \\
& 0=t_{0}<t_{1}<\ldots<t_{N}=T, \text { probability space }(\Omega, \mathcal{F}, \mathbb{P}) \text {, independent Brownian } \\
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& \mathcal{Y}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k} \text { satisfying } \mathcal{Y}_{0}^{\theta, m}=\theta_{1} \text { and } \forall n=0,1, \ldots, N-1: \\
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for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}: \quad \phi^{m}(\theta)=\left|\mathcal{Y}_{N}^{\theta, m}-g\left(\xi+W_{T}^{m}\right)\right|^{2}
$$

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\forall \theta \in \mathbb{R}^{\rho}, \omega \in \Omega: \quad \Phi^{m}(\theta, \omega)=\left(\nabla_{\theta} \phi^{m}\right)(\theta, \omega)
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and $\Theta=\left(\Theta^{(1)}\right.$,


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\forall \theta \in \mathbb{R}^{\rho}, \omega \in \Omega: \quad \Phi^{m}(\theta, \omega)=\left(\nabla_{\theta} \phi^{m}\right)(\theta, \omega)
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and $\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

$$
\forall m \in \mathbb{N}: \quad \Theta_{m}=\Theta_{m-1}-\gamma \cdot \Phi^{m}\left(\Theta_{m-1}\right) .
$$

Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$, $0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$, functions $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}$, $0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function $\mathcal{Y}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k}$ satisfying $\mathcal{Y}_{0}^{\theta, m}=\theta_{1}$ and $\forall n=0,1, \ldots, N-1$ :
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and $\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

$$
\forall m \in \mathbb{N}: \quad \Theta_{m}=\Theta_{m-1}-\gamma \cdot \Phi^{m}\left(\Theta_{m-1}\right) .
$$

We suggest for sufficiently large $N, \rho, m \in \mathbb{N}$ that

Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$, $0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$, functions $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}$, $0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function $\mathcal{Y}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k}$ satisfying $\mathcal{Y}_{0}^{\theta, m}=\theta_{1}$ and $\forall n=0,1, \ldots, N-1$ :
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\forall \theta \in \mathbb{R}^{\rho}, \omega \in \Omega: \quad \Phi^{m}(\theta, \omega)=\left(\nabla_{\theta} \phi^{m}\right)(\theta, \omega)
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$$
\forall m \in \mathbb{N}: \quad \Theta_{m}=\Theta_{m-1}-\gamma \cdot \Phi^{m}\left(\Theta_{m-1}\right) .
$$

We suggest for sufficiently large $N, \rho, m \in \mathbb{N}$ that $\Theta_{m}^{(1)} \approx u(0, \xi)$.

## Numerical simulations

> Implementations in Python using TensorFlow on a Macpook Pro 2.9 GHz (Intel i5, 16 GB RAM)

## Numerical simulations

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## 100-dimensional Allen-Cahn equation

Consider

$$
\frac{\partial u}{\partial t}(t, x)=\left(\Delta_{x} u\right)(t, x)+u(t, x)-[u(t, x)]^{3}
$$

(Allen-Cahn)
with $u(0, x)=\frac{1}{\left(2+0.4\|x\|^{2}\right)}$ for $t \in\left[0, \frac{3}{10}\right], x \in \mathbb{R}^{100}$.

(a) Relative $L^{1}$-error for $u\left(\frac{3}{10}, 0\right) \approx 0.0528$

(b) Approximative plot of $u(t, 0), 0 \leq t \leq \frac{3}{10}$

Deep BSDE solver $\left(N=20, \gamma=\frac{5}{10000}\right): L^{1}$-error: $0.3 \%$, Runtime: 647 seconds.

## 100-dimensional Hamiltonian-Jacobi-Bellmann equation

Consider

$$
\begin{equation*}
\frac{\partial u}{\partial t}(t, x)+\left(\Delta u_{x}\right)(t, x)-\lambda\left\|\left(\nabla_{x} u\right)(t, x)\right\|^{2}=0 \tag{HJB}
\end{equation*}
$$

with $u(1, x)=\frac{2}{\left(1+\|x\|^{2}\right)}, \lambda \geq 0$ for $t \in[0,1], x \in \mathbb{R}^{100}$.

(a) Relative $L^{1}$-error when $\lambda=1$

(b) Optimal cost against different $\lambda$

Deep BSDE solver ( $N=20, \gamma=\frac{1}{100}$ ) : $L^{1}$-error: $0.17 \%$, Runtime: 330 seconds.

## 100-dimensional nonlinear derivative pricing model

Duffie, Schroder, \& Skiadas 1996 AAP, Bender, Schweizer, \& Zhuo 2015 MF:

$$
\frac{\partial u}{\partial t}(t, x)+\bar{\mu}\left\langle x,\left(\nabla_{x} u\right)(t, x)\right\rangle_{\mathbb{R}^{d}}+\frac{\bar{\sigma}^{2}}{2} \sum_{i=1}^{d}\left|x_{i}\right|^{2} \frac{\partial^{2} u}{\partial x_{i}^{2}}(t, x)-Q(u(t, x)) u(t, x)-R u(t, x)
$$

with $u(1, x)=\min \left\{x_{1}, \ldots, x_{100}\right\}, \bar{\mu}=0.02, \bar{\sigma}=0.2$ for $t \in[0,1], x \in \mathbb{R}^{100}$.

(a) Relative $L^{1}$-error for $u(0,100, \ldots) \approx 57.3$

Deep BSDE solver $\left(N=40, \gamma=\frac{8}{1000}\right): L^{1}$-error: $0.46 \%$, Runtime: 617 seconds.

Numerics for forward stochastic differential equations

## Theorem (Gerencsér, J, \& Salimova 2017 PRSL A (minor revision))

Let $T \in(0, \infty), d \in\{2,3,4, \ldots\}, \delta \in \mathbb{R}^{d},\left(a_{N}\right)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy
$\lim _{N \rightarrow \infty} a_{N}=0$. Then there exist globally bounded $\mu, \sigma \in C^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

$$
\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi
$$

and every $N \in \mathbb{N}$ we have

- Dimension $d \geq 4$ and Euler scheme: Hairer, Hutzenthaler, \& J 2015 AOP
- Dimension $d>4$ : J, Müller-Gronbach \& Varoslavtseva 2016 CMS
- Weak convergence and $d \geq 4$ : Müller-Gronbach \& Yaroslavtseva 2016 SAA (to appear)
- Adantive anproximations and $d \geq$ 4: Yaroslavtseva 2016


## Theorem (Gerencsér, J, \& Salimova 2017 PRSL A (minor revision))

Let $T \in(0, \infty), d \in\{2,3,4, \ldots\}, \xi \in \mathbb{R}^{d},\left(a_{N}\right)_{N \in N} \subseteq \mathbb{R}$ satisfy
$\lim _{N \rightarrow \infty} a_{N}=0$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

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\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi,
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## Theorem (Gerencsér, J, \& Salimova 2017 PRSL A (minor revision))

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\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi,
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$$
\frac{\partial}{\partial t} x_{t}=\mu\left(x_{t}\right)+\sigma\left(x_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad x_{0}=\xi
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$$

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$$

and every $N \in \mathbb{N}$ we have

$$
\inf _{\inf _{1, \ldots,}, \ldots, s_{N} \in[0, T]} \inf _{\substack{\mathbb{R}^{N} \rightarrow \mathbb{R}^{d} \\ \text { measurable }}} \mathbb{E}\left[\left\|X_{T}-u\left(W_{s_{1}}, \ldots, W_{s_{N}}\right)\right\|\right] \geq a_{N} .
$$

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\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi,
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## Theorem (Hefter \& J 2017)

Let $T, \delta, \beta \in(0, \infty), \gamma, \xi \in[0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let
$W:[0, T] \times \Omega \rightarrow \mathbb{R}$ be a Brownian motion, let $X:[0, T] \times \Omega \rightarrow \mathbb{R}$ be a solution of

$$
\frac{\partial}{\partial t} X_{t}=\left(\delta-\gamma X_{t}\right)+\beta \sqrt{X_{t}} \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi .
$$

Then there exists a c $\in(0, \infty)$ such that for all $N \in \mathbb{N}$ we have

$$
\begin{equation*}
\inf _{\mathbb{R}^{N} \rightarrow \mathbb{R}}\left[X_{T}-u\left(W_{\frac{T}{N}}, W_{\frac{2 T}{N}}, \ldots, W_{T}\right)\right]^{]} \geq c \cdot N^{-\operatorname{mn}\left\{1, \frac{25}{\beta^{2}}\right\}} \tag{*}
\end{equation*}
$$

measurable
Deelstra \& Delbaen 1998 Appl Stoch Models Data Anal: Let
$Y^{N}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}, N \in \mathbb{N}$, satisfy for all $N \in \mathbb{N}, n \in\{0,1, \ldots, N-1\}$
that $Y_{0}^{N}=\xi$ and

$$
Y_{n+1}^{N}=Y_{n}^{N}+\left(\delta-\gamma Y_{n}^{N}\right) \frac{T}{N}+\beta \sqrt{\left[Y_{n}^{N}\right]^{+}}\left(W_{\frac{(n+1) T}{N}}-W_{\frac{n T}{N}}\right) .
$$

There exists a $c \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ we have

$$
\begin{equation*}
\mathbb{E}\left[\left|X_{T}-Y_{N}^{N}\right|\right] \leq c \cdot N^{-\frac{1}{2}} . \tag{}
\end{equation*}
$$

$\left(^{*}\right)$ disproves (**).

## Theorem (Hefter \& J 2017)

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$$

Then there exists a $c \in(0, \infty)$ such that for all $N \in \mathbb{N}$ we have

$$
\begin{equation*}
\inf _{u: \mathbb{R}^{N} \rightarrow \mathbb{R}} \mathbb{E}\left[\left\lvert\, X_{T}-u\left(W_{\frac{T}{N}}, W_{\frac{2 T}{N}}, \ldots, W_{T}\right)\right.\right] \geq c \cdot N^{-\operatorname{mn}}\left\{1, \frac{28}{B^{2}}\right\} . \tag{*}
\end{equation*}
$$

Deelstra \& Delbaen 1998 Appl Stoch Models Data Anal: Let $Y^{N}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}, N \in \mathbb{N}$, satisfy for all $N \in \mathbb{N}, n \in\{0,1, \ldots, N-1\}$ that $Y_{0}^{N}=\xi$ and

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$$

There exists a $c \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ we have

$$
\begin{equation*}
\mathbb{E}\left[\left|X_{T}-V_{N}^{N}\right|\right] \leq c \cdot N^{-\frac{1}{2}} \tag{}
\end{equation*}
$$

(*) disproves (**).

## Theorem (Hefter \& J 2017)

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$$

Then there exists a $c \in(0, \infty)$ such that for all $N \in \mathbb{N}$ we have $\inf _{u: \mathbb{N}_{N}} \mathbb{E}\left[\left\lvert\, X_{T}-u\left(W_{\frac{T}{N}}, W_{\frac{2 T}{N}}, \ldots, W_{T}\right)\right.\right] \geq 0 \cdot N^{-\operatorname{mn}}\left\{1, \frac{25}{B^{2}}\right\}$
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Let $T, \delta, \beta \in(0, \infty), \gamma, \xi \in[0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W:[0, T] \times \Omega \rightarrow \mathbb{R}$ be a Brownian motion,


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${ }^{(*)}$ disproves (**).

Thanks for your attention!

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## Pricing with default risk

\& Zhuo 2015 MF$) T=1, d=1, \xi=(100$,
$\frac{\partial}{\partial t} u(t, x)+f(x, u(t, x))+\bar{\mu} \sum_{i=1}^{d} x_{i}\left(\frac{\partial}{\partial x_{i}} u\right)(t, x)+\frac{\bar{\sigma}^{2}}{2} \sum_{i=1}^{d}\left|x_{i}\right|^{2}\left(\frac{\partial^{2}}{\partial x_{i}^{2}} u\right)(t, x)=0$ for $(t, x) \in[0, T) \times \mathbb{R}^{d}$. Relative arrors $\frac{1}{10} \sum^{10}\left|w_{p, \rho}(0,5)-\cdots\right|$ for $\rho \in\{1,2, \ldots, 7\}$ against runtime; $u(0, \xi) \approx \mathrm{v}=97.705$.


Pricing with default risk (Duffie, Schroder, \& Skiadas 1996 AAP, Bender, Schweizer, \& Zhuo 2015 MF)
for $(t, x) \in[0, T) \times \mathbb{R}^{d}$. Relative errors $\frac{1}{10|v|} \sum_{i=1}^{10}\left|u_{\rho, \rho}^{i}(0, \xi)-v\right|$ for $\rho \in\{1,2, \ldots, 7\}$ against runtime; $u(0, \xi) \approx \mathrm{v}=97.705$.


Pricing with default risk (Duffie, Schroder, \& Skiadas 1996 AAP, Bender, Schweizer, \& Zhuo 2015 MF) $T=1, d=1, \xi=(100, \ldots, 100) \in \mathbb{R}^{d}, u(T, x)=\min _{1 \leq i \leq d} x_{i}$, against runtime; $u(0, \xi) \approx \mathrm{v}=97.705$.


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$$
\begin{aligned}
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& \text { for }(t, x) \in[0, T) \times \mathbb{R}^{d} .\left[\frac{1}{10} \sum_{i=1}^{10}\left|\mathbf{U}_{\rho+1, \rho+1}^{i}(0, \xi)-\mathbf{U}_{\rho, \rho}^{i}(0, \xi)\right|\right] /\left[\frac{1}{10}\left|\sum_{i=1}^{10} \mathbf{U}_{7,7}^{i}(0, \xi)\right|\right] \text { for } \\
& \rho \in\{1,2, \ldots, 6\} \text { against runtime; } u(0, \xi) \approx 58.113 .
\end{aligned}
$$



Pricing with default risk Runtime for one realization of $\mathbf{U}_{6.6}^{1}(0, \xi)$ against dimension $d \in\{5,6, \ldots, 100\}$.


Pricing with default risk (Duffie, Schroder, \& Skiadas 1996 AAP, Bender, Schweizer, \& Zhuo 2015 MF)

Consider $\delta=\frac{2}{3}, R=\frac{2}{100}, \gamma^{h}=\frac{2}{10}, \gamma^{\prime}=\frac{2}{100}, \bar{\mu}=\frac{2}{100}, \bar{\sigma}=\frac{2}{10}, v^{h}, v^{\prime} \in(0, \infty)$ satisfy $v^{h}<v^{\prime}$, and assume for all $x \in \mathbb{R}^{d}, y \in \mathbb{R}$ that

$$
\mu(x)=\bar{\mu} x, \quad \sigma(x)=\bar{\sigma} \operatorname{diag}(x)
$$

and

$$
\begin{aligned}
& f(x, y)=-(1-\delta) y\left[\gamma^{h} \mathbb{1}_{\left(-\infty, v^{h}\right)}(y)+\gamma^{\prime} \mathbb{1}_{\left[v^{\prime}, \infty\right)}(y)\right. \\
&\left.+\left[\frac{\left(\gamma^{h}-\gamma^{\prime}\right)}{\left(v^{h}-v^{\prime}\right)}\left(y-v^{h}\right)+\gamma^{h}\right] \mathbb{1}_{\left[v^{h}, v^{\prime}\right)}(y)\right]-R y .
\end{aligned}
$$

- We consider $v^{h}=50, v^{\prime}=120$ in the case $d=1$.
- Bender et al. consider $v^{h}=54, v^{\prime}=90$ in the case $d=5$.
- We consider $v^{h}=47, v^{\prime}=65$ in the case $d=100$.

