Pentagon, tetrahedron and Yang–Baxter maps in non-commuting variables, and their geometric origin

Adam Doliwa
doliwa@matman.uwm.edu.pl

University of Warmia and Mazury, Olsztyn, Poland

Geometric and Algebraic Aspects of Integrability
London Mathematical Society EPSRC Durham Symposium

25th July – 4th August 2016

OUTLINE

1. GEOMETRIC PENTAGON AND TETRAHEDRON MAPS (WITH R. KASHAEV)
   - The normalization map
   - The Veblen map
   - Geometric tetrahedron map

2. 4D CONSISTENT SYSTEMS AND ZAMOLODCHIKOV'S CONDITION
   - Desargues maps and non-commutative KP systems
   - Multidimensional discrete conjugate nets

3. YANG–BAXTER MAPS
   - Periodic reduction of the non-commutative KP map
   - Central-product reduction

Outline

1. Geometric pentagon and tetrahedron maps (with R. Kashaev)
   - The normalization map
   - The Veblen map
   - Geometric tetrahedron map
2. 4D consistent systems and Zamolodchikov's condition
   - Desargues maps and non-commutative KP systems
   - Multidimensional discrete conjugate nets
3. Yang–Baxter maps
   - Periodic reduction of the non-commutative KP map
   - Central-product reduction

PENTAGON MAPS

$\mathcal{X}$ – a set, a map $S : \mathcal{X}^2 \to \mathcal{X}^2$ satisfying in $\mathcal{X}^3$ the relation

$$S_{12} \circ S_{13} \circ S_{23} = S_{23} \circ S_{12}$$

is called *a pentagon map* *[S. Zakrzewski 1992], [Semenov-Tian-Shanskii 1992]*

PENTAGON PROPERTY OF THE NORMALIZATION MAP

The following birational map $N\colon (x_1, x_2) \dashrightarrow (x_1', x_2')$

$$x_1' = (x_2 + x_1 - x_1 x_2)^{-1} x_1, \qquad x_2' = x_2 + x_1 - x_1 x_2,$$

satisfies the pentagonal condition

COROLLARY

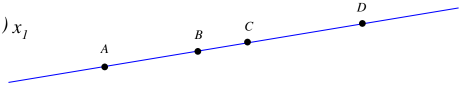
The inverse of N is given by

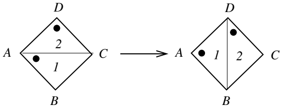
$$x_1 = x_2' x_1', \qquad x_2 = (1 - x_2' x_1')^{-1} x_2' (1 - x_1'),$$

and satisfies the reversed pentagonal condition

Given four collinear points A, B, C and D, consider two pairs of linear relations between their *non-homogeneous* coordinates

$$\phi_A - \phi_B = (\phi_C - \phi_B)\, x_1$$

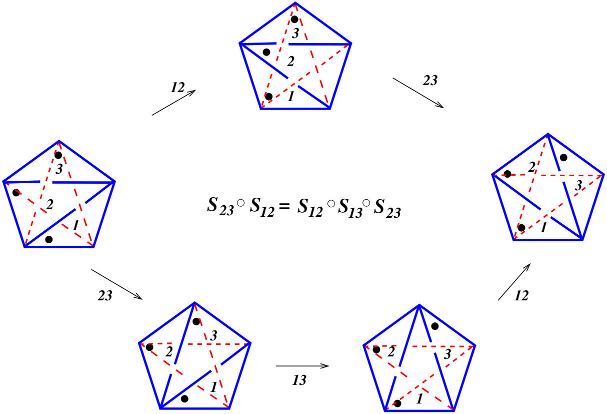




$$\begin{aligned} \phi_A &= \phi_C x_1 + \phi_B(1 - x_1) \\ \phi_D &= \phi_C x_2 + \phi_A(1 - x_2) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \phi_A &= \phi_D x_1' + \phi_B(1 - x_1') \\ \phi_D &= \phi_C x_2' + \phi_A(1 - x_2'). \end{aligned}$$

The normalization map is a consequence of that change of basis

Combinatorial description of the pentagonal condition

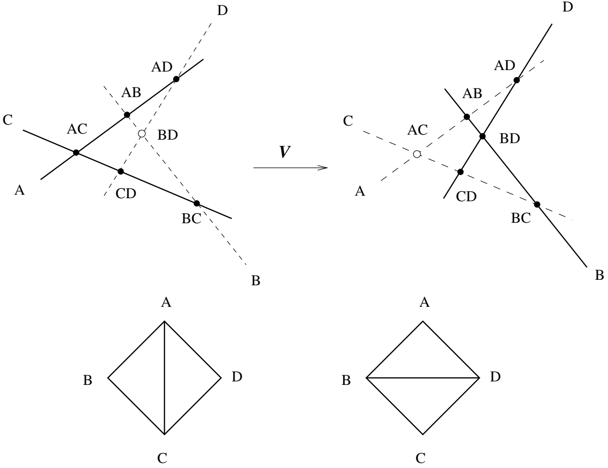


$$S_{23} \circ S_{12} = S_{12} \circ S_{13} \circ S_{23}$$

Linear problem for the Veblen map [AD, Sergeev 2014]

$$\phi_{AC} = \phi_{AB} x_1 + \phi_{AD}(1 - x_1) \qquad \phi_{BC} = \phi_{AB}\bar{x}_1 + \phi_{BD}(1 - \bar{x}_1)$$
$$\phi_{BC} = \phi_{AC} x_2 + \phi_{CD}(1 - x_2) \quad \text{and} \quad \phi_{BD} = \phi_{AD}\bar{x}_2 + \phi_{CD}(1 - \bar{x}_2).$$

The Veblen $(6_2, 4_3)$ configuration



PENTAGON PROPERTY OF THE VEBLEN MAP

The birational map $V\colon (x,y) \dashrightarrow (\bar{x},\bar{y})$, where

$$\bar{x}_1 = x_1 x_2, \qquad \bar{x}_2 = (1-x_1)x_2(1-x_1x_2)^{-1},$$

satisfies the reversed pentagonal condition

$$V_{23} \circ V_{13} \circ V_{12} = V_{12} \circ V_{23}$$

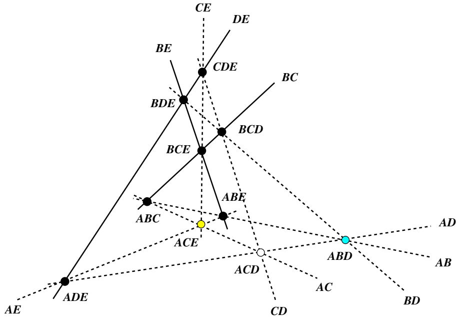
Notice that $V^{\mathrm{op}} = N^{-1}$

COROLLARY

The inverse of V is given by

$$x_1 = \bar{x}_1(\bar{x}_1 + \bar{x}_2 - \bar{x}_2\bar{x}_1)^{-1}, \qquad x_2 = \bar{x}_1 + \bar{x}_2 - \bar{x}_2\bar{x}_1,$$

and satisfies the pentagonal condition



Pentagonal property of the Veblen map is a consequence of the Desargues theorem
- lines are labeled by two-element subsets out of five-letter set
- points are labeled by three-element subsets
- contains five Veblen configurations labeled by subsets containing a fixed letter

A map $R : \mathcal{X}^3 \to \mathcal{X}^3$, which satisfies in $\mathcal{X}^6$ the relation

$$R_{123} \circ R_{145} \circ R_{246} \circ R_{356} = R_{356} \circ R_{246} \circ R_{145} \circ R_{123},$$

proposed on the quantum level by *[Zamolodchikov 1981]*, is called *tetrahedron map*

The birational map $R = P_{23} \circ V_{12} \circ N_{13}, : (x_1, x_2, x_3) \dashrightarrow (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, where

$$\tilde{x}_1 = [x_3 + x_1(1 - x_3)]^{-1} x_1 x_2, \qquad \tilde{x}_2 = x_3 + x_1(1 - x_3),$$
$$\tilde{x}_3 = 1 + (x_2 - 1)\left[(1 - x_1)x_3 + x_1(1 - x_2)\right]^{-1} (x_3 + x_1(1 - x_3)),$$

satisfies the tetrahedron condition

The inverse map $R^{-1} : (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \dashrightarrow (x_1, x_2, x_3)$, given explicitly by

$$x_1 = \tilde{x}_2 \tilde{x}_1 \left[\tilde{x}_3 + (1 - \tilde{x}_3)\tilde{x}_1\right]^{-1}, \qquad x_2 = \tilde{x}_3 + (1 - \tilde{x}_3)\tilde{x}_1,$$
$$x_3 = 1 + (\tilde{x}_3 + (1 - \tilde{x}_3)\tilde{x}_1)\left[\tilde{x}_3(1 - \tilde{x}_1) + (1 - \tilde{x}_2)\tilde{x}_1\right]^{-1} (\tilde{x}_2 - 1),$$

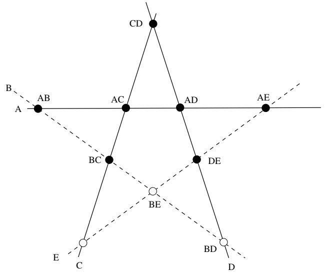
satisfies the tetrahedron condition as well

Geometry of the ten-term relation for the normalization and Veblen maps

Theorem *[Kashaev, Sergeev 1998]*

Given a solution N of the functional pentagon equation, and given a solution V of the reversed functional pentagon equation on the same set $\mathcal{X}$, then the map $R = P_{23} \circ V_{12} \circ N_{13}$ satisfies the Zamolodchikov tetrahedron equation, provided

$$V_{13} \circ N_{12} \circ V_{14} \circ N_{34} \circ V_{24} = N_{34} \circ V_{24} \circ N_{14} \circ V_{13} \circ N_{12}.$$



Start from seven points (black circles) of the star configuration $(10_2, 5_4)$ AND FOUR CORRESPONDING LINEAR RELATIONS there are two distinct ways to complete the configuration using the normalization and Veblen flips

Outline

DESARGUES MAPS [AD 2010]

Maps $\phi : \mathbb{Z}^N \to \mathbb{P}^M(\mathbb{D})$, such that the points $\phi(n)$, $\phi_{(i)}(n)$ and $\phi_{(j)}(n)$ are collinear, for all $n \in \mathbb{Z}^N$, $i \neq j$; here $\mathbb{D}$ is a division ring

NOTATION: $\phi_{(i)}(n_1, \dots, n_i, \dots, n_N) = \phi(n_1, \dots, n_i + 1, \dots, n_N)$

In non-homogeneous (affine) coordinates we have $\boldsymbol{\Phi} : \mathbb{Z}^N \to \mathbb{D}^M$

$$(\boldsymbol{\Phi}_{(j)} - \boldsymbol{\Phi}) = (\boldsymbol{\Phi}_{(i)} - \boldsymbol{\Phi}) B_{ij},$$

- the first part of the compatibility condition gives

$$B_{ij} B_{jk} = B_{ij},$$

which allows to introduce a potential $\sigma : \mathbb{Z}^N \to \mathbb{D}_*$ such that

$$B_{ij} = \sigma_{(i)} \sigma_{(j)}^{-1};$$

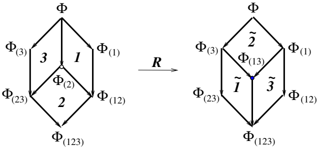
- the second part of the compatibility condition takes then the form

$$(\sigma_{(i)}^{-1} - \sigma_{(j)}^{-1})\sigma_{(ij)} + (\sigma_{(j)}^{-1} - \sigma_{(k)}^{-1})\sigma_{(jk)} + (\sigma_{(k)}^{-1} - \sigma_{(i)}^{-1})\sigma_{(ki)} = 0$$

known as the non-commutative discrete mKP system [Nijhoff, Capel 1990]

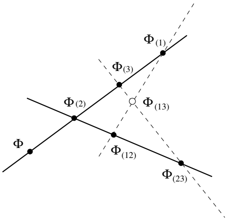
The linear problem for the tetrahedron map

$$\boldsymbol{\Phi}_{(2)} = \boldsymbol{\Phi} x_1 + \boldsymbol{\Phi}_{(1)}(1 - x_1), \quad \boldsymbol{\Phi}_{(23)} = \boldsymbol{\Phi}_{(2)} x_2 + \boldsymbol{\Phi}_{(12)}(1 - x_2), \quad \boldsymbol{\Phi}_{(3)} = \boldsymbol{\Phi} x_3 + \boldsymbol{\Phi}_{(2)}(1 - x_3)$$



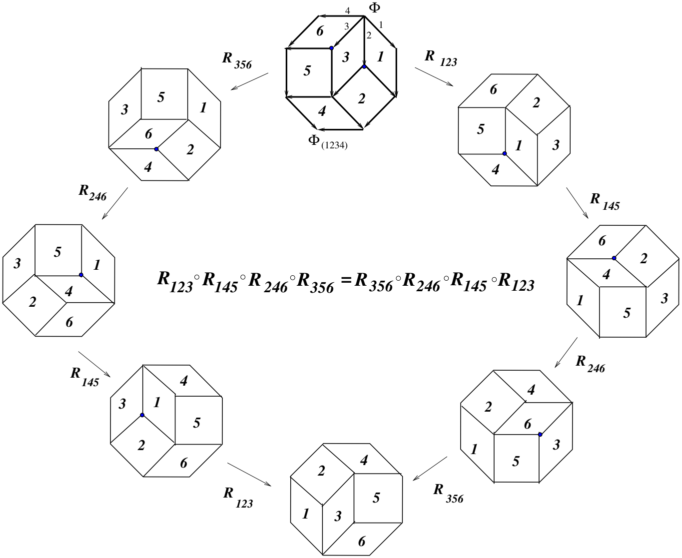
After the cube flip we arrive to three new linear relations

$$\boldsymbol{\Phi}_{(23)} = \boldsymbol{\Phi}_{(3)} \tilde{x}_1 + \boldsymbol{\Phi}_{(13)}(1 - \tilde{x}_1), \quad \boldsymbol{\Phi}_{(3)} = \boldsymbol{\Phi} \tilde{x}_2 + \boldsymbol{\Phi}_{(1)}(1 - \tilde{x}_2), \quad \boldsymbol{\Phi}_{(13)} = \boldsymbol{\Phi}_{(1)} \tilde{x}_3 + \boldsymbol{\Phi}_{(12)}(1 - \tilde{x}_3)$$



The relation between the Veblen configuration (the Menelaus theorem) and the discrete Schwarzian KP equation was known to *[Konopelchenko, Schief 2002]*

4D cube (tesseract) visualization of Zamolodchikov's condition



$$R_{123}\circ R_{145}\circ R_{246}\circ R_{356} = R_{356}\circ R_{246}\circ R_{145}\circ R_{123}$$

Multidimensional consistency of discrete conjugate nets and discrete Darboux equations *[AD, Santini 1997]*

A. Doliwa, P.M. Santini / Physics Letters A 233 (1997) 365–372 371

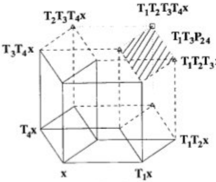


Fig. 6.

$$P_{1234} \ni T_1T_2T_3T_4\boldsymbol{x} = \bigcap_{i=1}^{4} T_iP_{1\hat{i}4} = \bigcap_{i=1, i\neq k}^{4} T_kT_iP_{1\hat{i}\hat{k}4}. \quad (25)$$

The same argument can be used to prove the compatibility of the construction for an arbitrary dimension N of the lattice. In the natural notation inherited from the example $N = 4$,

$$A_{ij}(n_1, n_2, \ldots, n_N), \quad i \neq j, \quad i, j = 1, \ldots, N, \quad (28)$$

which solve the MQL equation (11) and satisfy the following boundary conditions,

$$A_{ij}(0, \ldots, n_i, \ldots, n_j, \ldots, 0) = A_{ij}^{(0)}(n_i, n_j),$$
$$A_{ji}(0, \ldots, n_i, \ldots, n_j, \ldots, 0) = A_{ji}^{(0)}(n_i, n_j),$$
$$1 \leqslant i < j \leqslant N. \quad (29)$$

Note that the number of the arbitrary functions of two variables entering into the general solution of the MQL equation agrees with that of the continuous case [21].

Remark. All the geometric considerations of this section can be reformulated in terms of linear equations (for example, a three-dimensional subspace of a five-dimensional space is equivalent to a system of $(5-3)$ linear equations for 5 unknowns) and the corresponding theorems about their solutions.

JOURNAL OF INTEGRABLE SYSTEMS

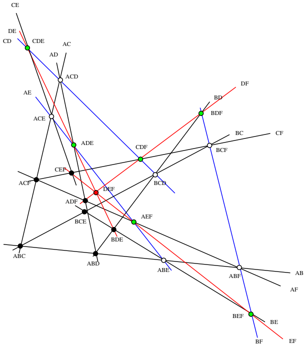
$$Q_{ij(k)} = Q_{ij} + Q_{ik(j)}Q_{kj}, \qquad i, j, k \quad \text{disctinct}$$

[Bogdanov, Konopelchenko 1995]

The corresponding solution of the functional tetrahedron equation was constructed by *[Bazhanov, Sergeev 2006]*

FURTHER COMMENTS ON DESARGUES MAPS AND MULTIDIMENSIONAL QUADRILATERAL LATTICES

- Desargues maps naturally are defined on the root lattice $Q(A_N)$ *[AD 2011]*
- theory of K dimensional discrete conjugate nets is equivalent to theory of $2K-1$ dimensional Darboux maps *[AD 2010]*
- the discrete BKP *[Miwa 1982]* , and the discrete CKP *[Kashaev 1996]* equations in K dimensions are obtained as reductions of the discrete (A)KP *[Hirota 1981]* equations in $2K-1$ dimensions *[AD 2013]*



The $(20_3, 15_4)$ configuration as "linear" construction of the quadrilateral lattice

In homogeneous coordinates of the projective space, and in a suitable gauge

$$\boldsymbol{\Phi}_{(i)} - \boldsymbol{\Phi}_{(j)} = \boldsymbol{\Phi} U_{ij}, \qquad 1 \leq i \neq j,$$

whose compatibility is

$$U_{ij} + U_{ji} = 0, \quad U_{ij} + U_{jk} + U_{ki} = 0, \quad U_{ki} U_{kj(i)} = U_{kj} U_{ki(j)} \quad i, j, k \text{ distinct}$$

[Nimmo 2006]

When $\mathbb{D}$ is commutative then the functions U_{ij} can be parametrized in terms of a single potential $\tau : \mathbb{Z}^N \to \mathbb{D}$

$$U_{ij} = \frac{\tau \tau_{(ij)}}{\tau_{(i)} \tau_{(j)}}, \qquad 1 \leq i < j \leq N$$

and the nonlinear system reads *[Hirota 1981], [Miwa 1982]*

$$\tau_{(i)} \tau_{(jk)} - \tau_{(j)} \tau_{(ik)} + \tau_{(k)} \tau_{(ij)} = 0, \qquad 1 \leq i < j < k$$

Outline

1. Geometric pentagon and tetrahedron maps (with R. Kashaev)
   - The normalization map
   - The Veblen map
   - Geometric tetrahedron map
2. 4D consistent systems and Zamolodchikov's condition
   - Desargues maps and non-commutative KP systems
   - Multidimensional discrete conjugate nets
3. Yang–Baxter maps
   - Periodic reduction of the non-commutative KP map
   - Central-product reduction

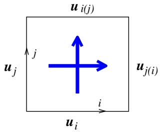
Let us distinguish the last variable $k = n_N$, denote also

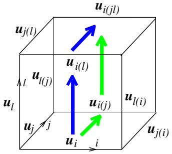
$$n = (n_1, \ldots, n_{N-1}), \quad \boldsymbol{\Phi}(n,k) = \boldsymbol{\Psi}_k(n), \quad U_{N,i}(n,k) = u_{i,k}(n)$$

which allows the rewrite a part (that with the distinguished variable) of the linear problem in the form *[Kajiwara, Noumi, Yamada 2002]*

$$\boldsymbol{\Psi}_{k+1} - \boldsymbol{\Psi}_{k(i)} = \boldsymbol{\Psi}_k u_{i,k}, \qquad i = 1, \ldots, N-1$$

$$u_{j,k} u_{i,k(j)} = u_{i,k} u_{j,k(i)}, \qquad u_{i,k(j)} + u_{j,k+1} = u_{j,k(i)} + u_{i,k+1}$$





The non-commutative KP map

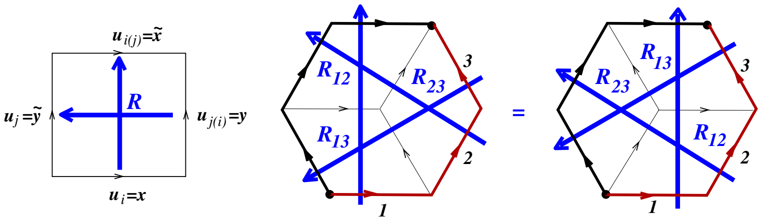
$$u_{i,k(j)} = (u_{i,k} - u_{j,k})^{-1} u_{i,k}(u_{i,k+1} - u_{j,k+1}), \qquad 1 \leq i \neq j \leq N,$$

is multidimensionaly consistent $\quad \boldsymbol{u}_i = (u_{i,k}) \quad \begin{cases} k \in \mathbb{Z} \\ k \in \mathbb{Z}_P, \quad u_{i,k+P} = u_{i,k} \end{cases}$

A map $\mathcal{R}\colon \mathcal{X}\times\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ satisfying the relation

$$\mathcal{R}_{12}\circ\mathcal{R}_{13}\circ\mathcal{R}_{23}=\mathcal{R}_{23}\circ\mathcal{R}_{13}\circ\mathcal{R}_{12}, \qquad \text{in} \quad \mathcal{X}\times\mathcal{X}\times\mathcal{X}$$

is called Yang–Baxter map *[Drinfeld 1992]*



Observation

The *companion map* of a three dimensionally consistent map gives rise to Yang–Baxter map *[Adler, Bobenko, Suris 2004]*

$$x_k y_k=\tilde{y}_k\tilde{x}_k, \qquad y_k+x_{k+1}=\tilde{x}_k+\tilde{y}_{k+1}$$

Problem: Find the companion map of the KP map

THEOREM

Given non-commuting variables $\boldsymbol{x} = (x_1, \dots, x_P)$, $\boldsymbol{y} = (y_1, \dots, y_P)$ define

$$\mathcal{X}_k = x_k x_{k+1} \dots x_{k+P}, \qquad \mathcal{Y}_k = y_k y_{k+1} \dots y_{k+P}$$

$$\mathcal{P}_k = \sum_{a=0}^{P-1} \left(\prod_{i=0}^{a-1} y_{k+i} \prod_{i=a+1}^{P-1} x_{k+i} \right), \qquad k = 1, \dots, P,$$

where subscripts should be read modulo P. If h_k is a solution of the Sylvester equation

$$h_k \mathcal{X}_k + \mathcal{P}_{k+1} = \mathcal{Y}_k h_k \qquad (h_k = \sum_{j=0}^{\infty} \mathcal{Y}_k^{-j-1} \mathcal{P}_{k+1} \mathcal{X}_k^j)$$

then

$$\mathcal{R}(\boldsymbol{x}, \boldsymbol{y}) = (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}), \qquad \tilde{x}_k = h_{k-1} x_k h_k^{-1}, \qquad \tilde{y}_k = h_{k-1}^{-1} y_k h_k,$$

is a Yang–Baxter map

Commutative case *[Kajiwara, Noumi, Yamada 2002], [Etingof 2003]*

Assume that the products $\alpha = \mathcal{X}_1 = x_1 x_2 \dots x_P$, $\beta = \mathcal{Y}_1 = y_1 y_2 \dots y_P$ are central, then:
(i) $\mathcal{X}_k$ and $\mathcal{Y}_k$ do not depend on k
(ii) the Yang–Baxter map $\mathcal{R}(\boldsymbol{x}, \boldsymbol{y}) = (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ simplifies to

$$\tilde{x}_k = \mathcal{P}_k x_k \mathcal{P}_{k+1}^{-1}, \qquad \tilde{y}_k = \mathcal{P}_k^{-1} y_k \mathcal{P}_{k+1}$$

(iii) the products α and β are conserved under the map $\mathcal{R}$

In the simplest case $P = 2$: $\alpha = x_1 x_2$, $\beta = y_1 y_2$ we put $x = x_1$, $y = y_1$ to get a parameter dependent reversible Yang–Baxter map $\mathcal{R}(\alpha, \beta) : (x, y) \mapsto (\tilde{x}, \tilde{y})$

$$\tilde{x} = \left(\alpha x^{-1} + y\right) x \left(x + \beta y^{-1}\right)^{-1},$$
$$\tilde{y} = \left(\alpha x^{-1} + y\right)^{-1} y \left(x + \beta y^{-1}\right),$$

which in the commutative case is equivalent to the F_{III} map in the list of

[Adler, Bobenko, Suris 2004]

THANK YOU

- A. Doliwa, Desargues maps and the Hirota-Miwa equation, Proc. R. Soc. A 466 (2010) 1177-1200
- A. Doliwa, S. M. Sergeev, The pentagon relation and incidence geometry, J. Math. Phys. **55** (2014) 063504
- A. Doliwa, Non-commutative rational Yang-Baxter maps, Lett. Math. Phys. **104** (2014) 299-309
- A. Doliwa, R. M. Kashaev, Non-commutative rational pentagon and tetrahedron relations, and Desargues maps, in preparation