Pöschl-Teller made relativistic

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Durham Symposium GAAI, 25 July–4 August 2016

Outline

1. Nonrelativistic 1D potential scattering

- As an introductory reminder, we consider self-adjoint Schrödinger operators of the form ($\hbar \equiv 1$)

$$H_0 = -d^2/dx^2, \quad H = -d^2/dx^2 + V(x),$$

with $V(x)$ real-valued.

- Two 'position space' Hilbert spaces occur:

$$\mathcal{H}_s \equiv L^2((0,\infty), dx), \quad \mathcal{H}_d \equiv L^2((-\infty,\infty), dx).$$

- With suitable assumptions on $V(x)$, we recall the connection of the wave operators

$$W_\pm = \lim_{t\to\pm\infty} e^{itH} e^{-itH_0},$$

from time-dependent scattering theory with time-independent scattering theory in terms of (improper) eigenfunctions

$$H\Psi = p^2\Psi, \quad p > 0,$$

with unitary asymptotics.

1A. Scattering on the half-line

- Assume $V(x)$ is smooth on $(0, \infty)$, vanishes quickly for $x \to \infty$, and satisfies

$$V(x) \to \infty, \quad x \to 0, \qquad V'(x) < 0, \quad x > 0.$$

- With Dirichlet b. c. at $x = 0$, the interacting and free evolutions $\exp(-itH)$ and $\exp(-itH_0)$ on $\mathcal{H}_s$ can be compared via the wave operators $W_\pm$. They are unitary, with the scattering encoded in the (position space) S-operator

$$S \equiv W_+^* W_- .$$

- This can be made more explicit by using the so-called incoming wave functions

$$H\Psi = p^2\Psi, \; p > 0, \quad \Psi(x,p) \sim u(p)e^{ixp} - e^{-ixp}, \quad x \to \infty,$$

with $u(p) =: \hat{S}_s(p)$ the unitary S-matrix $(|\hat{S}_s(p)| = 1)$.

- The sine transform

$$(\mathcal{F}_0 f)(x) \equiv \sqrt{\frac{1}{2\pi}} \int_0^\infty dp \left(e^{ixp} - e^{-ixp}\right) f(p), \quad f \in C_0^\infty((0,\infty)),$$

diagonalizes H_0 on $\hat{\mathcal{H}}_s \equiv L^2((0,\infty), dp)$ ('momentum space'):

$$H_0 \mathcal{F}_0 = \mathcal{F}_0 p^2.$$

- Letting

$$(\mathcal{F} f)(x) \equiv \sqrt{\frac{1}{2\pi}} \int_0^\infty dp \, \Psi(x,p) f(p), \quad f \in C_0^\infty((0,\infty)),$$

we get more generally a unitary operator from $\hat{\mathcal{H}}_s$ to $\mathcal{H}_s$ such that

$$H\mathcal{F} = \mathcal{F} p^2.$$

- We also have

$$\mathcal{F} = W_- \mathcal{F}_0, \quad \mathcal{F}\hat{S}^* = W_+ \mathcal{F}_0, \quad (\hat{S} f)(p) \equiv \hat{S}_s(p) f(p),$$

with $\hat{S} = \mathcal{F}_0^* S \mathcal{F}_0$ the momentum space scattering operator.

1B. Scattering on the line

- Assume $V(x)$ is smooth, even, vanishes quickly for $|x| \to \infty$, and satisfies $V'(x) > 0$ for $x > 0$. Such V have finitely many bound states, i. e.,

$$H\Psi_\ell = E_\ell \Psi_\ell, \; E_\ell < 0, \; \Psi_\ell \in \mathcal{H}_d = L^2(\mathbb{R}, dx), \; \ell = 0, \ldots, L-1.$$

- The wave operators $W_\pm$ exist and are isometric, with range equal to the orthogonal complement of the bound states. Thus, the position space S-operator $S = W_+^* W_-$ is unitary.
- A corresponding unitary S-matrix

$$\hat{S}_d(p) \equiv \begin{pmatrix} t(p) & r(p) \\ r(p) & t(p) \end{pmatrix}, \quad p > 0,$$

on the momentum space $\hat{\mathcal{H}}_d \equiv L^2((0,\infty), dp)^2$ arises as follows.

- Diagonalize H_0 and H via eigenfunction transforms

$$(\mathcal{F}_{(0)}f)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dp \begin{pmatrix} \Psi_{(0)}(x,p) \\ -\Psi_{(0)}(-x,p) \end{pmatrix} \cdot \begin{pmatrix} f_+(p) \\ f_-(p) \end{pmatrix},$$

with

$$H_{(0)}\Psi_{(0)} = p^2\Psi_{(0)}.$$

- For H_0 choose $\Psi_0(x,p) = \exp(ixp)$, so $\mathcal{F}_0$ amounts to the Fourier transform, with $\hat{f} \in L^2(\mathbb{R}, dp)$ yielding $(f_+, f_-) \in \hat{\mathcal{H}}_d$ via

$$f_+(p) \equiv \hat{f}(p), \quad f_-(p) \equiv -\hat{f}(-p), \quad p > 0.$$

- For H choose the incoming wave function $\Psi(x,p)$:

$$H\Psi = p^2\Psi, \; p > 0, \;\; \Psi(x,p) \sim \begin{cases} t(p)e^{ixp}, & x \to \infty, \\ e^{ixp} - r(p)e^{-ixp}, & x \to -\infty. \end{cases}$$

(So $\Psi(x,p)/t(p)$ is a Jost function.)

- Once more, we get $H_{(0)}\mathcal{F}_{(0)} = \mathcal{F}_{(0)}p^2$ and

$$\mathcal{F} = W_-\mathcal{F}_0, \mathcal{F}\hat{S}^* = W_+\mathcal{F}_0, (\hat{S}f)(p) \equiv \hat{S}_d(p)\begin{pmatrix} f_+(p) \\ f_-(p) \end{pmatrix},$$

  so that the scattering is encoded in the momentum space scattering operator

$$\hat{S} = \mathcal{F}_0^* S \mathcal{F}_0.$$

- Hence H is diagonalized as multiplication by $(p^2, p^2) \oplus (E_0, \ldots, E_{L-1})$ on $\hat{\mathcal{H}}_d \oplus$ Span(bound states).

- N. B. In both cases, the eigenfunction transforms yield a concrete realization of the spectral theorem. Scattering theory can be avoided by using the so-called Weyl/Titchmarsh/Kodaira approaches.

2. The Pöschl-Teller potentials

- We consider two explicit examples of the above potentials on the half-line and the line, namely,

$$V_s(x) \equiv g(g-1)/\sinh^2(x), \quad x \in (0,\infty), \quad g > 1,$$

and

$$V_d(x) \equiv -g(g-1)/\cosh^2(x), \quad x \in \mathbb{R}, \quad g > 1.$$

- Here, the suffix s stands for 'same', and d for 'different'. These potentials encode the interaction between two charged particles in their center-of-mass frame, with repulsion between same charges and attraction between different charges (as in electrodynamics).
- N. B. $V_d(x)$ arises from $V_s(x)$ by the analytic continuations $x \to x \pm i\pi/2$.

2A. Repulsive Pöschl-Teller

- The above incoming wave function $\Psi(x,p)$ involves the so-called conical function:

$$P^{1/2-g}_{ip-1/2}(\cosh x) \equiv \frac{(\sinh x)^{g-1/2}}{2^{g-1/2}\Gamma(g+1/2)}\psi_{\rm nr}(g;x,p),$$

$$\psi_{\rm nr}(g;x,p) \equiv {}_2F_1((g+ip)/2,(g-ip)/2,g+1/2;-\sinh^2(x)).$$

- These functions admit a variety of integral representations. Probably the simplest is

$$\psi_{\rm nr}(g;x,p) = \frac{2\Gamma(2g)}{2^g\Gamma(g+ip)\Gamma(g-ip)}\int_0^\infty dy\, \frac{\cos(yp)}{(\cosh y+\cosh x)^g},$$

which entails in particular

$$\psi_{\rm nr}(1;x,p) = \sin(xp)/p\sinh x.$$

- Setting

$$\Psi(x,p) \equiv -\frac{(2\sinh x)^g \Gamma(g)\Gamma(g-ip)}{\Gamma(2g)\Gamma(-ip)} \psi_{\mathrm{nr}}(g;x,p)$$

$$= -\frac{2(\sinh x)^g \Gamma(g)}{\Gamma(-ip)\Gamma(g+ip)} \int_0^\infty dy\, \frac{\cos(yp)}{(\cosh y + \cosh x)^g},$$

yields the announced incoming wave function:

$$\Psi(x,p) \sim u(p)e^{ixp} - e^{-ixp}, \quad x \to \infty,$$

where

$$u(p) = -\frac{\Gamma(ip)\Gamma(g-ip)}{\Gamma(-ip)\Gamma(g+ip)}.$$

- N. B. For $g = 1$ this gives the free solution

$$\Psi(x,p) = e^{ixp} - e^{-ixp}.$$

2B. Attractive Pöschl-Teller

- For $g \in (L, L+1]$ there are L bound states

$$\Psi_\ell(x) = (\cosh x)^{1-g} P_\ell(i \sinh x),$$

$$H\Psi_\ell = E_\ell \Psi_\ell, \quad E_\ell = -(g - \ell - 1)^2, \;\; \ell = 0, \ldots, L-1,$$

with $P_\ell(t)$ Gegenbauer polynomials of degree ℓ, satisfying

$$P_\ell(-t) = (-)^\ell P_\ell(t).$$

- The solution space to $H\Psi = p^2\Psi$, $p > 0$, is spanned by the two functions

$$(\cosh x)^g \psi_{\mathrm{nr}}(g; x \pm i\pi/2, p).$$

Therefore the desired incoming wave function $\Psi(x, p)$ is characterized by two p-dependent coefficients.

- Specifically, it reads

$$\Psi(x,p) = \frac{(2\cosh x)^g \Gamma(g)\Gamma(g-ip)}{2\Gamma(2g)\Gamma(-ip)\sinh(i\pi g - \pi p)}$$

$$\times \sum_{\delta=+,-} \delta \exp(\delta(i\pi g - \pi p)/2)\psi_{\mathrm{nr}}(g; x + \delta i\pi/2, p)$$

$$\sim \begin{cases} t(p)e^{ixp}, & x \to \infty, \\ e^{ixp} - r(p)e^{-ixp}, & x \to -\infty, \end{cases}$$

with

$$t(p) = \frac{\sinh(\pi p)}{\sinh(i\pi g - \pi p)}u(p), \quad r(p) = \frac{\sinh(i\pi g)}{\sinh(i\pi g - \pi p)}u(p).$$

- N. B. For $g = 1, 2, 3, \ldots$, we get $r(p) = 0$. Moreover, $g = 1$ yields the free solution

$$\Psi(x,p) = e^{ixp}.$$

3. Nonrelativistic hyperbolic Calogero-Moser systems

- The nonrelativistic N-particle Calogero-Moser Hamiltonian of hyperbolic type is given by

$$H_{\mathrm{nr}} = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\partial_{x_j}^2 + \frac{\mu^2 g(g-\hbar)}{4m}\sum_{1\leq j<k\leq N}\frac{1}{\sinh^2(\mu(x_j-x_k)/2)},$$

  with $\hbar > 0$ (Planck's constant), $m > 0$ (particle mass), $g \in \mathbb{R}$ (coupling constant), $\mu > 0$ (inverse length scale).
- Associated integrable system (N commuting PDOs):

$$H_1 = -i\hbar\sum_{j=1}^{N}\partial_{x_j} =: P_{\mathrm{nr}}, \quad H_2 = mH_{\mathrm{nr}},$$

$$H_k = \frac{(-i\hbar)^k}{k}\sum_{j=1}^{N}\partial_{x_j}^k + \mathrm{l.\ o.}, \quad k = 3, \ldots, N,$$

  where l.o. = lower order in partials.

- For $g \geq 0$ they can be promoted to commuting self-adjoint operators on

$$\mathcal{H}_s \equiv L^2(G_N, dx), \quad G_N \equiv \{z \in \mathbb{R}^N \mid z_N < \cdots < z_1\}.$$

Indeed, papers by Heckman and Opdam (specialized to A_{N-1}) yield a unitary joint eigenfunction transform that gives rise to diagonalization of the PDOs on $\hat{\mathcal{H}}_s = L^2(G_N, dp)$ as multiplication by

$$\hat{H}_k = \frac{1}{k} \sum_{j=1}^{N} p_j^k, \quad k = 1, \ldots, N.$$

- Harish-Chandra type asymptotics of the joint eigenfunction transform kernel yields the factorized S-matrix

$$\hat{S}_s(p) = \prod_{1 \leq j < k \leq N} u(p_j - p_k),$$

with $u(p)$ specified in Subsection 2A.

- This is a system of N particles with the same charge, and the account in 2A encodes the reduced $N = 2$ (center-of-mass) state of affairs.
- We can keep $x_1, \ldots, x_{N_+}$ real and take

$$x_{N_++1}, \ldots, x_N \to x_{N_++1} - i\pi/\mu, \ldots, x_N - i\pi/\mu,$$

  to get a system with N_+ positive charges and $N_- = N - N_+$ negative charges; the account in 2B encodes the case $N_+ = N_- = 1$.
- Expectation: The unequal charge case still gives a factorized S-matrix, with consistent factorization expressed by the Yang-Baxter equations for the 3-particle case. They can be thought of as pertaining to the 3-particle subspace of the fermion Fock space over $L^2(\mathbb{R}, dp) \otimes \mathbb{C}^2$.

- Specifically, the equations can be written

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12},$$

where the indices refer to the tensor legs of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, together with

$$S_{jk} \equiv \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & t & r & 0 \\ 0 & r & t & 0 \\ 0 & 0 & 0 & u \end{pmatrix}(p_j - p_k), \quad p \in G_3.$$

- This yields only two nontrivial relations, namely,

$$r_{12}t_{13}u_{23} = t_{23}u_{13}r_{12} + r_{23}r_{13}t_{12},$$

$$u_{12}r_{13}u_{23} = t_{23}r_{13}t_{12} + r_{23}u_{13}r_{12}.$$

Division by $u_{12}u_{13}u_{23}$ yields equations for t/u and r/u that are satisfied for the 'Pöschl-Teller' u, t, r in Section 2.

4. Relativistic hyperbolic Calogero-Moser systems

- The N commuting Hamiltonians (for equal charge) are the AΔOs (analytic difference operators)

$$H_k(x) = \sum_{|I|=k} \prod_{\substack{m\in I\\ n\notin I}} f_-(x_m - x_n) e^{-i\hbar\beta \sum_{m\in I} \partial_{x_m}} \prod_{\substack{m\in I\\ n\notin I}} f_+(x_m - x_n),$$

where $k = 1, \ldots, N$, $\beta > 0$, and

$$f_\pm(x)^2 = \sinh(\mu(x \pm i\beta g)/2))/\sinh(\mu x/2).$$

- Physical picture: $\beta = 1/mc$ and c =light speed;

$$H_{\mathrm{rel}} = mc^2[H_1(x) + H_1(-x)], \quad P_{\mathrm{rel}} = mc[H_1(x) - H_1(-x)],$$

$$B = -m \sum_{j=1}^{N} x_j,$$

are space-time translation and boost generators.

- They represent the Lie algebra of the Poincaré group in 2D:

$$[H_{\mathrm{rel}}, P_{\mathrm{rel}}] = 0, \quad [H_{\mathrm{rel}}, B] = i\hbar P_{\mathrm{rel}}, \quad [P_{\mathrm{rel}}, B] = i\hbar c^{-2} H_{\mathrm{rel}}.$$

- The nonrelativistic limit $c \to \infty$ yields

$$H_{\mathrm{rel}} - Nmc^2 \to H_{\mathrm{nr}}, \quad P_{\mathrm{rel}} \to P_{\mathrm{nr}},$$

and the Galilei Lie algebra

$$[H_{\mathrm{nr}}, P_{\mathrm{nr}}] = 0, \quad [H_{\mathrm{nr}}, B] = i\hbar P_{\mathrm{nr}}, \quad [P_{\mathrm{nr}}, B] = iN\hbar m\mathbf{1}.$$

- As before, we get a system with N_+ positive charges and $N_- = N - N_+$ negative charges by taking

$$x_n \to x_n - i\pi/\mu, \quad n = N_+ + 1, \ldots, N,$$

entailing sinh $\to$ cosh for different charges.

- Conjecture. For

$$\mu\beta g \in (0, \pi + \mu\beta\hbar),$$

the single-charge AΔOs $H_k(x)$ and their two-charge cousins can be promoted to commuting self-adjoint Hilbert space operators with a factorized S-matrix; for $\mu\beta g = \pi$ this yields the same 'physics' (scattering including bound states) as for the sine-Gordon quantum field theory

$$\ddot{\phi} - \phi'' = \sin\phi.$$

- N. B. In joint work with M. Hallnäs, joint eigenfunctions for the Hamiltonians $H_1(x), \ldots, H_N(x)$ have been recursively constructed with the aid of kernel functions; they do give rise to a factorized S-matrix $\prod_{j<k} u(y_j - y_k)$, with $u(y)$ specified below.

4A. The repulsive reduced $N = 2$ case

- To date no general Hilbert space theory for AΔOs exists. Worse yet, the solutions to a Schrödinger equation of the form

$$f(x)\Psi(x + is, p) + g(x)\Psi(x - is, p) = 2\cosh(sp)\Psi(x, p),$$

  with shift parameter $s > 0$ form an infinite-dimensional vector space whenever one nonzero solution $\Psi(x, p)$ exists.
- Example: The free case $f(x) = g(x) = 1$. Just multiply the obvious solution $\exp(ixp)$ by any function $m(x, p)$ that has is-periodicity in x to get another solution.
- Certain special AΔOs, however, have been promoted to self-adjoint Hilbert space operators. This hinges on the existence of special solutions to the Schrödinger equation that give rise to a unitary eigenfunction transform.

- For the reduced $N = 2$ case at hand, this transform involves the relativistic conical function. This conical function generalization has many distinct integral representations. The integrands are built from the hyperbolic gamma function $G(a_+, a_-; z)$, which is a generalization of the (rational) gamma function $\Gamma(z)$.
- In the present setting, $a_\pm$ can be viewed as length scales:

$$a_+ \equiv 2\pi/\mu, \quad \text{(imaginary period/interaction length)},$$

$$a_- \equiv \hbar/mc, \quad \text{(shift step size/Compton wave length)}.$$

- From now on, we use the notation

$$c_\delta(z) \equiv \cosh(\pi z/a_\delta), \; s_\delta(z) \equiv \sinh(\pi z/a_\delta), \; e_\delta(z) \equiv e^{\pi z/a_\delta},$$

where $\delta = +, -$; also, we define the average

$$a \equiv (a_+ + a_-)/2.$$

- The hyperbolic gamma function $G(z)$ can be defined as the meromorphic solution to one of the first order AΔEs

$$\frac{G(z + ia_\delta/2)}{G(z - ia_\delta/2)} = 2c_{-\delta}(z), \quad \delta = +, -, \quad a_+, a_- > 0,$$

which is uniquely determined by requiring $G(0) = 1$ and 'minimality'; the second AΔE is then satisfied as well.

- In the strip $|\mathrm{Im}\, z| < a$ it has the integral representation

$$G(z) = \exp\Big(i \int_0^\infty \frac{dy}{y}\Big(\frac{\sin 2yz}{2\sinh(a_+y)\sinh(a_-y)} - \frac{z}{a_+a_-y}\Big)\Big).$$

This entails absence of zeros and poles in this strip and the properties

$$G(a_-, a_+; z) = G(a_+, a_-; z), \quad (\text{modular invariance}),$$

$$G(-z) = 1/G(z), \quad (\text{reflection equation}),$$

$$\overline{G(z)} = G(-\bar{z}).$$

- The simplest and most revealing representation of the relativistic conical function is given by

$$\mathcal{R}(a_+, a_-, b; x, y) = \sqrt{\frac{1}{a_+ a_-}\frac{G(2ib - ia)}{G(ib - ia)^2}}$$

$$\times \int_{\mathbb{R}} dz \prod_{\delta=+,-} \frac{G(z + \delta(x - y)/2 - ib/2)}{G(z + \delta(x + y)/2 + ib/2)}.$$

Here, b and y are the coupling constant and spectral parameter, related to the previous parameters by

$$b = \beta g (= g/mc), \quad y = \beta p/\mu.$$

- From this one reads off evenness in x and y and the properties

$$\mathcal{R}(a_-, a_+, b; x, y) = \mathcal{R}(a_+, a_-, b; x, y), \quad (\text{modular invariance}),$$

$$\mathcal{R}(a_+, a_-, b; y, x) = \mathcal{R}(a_+, a_-, b; x, y), \quad (\text{self} - \text{duality}).$$

- The $\mathcal{R}$-function is meromorphic for $b, x, y \in \mathbb{C}$ and $\mathrm{Re}\, a_+, \mathrm{Re}\, a_- > 0$. It satisfies the four AΔEs

$$A_\delta(x)\mathcal{R}(x,y) = 2c_\delta(y)\mathcal{R}(x,y), \;\; A_\delta(y)\mathcal{R}(x,y) = 2c_\delta(x)\mathcal{R}(x,y),$$

$$A_\delta(z) \equiv \frac{s_\delta(z+ib)}{s_\delta(z)} \exp(ia_{-\delta} d/dz) + (z \to -z),$$

where $\delta = +, -$.

- The AΔO $A_+(x)$ is related to the above (reduced) $N = 2$ Hamiltonian H_{rel} by a similarity transformation involving the generalized Harish-Chandra c-function

$$c(z) \equiv G(z + ia - ib)/G(z + ia).$$

- Introducing the weight and scattering functions

$$w(z) \equiv 1/c(z)c(-z), \quad u(z) \equiv -c(z)/c(-z),$$

(with $w(z)$ having a double zero for $z = 0$), this relation is given by

$$H_{\mathrm{rel}} = C^{st} H_+(x), \;\; H_\pm(z) \equiv w(z)^{1/2} A_\pm(z) w(z)^{-1/2}.$$

- The function

$$\Psi(x,y) \equiv -\frac{G(ib-ia)}{G(2ib-ia)} \frac{w(x)^{1/2}}{c(-y)} \mathcal{R}(x,y),$$

satisfies $H_{\pm}(x)\Psi(x,y) = 2c_{\pm}(y)\Psi(x,y)$ and

$$\Psi(x,y) \sim u(y)\exp(i\pi xy/a_+a_-) - \exp(-i\pi xy/a_+a_-), \quad x \to \infty.$$

- Setting

$$H_{0,\pm}(x) \equiv \exp(ia_{\mp}d/dx) + \exp(-ia_{\mp}d/dx),$$

$$\Psi_0(x,y) \equiv \exp(i\pi xy/a_+a_-) - \exp(-i\pi xy/a_+a_-),$$

one clearly gets $H_{0,\pm}(x)\Psi_0(x,y) = 2c_{\pm}(y)\Psi_0(x,y)$.
- The sine transform $\mathcal{F}_0$ with kernel $(2a_+a_-)^{-1/2}\Psi_0(x,y)$ can now be used to reinterpret the AΔOs $H_{0,\pm}(x)$ as self-adjoint operators on $\mathcal{H}_s = L^2((0,\infty), dx)$, namely as pullbacks of the self-adjoint operators of multiplication by $2c_{\pm}(y)$ on $\hat{\mathcal{H}}_s = L^2((0,\infty), dy)$ under the unitary $\mathcal{F}_0$.

- Provided $b \in [0, 2a]$, the transform $\mathcal{F}$ with kernel $(2a_+ a_-)^{-1/2}\Psi(x, y)$ yields a unitary operator $\hat{\mathcal{H}}_s \to \mathcal{H}_s$. (It equals $\mathcal{F}_0$ for $b = a_\pm$.) The AΔOs $H_\pm(x)$ can then be viewed as commuting self-adjoint operators on $\mathcal{H}_s$, defined by $\mathcal{F} 2c_\pm(\cdot)\mathcal{F}^*$.
- These transforms are related to the wave operators

$$W_\pm = \lim_{t\to\pm\infty} \exp(itH_\delta)\exp(-itH_{0,\delta}), \quad \delta = +, -,$$

in the same way as in the nonrelativistic setting.
- In particular, the scattering operator on $\hat{\mathcal{H}}_s$ is given by

$$(\hat{S}f)(y) = \hat{S}_s(y)f(y), \quad \hat{S}_s(y) \equiv u(y),$$

with

$$u(y) = -\frac{G(y + ia - ib)G(y - ia + ib)}{G(y + ia)G(y - ia)}.$$

4B. The attractive reduced $N = 2$ case

- Reminder:

$$s_+(x) = \sinh(\pi x/a_+), \;\; a_+ = 2\pi/\mu, \;\; a_- = \hbar/mc, \;\; y = p/mc\mu.$$

- The repulsive (same charge) and attractive (different charge) AΔOs are given by

$$A_s(x) \equiv A_+(x) = \frac{s_+(x+ib)}{s_+(x)} \exp(ia_- d/dx) + (x \to -x),$$

$$A_d(x) \equiv A_+(x - ia_+/2) = \frac{c_+(x+ib)}{c_+(x)} \exp(ia_- d/dx) + (x \to -x).$$

- Setting

$$\tilde{c}(x) \equiv c(x - ia_+/2), \quad \tilde{w}(x) \equiv 1/\tilde{c}(x)\tilde{c}(-x) > 0, \;\; \forall x \in \mathbb{R},$$

the corresponding Hamiltonian is

$$H_d(x) \equiv \tilde{w}(x)^{1/2} A_d(x) \tilde{w}(x)^{-1/2}.$$

For $b = a_-$ it equals $e^{ia_- d/dx} + e^{-ia_- d/dx}$.

- N. B. The x-shift $\mathcal{R}(x - ia_+/2, y)$ entails that modular invariance and self-duality break down. As a result, $A_d(x)$ has no natural 'modular partner', and we might as well trade the spectral variable y (a position) for p (a momentum). For brevity, we stick to y.
- In fact, we get two distinct eigenfunctions

$$A_d(x)\mathcal{R}(x \pm ia_+/2, y) = 2c_+(y)\mathcal{R}(x \pm ia_+/2, y),$$

entailing

$$H_d(x)\tilde{w}(x)^{1/2}\mathcal{R}(x \pm ia_+/2, y) = 2c_+(y)\tilde{w}(x)^{1/2}\mathcal{R}(x \pm ia_+/2, y).$$

- Snag. These $H_d(x)$-eigenfunctions remain eigenfunctions when multiplied by any function $m(x, y)$ that is ia_--periodic in x. There are no general results ensuring that a particular choice yields a function $\Psi(x, y)$ that can serve as the kernel of a unitary eigenfunction transform.

- The linear combination

$$\Psi(x,y) \equiv \frac{G(ib-ia)}{G(2ib-ia)} \frac{\tilde{w}(x)^{1/2}}{2s_-(ib-y)c(-y)}$$

$$\times \prod_{\delta=+,-} \delta e_-(\delta(ib-y)/2)\mathcal{R}(x+\delta ia_+/2, y),$$

has coefficients ensuring unitary asymptotics:

$$\Psi(x,y) \sim \begin{cases} t(y)e^{i\pi xy/a_+a_-}, & \mathrm{Re}\, x \to \infty, \\ e^{i\pi xy/a_+a_-} - r(y)e^{-i\pi xy/a_+a_-}, & \mathrm{Re}\, x \to -\infty, \end{cases}$$

with

$$t(y) \equiv \frac{s_-(y)}{s_-(ib-y)}u(y), \quad r(y) \equiv \frac{s_-(ib)}{s_-(ib-y)}u(y).$$

- N. B. The triple u, t, r satisfies the Yang-Baxter equations; note also $r = 0$ for $b = (L+1)a_-$, $L = 0, 1, 2, \ldots$.

- In joint work with S. Haworth we have shown that the transform

$$(\mathcal{F}f)(x) = \frac{1}{\sqrt{2a_+a_-}} \int_0^\infty dy \begin{pmatrix} \Psi(x,y) \\ -\Psi(-x,y) \end{pmatrix} \cdot \begin{pmatrix} f_+(y) \\ f_-(y) \end{pmatrix},$$

yields a unitary operator

$$\mathcal{F} : \hat{\mathcal{H}}_d \equiv L^2((0,\infty), dy)^2 \to \mathcal{H}_d \equiv L^2(\mathbb{R}, dx),$$

provided $b \in [0, a_-]$. Also, $\Psi(x,y)$ equals $e^{i\pi xy/a_+a_-}$ for $b = a_-$, so then $\mathcal{F}$ amounts to the Fourier transform $\mathcal{F}_0$.
- For $b \in (a_-, a_- + a_+/2)$ the transform is isometric. Its range is the orthogonal complement of $L \geq 1$ bound states

$$\Psi_\ell(x) = \frac{c_+(x)}{\tilde{w}(x)^{1/2}} Q_\ell(is_+(x)), \quad \ell = 0, \ldots, L-1,$$

with $Q_\ell(t)$ q-Gegenbauer polynomials of degree ℓ and parity $(-)^\ell$.

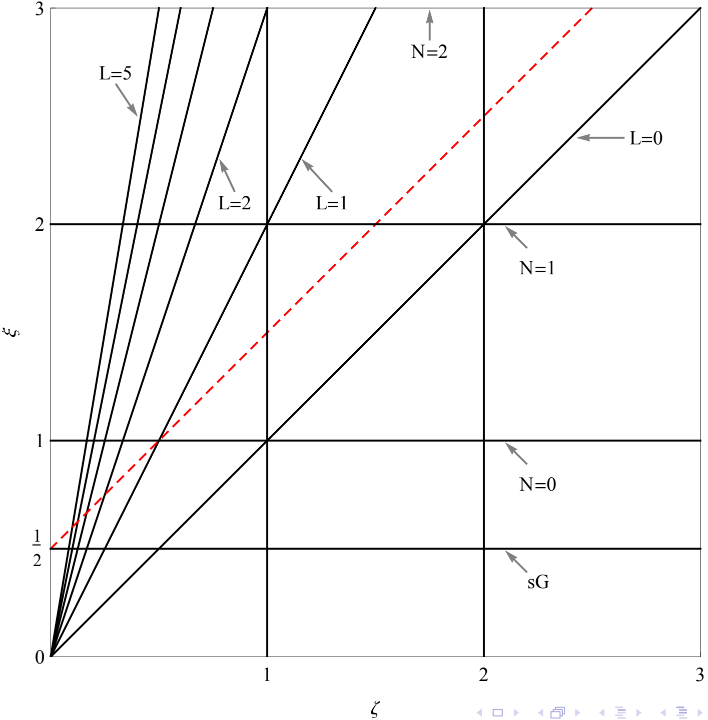
- The transforms $\mathcal{F}_{(0)}$ are related to the wave operators $W_{\pm}$ as before, and serve to associate a self-adjoint operator on $\mathcal{H}_d$ to the AΔO $H_d(x)$, namely the pullback of multiplication by $(2c_+(y), 2c_+(y))$ on $\hat{\mathcal{H}}_d$.
- For a fixed $b \in [0, a_- + a_+/2)$, the bound state number L is the smallest integer such that $b \leq (L+1)a_-$. For $b > a_-$ we have $H_d\Psi_\ell = E_\ell\Psi_\ell$, with

$$E_\ell = 2c_+(i(b - (\ell+1)a_-)) \in (0, 2), \quad \ell = 0, \ldots, L-1.$$

- Setting

$$\xi \equiv b/a_+, \quad \zeta \equiv a_-/a_+,$$

the following plot can be viewed as a phase diagram. The red line denotes the transition to the ‘unphysical’ regime (breakdown of isometry and self-adjointness). On the lines $\xi = (L+1)\zeta$, $L = 0, 1, \ldots$, the reflection vanishes. Also, sG stands for the sine-Gordon line $\xi = 1/2$. The nonrelativistic limit arises by setting $\xi = \lambda\zeta$, $\lambda = g/\hbar$ fixed, and letting $\zeta \to 0$.



L=5
L=2
L=1
L=0
N=2
N=1
N=0
sG
ξ
ζ
0
$\frac{1}{2}$
1
2
3

5. Some references

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