GL(2,R) geometry and integrable systems

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Introduction

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Plan:

1. Definitions
2. Motivations and examples
3. GL(2,R)-structures and complex geometry
4. Diemnsion 4

Definitions of GL(2,R)-structures

Let M be a manifold of dimension $k+1$.

Three equivalent definitions:

1. An isomorphism
$$TM \simeq S^k(E)$$
where E is a rank-2 bundle over M. Fixing a basis $(x,y) \in E_p$
$$T_pM \simeq H_k(\mathbb{R}^2)$$
where $H_k(\mathbb{R}^2)$ is the space of homogeneous polynomials in two variables and of order k.
2. A reduction of the frame bundle to a $GL(2,\mathbb{R})$-subbundle $B(E)$ (where $GL(2,\mathbb{R})$-acts irreducibly).
3. A field of rational normal curves $p \mapsto C(E)_p \subset P(T_pM)$
$$C(E) = \{(sx+ty)^k\}.$$

Motivations

Why $GL(2,\mathbb{R})$-structures are interesting:

1. Natural geometric structures on solutions spaces of ODEs (generalisation of 3-dim conformal Lorentzian metric) [Bryant, Dunajski-Tod, Nurowski, Doubrov].
2. Can appear as characteristic varieties of PDEs, e.g. Veronese hierarchy on the next slide, or recent results by [Ferapontov-Kruglikov].

Veronese hierarchy

Solutions $w : \mathbb{R}^{k+1} \to R$ to

$$(a_i - a_j)\partial_0 w \partial_i \partial_j w + a_j \partial_i w \partial_j \partial_0 w - a_i \partial_j w \partial_i \partial_0 w = 0, \qquad i, j = 1, \ldots, k$$

where a_i are distinct numbers are in a one to one correspondence with

1. Hyper-CR Einstein-Weyl structures in dim 3 (i.e. for $k = 2$) [Dunajski-K.].
2. Veronese webs on $\mathbb{R}^{k+1}$ for arbitrary k. The Veronese webs are special 1-parameter families of corank-1 foliations introduced by Gelfand and Zakharevich in connection to bi-Poisson systems on odd-dimensional manifolds.

A characteristic variety of the system is the null cone $C(E)$ of a $GL(2, \mathbb{R})$-structure.

Integrability

Fix a point V in the null cone $C(E)_p$. Let

$$\mathrm{span}\{V\} = D_1(V) \subset D_2(V) \subset \ldots \subset D_k(V) \subset T_pM$$

be a sequence of osculating spaces of $C(E)_p$ at V, where $dim D_i(V) = i$.

Definitions:

1. $D_i(V)$ is called α_i-plane of a $GL(2,\mathbb{R})$-structure.
2. A submanifold $N \subset M$ is called α_i-manifold if each T_pN is an α_i-plane.
3. A structure is α_i-integrable if all α_i-planes are tangent to α_i-manifolds.

Properties

1. α_i-integrability implies α_j-integrability for $j < i$.
2. α_k-integrability is equivalent to α_{k-1}-integrability (this follows from the geometry of Goursat flags).
3. Veronese webs are α_i-integrable for any i.

Connections to ODEs

Theorem

A $GL(2,\mathbb{R})$-structure in dimension $k+1$ comes from an ODE of order $k+1$ with the vanishing Wünschmann invariants if and only if it is α_k-integrable.

Remarks:

1. The Wünschamnn invariants are the basic contact invariants of ODEs. An ODE of order $k+1$ has $k-1$ Wünschmann invariants.
2. If one wants to make α_k-manifolds totally geodesic w.r.t. some connection then additional point invariants appear - this generalizes the Cartan invariant for third-order ODEs and the corresponding Einstein-Weyl structures.

We shall consider later the $\alpha_{\frac{k+1}{2}}$-integrability for even-dimensional manifolds.

Theorem (K.-Mettler)

Let $TM \simeq S^k(E)$ be a $GL(2,\mathbb{R})$-structure on even-dimensional manifold M and assume that a $GL(2,\mathbb{R})$-connection on $B(E)$ is defined by a 1-form $\phi = (\phi^i_j)_{i,j=1,2}$ with values in $gl(2,\mathbb{R})$. Then there is a canonical almost-complex structure J_ϕ on the quotient bundle

$$B(E)/CO(2,\mathbb{R})$$

whose $(1,0)$-forms pullback to $B(E)$ to become linear combinations of the forms $\xi^{k,0},\ldots,\xi^{k,\lfloor\frac{k}{2}\rfloor}$ and

$$\zeta = (\phi^1_2 + \phi^2_1) + \mathrm{i}(\phi^2_2 - \phi^1_1),$$

where $\xi^{k,0},\ldots,\xi^{k,\lfloor\frac{k}{2}\rfloor}$ are certain complex valued forms composed from the soldering form.

In dimension 4:

$$\xi^{3,1} = \frac{1}{4}(3\omega^0 + \omega^2 + \mathrm{i}(\omega^1 + 3\omega^3)), \qquad \xi^{3,0} = \frac{1}{4}(\omega^0 - \omega^2 + \mathrm{i}(\omega^1 - \omega^3)).$$

In dimension 6:

$$\begin{aligned}
\xi^{5,2} &= \frac{1}{76}(10\omega^0 + 7\omega^3 + 12\omega^5 + \mathrm{i}(12\omega^2 + 7\omega^4 + 10\omega^6)),\\
\xi^{5,1} &= \frac{1}{76}(5\omega^0 - 6\omega^3 - 13\omega^5 + \mathrm{i}(13\omega^2 + 6\omega^4 - 5\omega^6)),\\
\xi^{5,0} &= \frac{1}{76}(\omega^0 - 5\omega^3 + 5\omega^5 + \mathrm{i}(5\omega^2 - 5\omega^4 + \omega^6)).
\end{aligned}$$

Complex structure on $H_k(R^2)$

The first step in the proof is a construction of a complex structure on the space of polynomials. $H_k(R^2)$ decomposes into 2-dimensional subspaces, invariant w.r.t. $CO(2,\mathbb{R})$. The polynomials are

$$H_{k-2i} = \mathrm{span}\{\Re((x+\mathrm{i}y)^{k-i}(x-\mathrm{i}y)^i), \quad \Im((x+\mathrm{i}y)^{k-i}(x-\mathrm{i}y)^i)\}.$$

On H_j we defined a complex structure by formula

$$J_j = \sqrt[j]{J}$$

where $J \in CO(2,\mathbb{R})$ is the standard complex structure $(x,y) \mapsto (-y,x)$.

The construction gives $(x+\mathrm{i}y)^{k-i}(x-\mathrm{i}y)^i$ as $(1,0)$-vectors.

Integrability

The torsion T and curvature C of ϕ in the presence of J_ϕ decompose to parts $T^{(2,0)}$, $T^{(1,1)}$, $T^{(0,2)}$ and $C^{(2,0)}$ $C^{(1,1)}$, $C^{(0,2)}$.

Theorem (K.-Mettler)

The almost-complex structure J_ϕ on $B(E)/CO(2,\mathbb{R})$ is integrable if and only if $T^{(0,2)} = 0$ and $C^{(0,2)} = 0$.

Remark: If $T^{(1,1)} = 0$ and $T^{(0,2)} = 0$ then $C^{(0,2)} = 0$. In particular, if ϕ is torsion-free then J_ϕ is integrable.

Canonical connection

One can define a canonical connection for a $GL(2,\mathbb{R})$-structure. The corresponding almost-complex structure will be called canonical.

Let $g_k \subset gl(k+1,\mathbb{R})$ be the standard subalgebra isomorphic to $gl(2,\mathbb{R})$ corresponding to the irreducible action on $H_k(\mathbb{R}^2)$. Define

$$g_k^\perp = \{\psi \in gl(k+1,\mathbb{R}) \mid \mathrm{tr}(\eta \circ \psi) = 0 \quad \forall \eta \in g_k\}.$$

Theorem

Let $TM \simeq S^k(E)$ be a $GL(2,\mathbb{R})$-structure on a manifold M of dimension $k+1 > 3$. There is a unique $GL(2,\mathbb{R})$-connection $\phi = (\phi^i_j)_{i,j=1,2}$ with values in $gl(2,\mathbb{R})$ such that $\Theta_\chi(X,.) \in g_k^\perp$ for any $\chi \in B(E)$ and $X \in T_\chi B(E)$, where Θ is the torsion 2-form of ψ.

Remark: In dimension 4 this coincides with the Bryant connection.

Twistor space

Theorem

Let $TM \simeq S^k(E)$ be a $GL(2,\mathbb{R})$-structure on even-dimensional manifold M and assume that the almost-complex structure J_ϕ defined by a $GL(2,\mathbb{R})$-connection ϕ is integrable. Then, the $GL(2,\mathbb{R})$-structure is $\alpha_{\frac{k+1}{2}}$-integrable.

Remarks:

1. In dimension 4 we get that a $GL(2,\mathbb{R})$-structure is torsion-free if and only if the canonical almost-complex structure is integrable.
2. If a structure is $\alpha_{\frac{k+1}{2}}$-integrable then there is a well defined $\frac{k+3}{2}$-dimensional (real) twistor space. (This twistor space can be "glued" to $B(E)/C(2,\mathbb{R})$ to get (complex) twistor space.)

Holomorphic sections

$B(E)/CO(2,\mathbb{R}) \simeq P(E^{\mathbb{C}}) \setminus P(E)$ and any point

$$[z] = [x + iy] \in P(E^{\mathbb{C}}) \setminus P(E)$$

defines the following subspace in $T^{\mathbb{C}}M \simeq S^k(E^{\mathbb{C}})$

$$\mathrm{span}\{z^k, z^{k-1}\bar{z}, \ldots, z^{\frac{k+1}{2}}\bar{z}^{\frac{k-1}{2}}\}$$

This defines a complex structure on T_pM.

Remark: Holomorphic sections of $B(E)/CO(2,\mathbb{R})$ give complex structures on M.

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Now, we would like to describe all integrable $GL(2,\mathbb{R})$-structures in a convenient way.

Dimension 4

Theorem (K.-Mettler)

Any integrable $GL(2,\mathbb{R})$-structure in dim 4 can be put in the form $C(E) = \{\, s^3V_0 + s^2tV_1 + st^2V_2 + t^3V_3 |\, s,t \in \mathbb{R}\}$ where

$$V_0 = \partial_3, \qquad V_1 = \partial_2 + 9A\partial_3, \qquad V_2 = \partial_1 + 3A\partial_2 + B\partial_3,$$
$$V_3 = \partial_0 + A\partial_1 + C\partial_2 + D\partial_3,$$

and A, B, C, D are functions satisfying the following system

$$V_2(D) - V_3(B) - BV_2(A) - 9AV_2(C) + 27AV_3(A) + 27A^2V_2(A) = 0$$
$$3V_2(C) + 9V_3(A) - 2V_1(D) - 9AV_2(A) + 2BV_1(A) + 18AV_1(C) - 54A^2V_1(A) = 0$$
$$3V_2(A) - 6V_1(C) + 3V_0(D) + 18AV_1(A) - 27AV_0(C) + 81A^2V_0(A) - 3BV_0(A) = 0$$
$$3V_1(A) + 9V_0(C) - 2V_0(B) + 27AV_0(A) = 0$$

Remarks:

1. First step in the proof: write down a structure in terms of the corresponding ODE.
2. A priori there are 8 equations (components of the Bryant torsion) however half of them is void.
3. The system has a Lax representation $[L_0, L_1] = 0$.
4. I do not know how to construct a similar system describing the structures in higher dimensions.

Thank you!