A construction of commuting systems of integrable symplectic birational maps

Yuri B. Suris

(Technische Universität Berlin)

Joint work with Matteo Petrera

LMS – EPSRC Durham Symposium
"Geometric and Algebraic Aspects of Integrability", 03.08.2016

Discretization in
Geometry and Dynamics
SFB Transregio 109

Integrable systems – a playground of algebraic geometry. A nice example: QRT maps introduced in

- G.R.W. Quispel, J.A.G. Roberts, C.J. Thompson. *Integrable mappings and soliton equations II*, Physica D **34** (1989) 183–192,

prompted the treatise

- J.J. Duistermaat. *Discrete Integrable Systems. QRT Maps and Elliptic Surfaces*, Springer, 2010, xii+627 pp.

A goal and the hope of the present study – to produce a rich supply of examples attracting the attention of the algebraic geometers.

Kahan's discretization scheme

Invented in:

- W. Kahan. *Unconventional numerical methods for trajectory calculations* (Unpublished lecture notes, 1993).

He wrote:

> "*I have used these methods for 24 years without quite understanding why they work so well as they do, when they work.*"

Bilinear discretization of quadratic vector fields

Take an arbitrary system with a quadratic vector field:

$$\dot{x} = f(x) = Q(x) + Bx + c,$$

where $B \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$, each component of $Q : \mathbb{R}^n \to \mathbb{R}^n$ a *quadratic* form. Discretization:

$$\frac{\widetilde{x} - x}{\epsilon} = Q(x, \widetilde{x}) + \frac{1}{2}B(x + \widetilde{x}) + c,$$

with $Q(x, \widetilde{x})$ corresponding symmetric *bilinear* function:

$$Q(x, \widetilde{x}) = \frac{1}{2}\big(Q(x + \widetilde{x}) - Q(x) - Q(\widetilde{x})\big).$$

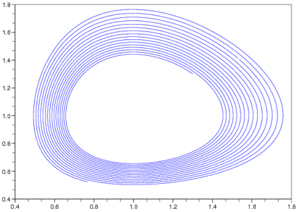
Equations for $\widetilde{x}$ always linear, map $\widetilde{x} = \Phi_f(x, \epsilon)$ rational and reversible:

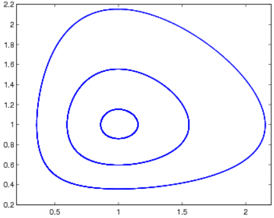
$$\Phi_f^{-1}(x, \epsilon) = \Phi_f(x, -\epsilon),$$

thus *birational*.

Example: Lotka-Volterra system

$$\left\{\begin{array}{l}\dot{x}=x(1-y),\\ \dot{y}=y(x-1)\end{array}\right. \quad \rightsquigarrow \quad \left\{\begin{array}{l}\widetilde{x}-x=\epsilon(x+\widetilde{x}-\widetilde{x}y-x\widetilde{y}),\\ \widetilde{y}-y=-\epsilon(y+\widetilde{y}-\widetilde{x}y-x\widetilde{y}).\end{array}\right.$$





Left: one orbit of explicit Euler with $\epsilon=0.01$; right: three orbits of Kahan's discretization with $\epsilon=0.01$.

Non-spiralling behavior explained by invariance of Poisson structure:

- J.M. Sanz-Serna. *An unconventional symplectic integrator of W. Kahan*, Appl. Numer. Math. **16** (1994), 245–250.

Hirota-Kimura's discretization scheme

- R.Hirota, K.Kimura. *Discretization of the Euler top.* J. Phys. Soc. Japan **69** (2000) 627–630,
- K.Kimura, R.Hirota. *Discretization of the Lagrange top.* J. Phys. Soc. Japan **69** (2000) 3193–3199.

Apparently unaware of Kahan's work, applied this method to two famous integrable systems.

Resulting maps integrable (possess 2, resp. 4, independent integrals of motion; solutions in terms of elliptic functions).

Example: Hirota-Kimura's discrete time Euler top

$$\begin{cases} \dot{x}_1 = \alpha_1 x_2 x_3, \\ \dot{x}_2 = \alpha_2 x_3 x_1, \\ \dot{x}_3 = \alpha_3 x_1 x_2, \end{cases} \qquad \rightsquigarrow \qquad \begin{cases} \widetilde{x}_1 - x_1 = \epsilon\alpha_1(\widetilde{x}_2 x_3 + x_2 \widetilde{x}_3), \\ \widetilde{x}_2 - x_2 = \epsilon\alpha_2(\widetilde{x}_3 x_1 + x_3 \widetilde{x}_1), \\ \widetilde{x}_3 - x_3 = \epsilon\alpha_3(\widetilde{x}_1 x_2 + x_1 \widetilde{x}_2). \end{cases}$$

Features:

- Equations are linear w.r.t. $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)^{\mathrm{T}}$:

$$A(x,\epsilon)\widetilde{x} = x, \qquad A(x,\epsilon) = \begin{pmatrix} 1 & -\epsilon\alpha_1 x_3 & -\epsilon\alpha_1 x_2 \\ -\epsilon\alpha_2 x_3 & 1 & -\epsilon\alpha_2 x_1 \\ -\epsilon\alpha_3 x_2 & -\epsilon\alpha_3 x_1 & 1 \end{pmatrix},$$

result in explicit rational map:

$$\widetilde{x} = \Phi_f(x,\epsilon) = A^{-1}(x,\epsilon)x.$$

- Explicit formulas rather messy:

$$\begin{cases} \widetilde{x}_1 = \dfrac{x_1 + 2\epsilon\alpha_1 x_2 x_3 + \epsilon^2 x_1(-\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\ \widetilde{x}_2 = \dfrac{x_2 + 2\epsilon\alpha_2 x_3 x_1 + \epsilon^2 x_2(\alpha_2\alpha_3 x_1^2 - \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\ \widetilde{x}_3 = \dfrac{x_3 + 2\epsilon\alpha_3 x_1 x_2 + \epsilon^2 x_3(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 - \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \end{cases}$$

where

$$\begin{aligned} \Delta(x,\epsilon) &= \det A(x,\epsilon) \\ &= 1 - \epsilon^2(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2) - 2\epsilon^3\alpha_1\alpha_2\alpha_3 x_1 x_2 x_3. \end{aligned}$$

- Reversibility:

$$\Phi_f^{-1}(x,\epsilon) = \Phi_f(x,-\epsilon).$$

(Try to see reversibility directly from explicit formulas!)

- Two independent integrals:

$$I_1(x,\epsilon) = \frac{1-\epsilon^2\alpha_2\alpha_3 x_1^2}{1-\epsilon^2\alpha_3\alpha_1 x_2^2}, \quad I_2(x,\epsilon) = \frac{1-\epsilon^2\alpha_3\alpha_1 x_2^2}{1-\epsilon^2\alpha_1\alpha_2 x_3^2}.$$

- Invariant volume form:

$$\omega = \frac{dx_1 \wedge dx_2 \wedge dx_3}{\phi(x)}, \quad \phi(x) = (1-\epsilon^2\alpha_i\alpha_j x_k^2)^2$$

and bi-Hamiltonian structure found in:

  - M. Petrera, Yu. Suris. *On the Hamiltonian structure of the Hirota-Kimura discretization of the Euler top.* Math. Nachr. **283** (2010) 1654–1663.

The unreasonable integrability of KHK

- M. Petrera, A. Pfadler, Yu.B. Suris. *On integrability of Hirota-Kimura type discretizations. Experimental study of the discrete Clebsch system*. Experimental Math. **18** (2009), 223–247,
- M. Petrera, A. Pfadler, Yu.B. Suris *On integrability of Hirota-Kimura type discretizations*. Regular Chaotic Dyn. **16** (2011), 245–289.

Integrability of KHK discretization for an amazingly long list of examples, including:

- Reduced Nahm equations ($n = 2$)
- Periodic Volterra chain ($n = 3, 4$)
- Dressing chain ($n = 3$)
- Three wave system ($n = 6$)
- Kirchhoff and Clebsch cases of rigid body in an ideal fluid ($n = 6$)
- $SO(4)$ Euler top ($n = 6$)

KHK applied to canonical Hamiltonian systems

- E. Celledoni, R.I. McLachlan, B. Owren, G.R.W. Quispel. *Geometric properties of Kahan's method*, J. Phys. A, **46** (2013), 025201, 12 pp.

Let $f(x) = J\nabla H(x)$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, $H : \mathbb{R}^{2n} \to \mathbb{R}$ a cubic polynomial. Then:

- map $\Phi_f(x, \epsilon) : x \mapsto \widetilde{x}$ admits a rational integral of motion

$$\widetilde{H}(x, \epsilon) = \frac{1}{6\epsilon}\, x^{\mathrm{T}} J^{-1} \widetilde{x} = H(x) + O(\epsilon^2).$$

- map $\Phi_f(x, \epsilon) : x \mapsto \widetilde{x}$ admits an invariant volume form:

$$\frac{dx_1 \wedge \ldots \wedge dx_n}{\det(I - \epsilon f'(x))}.$$

These results are not related to integrability!

Missing before the present work

- Conceptual (structure-clarifying) proof of integrability in any of numerous examples;
- Invariant symplectic or Poisson structure in any example in dim ≥ 4.

Both achieved here:

- M. Petrera, Yu.B. Suris. *A construction of a large family of commuting pairs of integrable symplectic birational 4-dimensional maps.* `arXiv:1606.08238 [nlin.SI]`
- M. Petrera, Yu.B. Suris. *A construction of commuting systems of integrable symplectic birational maps.* `arXiv:1607.07085 [nlin.SI]`

Main features of the novel construction

A family of completely integrable systems $\dot{x} = J\nabla H_0(x)$ in $\mathbb{R}^{2m}$ with m cubic integrals $(H_0, \ldots, H_{m-1})$ in involution, for which KHK discretization has following properties:

- The map $\Phi_{J\nabla H_0}$ is symplectic w.r.t. a perturbation of the canonical symplectic structure on $\mathbb{R}^{2m}$;
- The map $\Phi_{J\nabla H_0}$ has m rational integrals $\widetilde{H}_0(x,\epsilon), \ldots,$ $\widetilde{H}_{m-1}(x,\epsilon)$ in involution;
- The maps $\Phi_{J\nabla H_i}$ do not commute; however, there exist 2^{m-1} linear combinations $J\nabla K = \sum_{i=0}^{m-1} \alpha_i J\nabla H_i$ such that $\Phi_{J\nabla H_0}$ commutes with $\Phi_{J\nabla K}$.

Construction of functions in involution

Observation. Let A be a constant $2m \times 2m$ matrix, and suppose that functions $H_0(x)$, $H_1(x)$ satisfy

$$\nabla H_1 = A \nabla H_0.$$

If the matrix A is *skew-Hamiltonian*,

$$JA = A^{\mathrm{T}} J = -(JA)^{\mathrm{T}},$$

then H_0, H_1 are in involution.

Proof.

$$\{H_0, H_1\} = (\nabla H_0)^{\mathrm{T}} J \nabla H_1 = (\nabla H_0)^{\mathrm{T}} JA \nabla H_0.$$

Applicability of construction

Differential equations $\nabla H_1 = A\nabla H_0$ for H_1 are solvable if and only if H_0 satisfies

$$A(\nabla^2 H_0) = (\nabla^2 H_0)A^{\mathrm{T}},$$

(where $\nabla^2 H_0$ is the Hesse matrix of H_0). Then $H_1(x)$ satisfies the same conditions:

$$A(\nabla^2 H_1) = (\nabla^2 H_1)A^{\mathrm{T}}.$$

Proposition. The linear space of homogeneous polynomials $H_0(x_1, \ldots, x_{2m})$ of deg $= 3$ satisfying this system of 2nd order PDEs, has dimension $4m$.

Construction of completely integrable systems

Take a non-degenerate skew-Hamiltonian matrix A and a generic cubic polynomial $H_0(x)$ satisfying

$$A(\nabla^2 H_0) = (\nabla^2 H_0)A^{\mathrm{T}}.$$

Define $H_i \;\; (i = 1, \ldots, m-1)$ by

$$\nabla H_i = A \nabla H_{i-1}.$$

Then $(H_0, \ldots, H_{m-1})$ is a completely integrable Hamiltonian system.

Characteristic properties

Vector fields $f_i(x) = J\nabla H_i(x)$ satisfy:

$$(f_i'(x))^{\mathrm{T}} J + J f_i'(x) = 0 \quad \text{(Hamiltonian)},$$

$$f_i'(x) f_j(x) = f_j'(x) f_i(x) \quad \text{(commute)},$$

$$f_i'(x) f_j'(x) = f_j'(x) f_i'(x),$$

and

$$A^{\mathrm{T}} f_i'(x) = f_i'(x) A^{\mathrm{T}}.$$

Associated vector fields

Definition. Let the skew-Hamiltonian matrix

$$B = \sum_{i=0}^{m-1} \alpha_i A^i$$

satisfy

$$B^2 = I.$$

Then the vector field

$$g(x) = JB\nabla H_0(x) = B^{\mathrm{T}} J\nabla H_0(x) = B^{\mathrm{T}} f_0(x)$$

is called *associated* to $f_0(x)$. Vector field $g(x)$ is Hamiltonian:

$$g(x) = J\nabla K(x),$$

with the Hamilton function

$$K(x) = \sum_{i=0}^{m-1} \alpha_i H_i(x).$$

Associated vector fields

This defines an equivalence relation on the set of vector fields $J\nabla H(x)$ with Hamilton functions $H(x)$ satisfying

$$A(\nabla^2 H) = (\nabla^2 H)A^{\mathrm{T}}.$$

Lemma. If vector field $g(x)$ is associated to $f_0(x)$ via the matrix B, then the following identities hold true:

$$g'(x)g(x) = f_0'(x)f_0(x),$$

$$(g'(x))^2 = (f_0'(x))^2.$$

As a corollary,

$$\det(I - \epsilon g'(x)) = \det(I - \epsilon f_0'(x))$$

(this is the common denominator of $\Phi_g(x)$, resp. of $\Phi_{f_0}(x)$).

Construction of associated vector fields

A generic $2m \times 2m$ skew-Hamiltonian matrix A has m distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ of algebraic multiplicity 2:

$$\det(A - \lambda I) = (\lambda - \lambda_1)^2 \cdots (\lambda - \lambda_m)^2,$$

and of geometric multiplicity 2 (i.e., is diagonalizable). The latter follows from existence of a symplectic similarity transformation

$$SAS^{-1} = \begin{pmatrix} W & 0 \\ 0 & W^{\mathrm{T}} \end{pmatrix}, \quad S^{\mathrm{T}} J S = J.$$

Construction of associated vector fields

For any $P \subset \{1, \ldots, m\}$, define polynomial $B_P(\lambda)$ by

$$B_P(\lambda_i) = \begin{cases} 1, & i \in P, \\ -1, & i \notin P. \end{cases}$$

Then $B_P = B_P(A)$ is skew-Hamiltonian and satisfies $B_P^2 = I$.

This defines 2^m (or, better, 2^{m-1}, if considered up to sign) associated vector fields for any $f_0(x)$ from our class.

Main results. I

Theorem 1. Let f_0 and g be two associated vector fields, via the skew-Hamiltonian matrix B. Then the KHK maps Φ_{f_0} and Φ_g commute:

$$\Phi_{f_0} \circ \Phi_g = \Phi_g \circ \Phi_{f_0}.$$

Theorem 2. Let f_0 and g be two associated vector fields, via the skew-Hamiltonian matrix B. Then the maps $\Phi_{f_0} : x \mapsto \widetilde{x}$ and $\Phi_g : x \mapsto \widehat{x}$ share two functionally independent conserved quantities

$$\widetilde{H}_0(x, \epsilon) = \frac{1}{6\epsilon} x^{\mathrm{T}} J^{-1} \widetilde{x} = \frac{1}{6\epsilon} x^{\mathrm{T}} J^{-1} \big(I - \epsilon f_0'(x)\big)^{-1} x$$

and

$$\widetilde{K}(x, \epsilon) = \frac{1}{6\epsilon} x^{\mathrm{T}} J^{-1} \widehat{x} = \frac{1}{6\epsilon} x^{\mathrm{T}} J^{-1} \big(I - \epsilon g'(x)\big)^{-1} x.$$

Main results. II

Theorem 3.
The rational functions $\widetilde{H}_0(x,\epsilon)$, $\widetilde{K}(x,\epsilon)$ are related by the same differential equation as the cubic polynomials $H_0(x)$, $K(x)$:

$$\nabla\widetilde{K}(x,\epsilon) = B\nabla\widetilde{H}_0(x,\epsilon).$$

As a consequence, they satisfy the same 2nd order differential equations

$$A(\nabla^2 H) = (\nabla^2 H)A^{\mathrm{T}}$$

as the polynomials $H_0(x)$, $K(x)$.

Main results. III

Theorem 4. The map Φ_{f_0} is Poisson (symplectic) with respect to the brackets with the Poisson tensor $\Pi(x)$ given by

$$\begin{aligned}\Pi(x) &= J - \epsilon^2 (f_0'(x))^2 J \\ &= \big(1 - \epsilon^2 q_0(x)\big) J - \sum_{i=1}^{m-1} \epsilon^2 q_i(x) (A^{\mathrm{T}})^i J.\end{aligned}$$

If vector field $g(x)$ is associated to $f_0(x)$ then Φ_g is Poisson with respect to the same bracket.

Open problems

- Are our systems, both continuous and discrete time, algebraically completely integrable? In other words, are their invariant manifolds (affine parts of) Abelian varieties? Recall: in the continuous time case, they are intersections of m cubic hypersurfaces in $\mathbb{R}^{2m}$.
- Is anything similar possible for Hamiltonian systems with non-constant Poisson tensors?