Deformations of Poisson and bi-Hamiltonian structures on formal loop spaces

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Our recent results:

- with H. Posthuma, S. Shadrin:
  - “Bihamiltonian cohomology of the KdV brackets”, Comm. Math. Phys. (2016).
  - “Bihamiltonian cohomology of scalar Poisson brackets of hydrodynamic type”, Bull. London Math. Soc. (2016).
  - “Deformations of semisimple Poisson brackets of hydrodynamic type are unobstructed”, preprint (2015).
- with M. Casati, S. Shadrin:
  - “Poisson cohomology of scalar multidimensional Dubrovin-Novikov brackets”, preprint (2015).

Outline

Outline

KdV

The Korteweg - de Vries equation

$$u_t = uu_x + \epsilon^2 u_{xxx}$$

has bihamiltonian formulation

$$u_t(x) = \{u(x), H_1\}_1 = \{u(x), H_0\}_2$$

with compatible Poisson brackets

$$\{u(x), u(y)\}_1 = \delta'(x-y),$$
$$\{u(x), u(y)\}_2 = u(x)\delta'(x-y) + \frac{1}{2}u'(x)\delta(x-y) + \frac{3}{2}\epsilon^2\delta'''(x-y).$$

[Gardner-Zakharov-Faddeev '71, Magri '78]

General problem

Scalar case $N = 1$

Classify (bi)hamiltonian structures of the form

$$\{u(x), u(y)\} = \{u(x), u(y)\}^0 + \\ + \sum_{m\geqslant 2} \epsilon^m \sum_{l=0}^{m+1} A_{m,l}(u; u_x, \dots)\delta^{(l)}(x-y)$$

under Miura type transformations

$$u(x) \to u(x) + \epsilon f_1(u; u_x) + \epsilon^2 f_2(u; u_x, u_{xx}) + \dots$$

where $A_{m,l}$, f_i are differential polynomials.

[Dubrovin-Zhang'01]

Outline

Poisson brackets of Dubrovin-Novikov type

Leading order:

$$\{u^i(x), u^j(y)\}^0 = g^{ij}(u(x))\delta'(x-y) + \Gamma^{ij}_k(u(x))u^k_x(x)\delta(x-y),$$

is a Poisson structure iff

- g^{ij} flat contravariant metric,
- Γ^{ij}_k Christoffel symbols of g^{ij}.

[Dubrovin-Novikov'83]

Local multivectors

- In finite dimensions: the space Λ^* of multivectors on a manifold M is endowed with the Schouten-Nijenhuis bracket

$$[,] : \Lambda^p \times \Lambda^q \to \Lambda^{p+q-1}$$

- On a formal loop space $\mathcal{L}M = \{S^1 \to M\}$: one considers the space Λ^*_{loc} of local multivectors of the form (for $M = \mathbb{R}$)

$$\sum_{p_2 \cdots p_k \geqslant 0} B_{p_2 \dots p_k}(u(x); u_x(x), u_{xx}(x), \dots) \delta^{(p_2)}(x - x_2) \cdots \delta^{(p_k)}(x - x_k)$$

which is closed under a suitably defined Schouten-Nijenhuis bracket

$$[,] : \Lambda^p_{loc} \times \Lambda^q_{loc} \to \Lambda^{p+q-1}_{loc}$$

Poisson cohomology and deformations

- A bivector $P \in \Lambda^2_{loc}$ is a Poisson structure iff $[P, P] = 0$
  $\Longrightarrow d_P := [P, \cdot] : \Lambda_{loc} \to \Lambda_{loc}$ is a differential $d_P^2 = 0$.

- Let $P \in \Lambda^2_{loc}$ Poisson bivector. The Poisson cohomology of P is

$$H(\Lambda_{loc}, d_P) = \frac{\operatorname{Ker} d_P}{\operatorname{Im} d_P}.$$

- The Poisson cohomology $H(\Lambda_{loc}, d_P)$ of a Poisson structure of DN type P vanishes in positive degree.

  [Getzler'02, Degiovanni-Magri-Sciacca'05, Liu-Zhang'11]

- All deformations of a single Poisson structure of DN type are trivial under Miura transformations.

Outline

Deformations of bihamiltonian structure

Recent developments:

- Classification of deformations of dKdV bihamiltonian structure up to $O(\epsilon^4)$ [Lorenzoni'02] ...$O(\epsilon^8)$ [Arsie-Lorenzoni'11]
- Quasitriviality of dKdV deformations; reformulation as a double complex [Barakat'08]
- Computation of $BH^1(\hat{\mathcal{F}})$, $BH^2(\hat{\mathcal{F}})$, central invariants [Liu-Zhang'05, Dubrovin-Liu-Zhang'06]
- Computation of $BH^3(\hat{\mathcal{F}})$ for dKdV Poisson pencil: existence of deformation of dKdV Poisson pencil corresponding to infinitesimal deformations. Conjectured vanishing of $BH^{\geqslant 4}(\hat{\mathcal{F}})$. [Liu-Zhang'13]

Deformations of bihamiltonian structure

- Deformation theory of a Poisson pencil P_1, P_2 of hydrodynamic type is governed by bihamiltonian cohomology groups

$$BH(\Lambda_{loc}, d_1, d_2) = \frac{\mathrm{Ker}\, d_1 \cap \mathrm{Ker}\, d_2}{\mathrm{Im}\, d_1 d_2}$$

where $d_i = [P_i, \cdot]$.

- Infinitesimal deformations ($O(\epsilon^3)$) are classified by $BH^2(\Lambda_{loc})$, i.e., by central invariants

$$c_i(u) = \frac{1}{3(f^i(u))^2} \left(A^{ii}_{2,3;2} - u^i A^{ii}_{2,3;1} + \sum_{k \neq i} \frac{(A^{ij}_{1,2;2} - u^i A^{ij}_{1,2;1})^2}{f^k(u)(u^k - u^i)} \right).$$

[Liu-Zhang'05, Dubrovin-Liu-Zhang'06]

Existence of deformations

- Given an infinitesimal deformation of a Poisson pencil of DN type, is it possible to extend it to a full dispersive Poisson pencil ?

  Main Theorem [C-Posthuma-Shadrin'15]
  The deformations of any semisimple Poisson pencil of DN type are unobstructed.

- Previously known for the dKdV Poisson pencil. [Liu-Zhang'13]
- Sufficient to show that $BH^3_{\geqslant 5}(\Lambda_{loc}, d_1, d_2)$ vanishes.

Our results:

1 We compute the full bihamiltonian cohomology of the dispersionless KdV Poisson pencil:

Theorem [C-Posthuma-Shadrin'14]

The bihamiltonian cohomology of the dispersionless KdV Poisson pencil is given by

$$BH^p_d(\Lambda_{loc}, d_1, d_2) \cong \begin{cases} C^\infty(\mathbb{R}) & \text{for } (p,d) = (1,1), (2,1), (2,3), (3,3) \\ \mathbb{R} & \text{for } (p,d) = (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

2 We generalize the above result, computing the full bihamiltonian cohomology of general scalar Poisson pencil of hydrodynamic type. [C-Posthuma-Shadrin'15-a]

Our results:

3 We show that the bihamiltonian cohomology of a semisimple Poisson pencil of hydrodynamic type with n dependent variables vanishes but for a finite number of bi-degrees:

Theorem [C-Posthuma-Shadrin'15-b]

The bihamiltonian cohomology $BH^p_d(\Lambda_{loc}, d_1, d_2)$ vanishes for all bi-degrees (p, d) with $d \geqslant 2$, unless

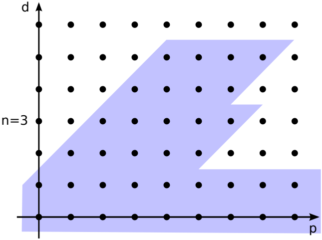
$$d = 2, \dots, n, \quad p = d, \dots, d + n,$$

$$d = n + 1, n + 2, \quad p = d, \dots, d + n - 1.$$

For example, in the $n = 3$ case, we claim the bihamiltonian cohomology

$$BH^p_d(\Lambda_{loc}, d_1, d_2)$$

vanishes in all bi-degrees but those highlighted.



In particular, this implies the vanishing of $BH^3_{\geqslant 5}(\Lambda_{loc})$ which in turn implies the vanishing of the obstructions.

Outline

Supervariables formalism

[Liu-Zhang'13]

Consider the space

$$\hat{\mathcal{A}} := C^\infty(\mathbb{R})[[u^1, u^2, \dots; \theta, \theta^1, \dots]]$$

of formal series

$$f(u; u^1, u^2, \dots; \theta, \theta^1, \dots) \in \hat{\mathcal{A}}$$

in the commuting variables $u^1, u^2, \dots$ and in the anticommuting variables $\theta, \theta^1, \theta^2, \dots$.

- x-derivative: $\partial = \sum_{s\geqslant 0} \left(u^{s+1} \frac{\partial}{\partial u^s} + \theta^{s+1} \frac{\partial}{\partial \theta^s} \right) : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$
- two gradations:

$$\hat{\mathcal{A}}^p_d = \text{homogeneous component with degree } \begin{cases} p \text{ in } \theta, \theta^1, \dots \\ d \text{ in } x\text{-derivatives.} \end{cases}$$

Supervariables formalism

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- two gradations:

$$\hat{\mathcal{A}}^p_d = \text{homogeneous component with degree } \begin{cases} p \text{ in } \theta, \theta^1, \dots \\ d \text{ in } x\text{-derivatives.} \end{cases}$$

- Let $\hat{\mathcal{F}} := \frac{\hat{\mathcal{A}}}{\partial\hat{\mathcal{A}}}$ and denote the projection map $\int : \hat{\mathcal{A}} \to \hat{\mathcal{F}}$.
- $\Lambda^p_{loc} \cong \hat{\mathcal{F}}^p$
- The Schouten-Nijenhuis bracket is

$$[,] : \hat{\mathcal{F}}^p \times \hat{\mathcal{F}}^q \to \hat{\mathcal{F}}^{p+q-1}$$

$$[P, Q] = \int \left(\delta^\bullet P \delta_\bullet Q + (-1)^p \delta_\bullet P \delta^\bullet Q\right)$$

$$\delta^\bullet = \sum_{s\geqslant 0} (-\partial)^s \frac{\partial}{\partial\theta^s}, \quad \delta_\bullet = \sum_{s\geqslant 0} (-\partial)^s \frac{\partial}{\partial u^s}$$

- A bivector $P \in \hat{\mathcal{F}}^2$ is a Poisson structure iff $[P, P] = 0$.
- By (graded) Jacobi identity $d_P := [P, \cdot] : \hat{\mathcal{F}} \to \hat{\mathcal{F}}$ is a differential $d_P^2 = 0$.

- It is more convenient to work in $\hat{\mathcal{A}}$ rather than in $\hat{\mathcal{F}}$.

[Liu-Zhang'13]

- For any $P \in \hat{\mathcal{F}}^2$, let $d_P = [P, \cdot]$, there exists a map D_P s.t. the diagram commutes

$$\begin{array}{ccc} \hat{\mathcal{A}} & \xrightarrow{D_P} & \hat{\mathcal{A}} \\ \Big\downarrow \int & & \Big\downarrow \int \\ \hat{\mathcal{F}} & \xrightarrow{d_P} & \hat{\mathcal{F}} \end{array}$$

which is given by

$$D_P = \sum_{s \geqslant 0} \left(\partial^s(\delta^\bullet P) \frac{\partial}{\partial u^s} + \partial^s(\delta_\bullet P) \frac{\partial}{\partial \theta^s} \right)$$

- The short exact sequence of complexes above gives rise to a long exact sequence in cohomology that allow to recover the cohomology of $\hat{\mathcal{F}}$ from the cohomology of $\hat{\mathcal{A}}$.

Barakat-Liu-Zhang lemma

Let us consider the related polynomial complex

$$(\hat{\mathcal{F}}[\lambda], d_\lambda), \qquad d_\lambda = d_2 - \lambda d_1.$$

For almost all (p, d) the bihamiltonian cohomology groups are isomorphic to the cohomology groups of the corresponding polynomial complex i.e.

$$BH^p_d(\hat{\mathcal{F}}, d_1, d_2) \cong H^p_d(\hat{\mathcal{F}}[\lambda], d_\lambda)$$

for $p, d \geqslant 0$ excluding $(p, d) = (0, 0), (1, 0), (1, 1), (2, 1)$.

[Barakat'08, Liu-Zhang'13]

KdV case

- The dispersionless KdV Poisson bivectors are represented by the elements in $\hat{\mathcal{F}}$

$$P_1 = \frac{1}{2}\int \theta\theta^1, \qquad P_2 = \frac{1}{2}\int u\theta\theta^1.$$

- The differentials on $\hat{\mathcal{F}}$ induced by the Schouten bracket are

$$d_i = d_{P_i} = [P_i, \cdot], \quad i = 1, 2.$$

- The corresponding differentials on $\hat{\mathcal{A}}$ are

$$D_1 = \sum_{s\geqslant 0} \theta^{s+1}\frac{\partial}{\partial u^s},$$

$$D_2 = \sum_{s\geqslant 0}\left(\partial^s(u\theta^1 + \frac{1}{2}u_1\theta)\frac{\partial}{\partial u^s} + \partial^s(\frac{1}{2}\theta\theta^1)\frac{\partial}{\partial\theta^s}\right).$$

Main problem

Our main problem is to compute the cohomology of the complex

$$(\hat{\mathcal{A}}[\lambda], D_\lambda)$$

where

$$\hat{\mathcal{A}} = C^\infty(\mathbb{R})[[u^1, u^2, \dots; \theta, \theta^1, \dots]]$$

and

$$D_\lambda = \sum_{s \geqslant 0} \left[\partial^s((u - \lambda)\theta^1 + \frac{1}{2} u^1 \theta) \frac{\partial}{\partial u^s} + \partial^s(\frac{1}{2}\theta\theta^1) \frac{\partial}{\partial \theta^s} \right].$$

A filtration of $\hat{\mathcal{A}}[\lambda]$

We define a filtration of $\hat{\mathcal{A}}[\lambda]$

$$F^i \hat{\mathcal{A}}_d[\lambda] = \hat{\mathcal{A}}_d^{(d-i)}[\lambda]$$

by imposing the upper bound $d - i$ on the highest derivative appearing in homogeneous component of standard degree d.

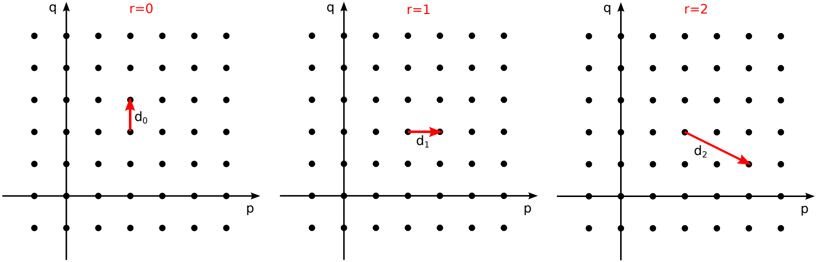
This filtration is bounded:

$$0 = F^{d+1}\hat{\mathcal{A}}_d[\lambda] \subset \cdots \subset F^{i+1}\hat{\mathcal{A}}_d[\lambda] \subset F^i\hat{\mathcal{A}}_d[\lambda] \subset \cdots \subset F^0\hat{\mathcal{A}}_d[\lambda] = \hat{\mathcal{A}}_d[\lambda].$$

We associate with this filtration a spectral sequence $E_r^{p,q}$.

Filtrations and spectral sequences

A (cohomological type) spectral sequence is a family of differential $\mathbb{Z}$-bigraded vector spaces $(E_r^{*,*}, d_r)$ with differentials d_r of bidegree $(r, 1-r)$



such that for all $p, q \in \mathbb{Z}$ and all $r \geqslant 0$

$$E_{r+1}^{pq} \cong H^{pq}(E_r^{*,*}, d_r) := \frac{\mathrm{Ker}(d_r : E_r^{pq} \to E_r^{p+r,q-r+1})}{\mathrm{Im}(d_r : E_r^{p-r,q+r-1} \to E_r^{pq})}.$$

(C, d) - filtered $\mathbb{Z}$-graded differential complex

- $F^i C$, $i \in \mathbb{Z}$ - decreasing filtration of (C, d)

$$\cdots \subset F^{i+1} \subset F^i C \subset \cdots \subset C$$

- $d(F^i C) \subset F^i C$ - filtration is preserved by differential

With a filtered Z-graded differential complex one associates a spectral sequence $(E_r^{*,*}, d_r)$ with

$$E_0^{p,q} = gr^p C^{p+q}$$

$$E_1^{p,q} = \frac{d^{-1}(F^{p+1} C^{p+q+1}) \cap F^p C^{p+q}}{d(F^p C^{p+q-1}) + F^{p+1} C^{p+q}},$$

with differentials d_0, d_1 induced by d on the quotients.

The cohomology of a filtered graded complex (C, d) inherits a filtration, where $F^i H(C, d)$ is given by the image of $H(F^i C, d)$ in $H(C, d)$ under the inclusion map.

Theorem

The spectral sequence associated with a bounded filtration converges to $H(C, d)$, i.e.,

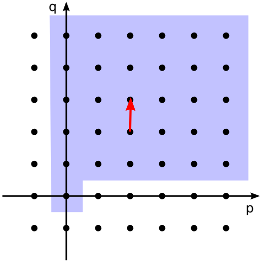
$$E_\infty^{p,q} \cong \frac{F^p H^{p+q}(C, d)}{F^{p+1} H^{p+q}(C, d)}$$

A filtration $F^* C$ is bounded if for each degree p there are integers s and t such that

$$0 = F^s C^p \subset \cdots \subset F^{i+1} C^p \subset F^i C^p \subset \cdots \subset F^t C^p = C^p.$$

Lemma: The zeroth page $E_0^{*,*}$ of the spectral sequence

$$E_0^{pq} = gr^p \hat{\mathcal{A}}_{p+q}[\lambda] \cong \hat{\mathcal{A}}_{p+q}^{[q]}[\lambda]$$

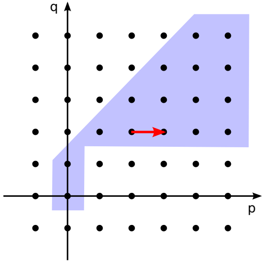


$$d_0 : E_0^{p,q} \to E_0^{p,q+1}$$

$$d_0 = \left((u-\lambda)\theta^{q+1} + \frac{1}{2}u^{q+1}\theta\right)\frac{\partial}{\partial u^q} + \frac{1}{2}\theta\theta^{q+1}\frac{\partial}{\partial \theta^q}$$

Lemma: The first page $E_1^{*,*}$

$$E_1^{p,q} = \begin{cases} \mathbb{R}[\lambda], & p = q = 0 \\ \frac{C^\infty(\mathbb{R})}{\mathbb{R}[u]}\theta\theta^1, & p = 0, q = 1 \\ \hat{\mathcal{A}}_p^{[q-1]}\theta\theta^q & p \geqslant 1, q \geqslant 2. \end{cases}$$



$$d_1 : E_1^{p,q} \to E_1^{p+1,q}$$

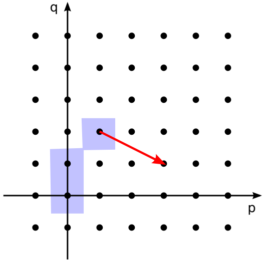
$$d_1(f\theta\theta^q) = \left((D_\lambda(f))_{\lambda=u} + \frac{q-2}{2}\theta^1 f \right) \theta\theta^q$$

Lemma: The second page $E_2^{*,*}$

Important: The following operator is a contracting homotopy of d_1 for $p \geqslant 1$, $q \geqslant 2$ and $(p,q) \neq (1,2)$

$$\left(\sum_{s \geqslant 1} \frac{s+2}{2} u^s \frac{\partial}{\partial u^s} + \sum_{s \geqslant 0} \frac{s-1}{2} \theta^s \frac{\partial}{\partial \theta^s} \right)^{-1} \frac{\partial}{\partial \theta^1}$$

$$E_2^{p,q} = \begin{cases} \mathbb{R}[\lambda] & p=0, q=0, \\ \frac{C^\infty(\mathbb{R})}{\mathbb{R}[u]} \theta\theta^1 & p=0, q=1 \\ C^\infty(\mathbb{R})\theta\theta^1\theta^2 & p=1, q=2 \\ 0 & \text{else.} \end{cases}$$



$$d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$$

The differential d_2 is zero $\to$ the spectral sequence stabilizes

Main proposition

By the convergence theorem for spectral sequences we have

$$E_2^{p,q} = E_\infty^{p,q} \cong \frac{F^p H_{p+q}(\hat{\mathcal{A}}[\lambda], D_\lambda)}{F^{p+1} H_{p+q}(\hat{\mathcal{A}}[\lambda], D_\lambda)}$$

and because the filtration is bounded we have

$$F^0 H_n(\hat{\mathcal{A}}[\lambda], D_\lambda) = H_n(\hat{\mathcal{A}}[\lambda], D_\lambda), \quad F^n H_n(\hat{\mathcal{A}}[\lambda], D_\lambda) = 0.$$

Proposition

The cohomology of the polynomial complex $(\hat{\mathcal{A}}[\lambda], D_\lambda)$ is

$$H(\hat{\mathcal{A}}[\lambda], D_\lambda) = \mathbb{R}[\lambda] \oplus (C^\infty(\mathbb{R})/\mathbb{R}[u])\theta\theta^1 \oplus C^\infty(\mathbb{R})\theta\theta^1\theta^2$$

Main result

By the long exact sequence argument and the Barakat lemma, we derive the bihamiltonian cohomology of $\hat{\mathcal{F}}$ from the cohomology of the complex $(\hat{\mathcal{A}}[\lambda], D_\lambda)$.

Theorem

The bihamiltonian cohomology of the dispersionless KdV Poisson pencil is given by

$$BH^p_d(\hat{\mathcal{F}}, d_1, d_2) \cong \begin{cases} C^\infty(\mathbb{R}) & \textit{for } (p,d) = (1,1), (2,1), (2,3), (3,3) \\ \mathbb{R} & \textit{for } (p,d) = (0,0) \\ 0 & \textit{otherwise.} \end{cases}$$

Remark: This result generalizes to the general scalar case.

Outline

Semisimple Poisson pencil of DN type

- Compatible Poisson brackets of DN type

$$\{w^i(x), w^j(y)\}_1^0, \quad \{w^i(x), w^j(y)\}_2^0$$

i.e., g_1^{ij}, g_2^{ij} flat pencil of metrics.

- Semisimple when $\det(g_2(w) - \lambda g_1(w)) = 0$ has pairwise distinct real roots in $\lambda = u^1(w), \ldots, u^n(w)$.

- $u^1, \ldots, u^n$ are canonical coordinates, i.e., the metrics are diagonal:

$$g_1^{ij} = f^i(u)\delta_{ij}, \quad g_2^{ij} = u^i f^i(u)\delta_{ij}.$$

Semisimple n-dimensional case

- Space of local multivectors:

$$\hat{\mathcal{F}} = \frac{\hat{\mathcal{A}}}{\partial \hat{\mathcal{A}}},$$

$$\hat{\mathcal{A}} = C^\infty(U)[[u^{i,1}, u^{i,2}, \dots; \theta_i^0, \theta_i^1, \theta_i^2, \dots]]$$

with $U \subset \mathbb{R}^n$.

- Poisson brackets $\{,\}_a^0$ as elements in $\hat{\mathcal{F}}^2$:

$$P_a = \frac{1}{2} \int \left(g_a^{ij} \theta_i^0 \theta_j^1 + \Gamma_{k,a}^{ij} u^{k,1} \theta_i \theta_j \right), \quad a = 1, 2.$$

- As before we associate to $P_a \in \hat{\mathcal{F}}^2$ a differential operator D_a on $\hat{\mathcal{A}}$, and define

$$D_\lambda = D_2 - \lambda D_1.$$

- Compute the cohomology

$$H(\hat{\mathcal{A}}[\lambda], D_\lambda).$$

Explicitly

$$D_\lambda = D(u^1 f^1, \dots, u^n f^n) - \lambda D(f^1, \dots, f^n)$$

where

$$D(f^1, \dots, f^n) = \sum_{s \geqslant 0} \partial^s \left(f^i \theta_i^1 \right) \frac{\partial}{\partial u^{i,s}}$$
$$+ \frac{1}{2} \sum_{s \geqslant 0} \partial^s \left(\partial_j f^i u^{j,1} \theta_i^0 + f^i \frac{\partial_i f^j}{f^j} u^{j,1} \theta_j^0 - f^j \frac{\partial_j f^i}{f^i} u^{i,1} \theta_j^0 \right) \frac{\partial}{\partial u^{i,s}}$$
$$+ \frac{1}{2} \sum_{s \geqslant 0} \partial^s \left(\partial_i f^j \theta_j^0 \theta_j^1 + f^j \frac{\partial_j f^i}{f^i} \theta_i^0 \theta_j^1 - f^j \frac{\partial_j f^i}{f^i} \theta_j^0 \theta_i^1 \right) \frac{\partial}{\partial \theta_i^s}$$
$$+ \frac{1}{2} \sum_{s \geqslant 0} \partial^s \left(f^j \frac{\partial_i f^l}{f^l} \frac{\partial_i f^l}{f^l} u^{l,1} \theta_l^0 \theta_j^0 - f^l \frac{\partial_i f^l}{f^l} \frac{\partial_l f^j}{f^j} u^{j,1} \theta_l^0 \theta_j^0 \right.$$
$$+ f^l \frac{\partial_l f^i}{f^i} \frac{\partial_i f^j}{f^j} u^{j,1} \theta_l^0 \theta_j^0 - \frac{f^l f^j}{f^i} \frac{\partial_l f^i}{f^i} \frac{\partial_j f^i}{f^i} u^{i,1} \theta_l^0 \theta_j^0$$
$$+ f^l \frac{\partial_l f^i}{f^i} \frac{\partial_l f^j}{f^j} u^{j,1} \theta_j^0 \theta_i^0 - f^j \frac{\partial_l f^i}{f^i} \frac{\partial_j f^l}{f^l} u^{l,1} \theta_j^0 \theta_i^0$$
$$\left. + f^l \frac{\partial_l f^i}{f^i} \frac{\partial_j f^l}{f^l} u^{j,1} \theta_l^0 \theta_i^0 + f^l \frac{\partial_l f^i}{f^i} \frac{\partial_l f^j}{f^j} u^{j,1} \theta_l^0 \theta_i^0 \right) \frac{\partial}{\partial \theta_i^s}.$$

Main result

Theorem

The cohomology $H^p_d(\hat{\mathcal{A}}[\lambda], D_\lambda)$ vanishes for all bi-degrees (p, d), unless

$$d = 0, \dots, n, \quad p = d, \dots, d + n, \qquad \textit{(case 1)}$$

$$d = 2, \dots, n + 2, \quad p = d, \dots, d + n - 1. \qquad \textit{(case 2)}$$

[C, Posthuma, Shadrin '15]

Simple observation

Let (C, d) be a cochain complex with a bounded decreasing filtration

$$\cdots \subset F^{i+1}C \subset F^{i}C \subset \cdots$$

and let (E_k, d_k) be the associated spectral sequence. Then

$$H^{\ell}(E_k, d_k) = 0 \quad \Longrightarrow \quad H^{\ell}(C, d) = 0.$$

First filtration

- The degree $\deg_u$ defined by

$$\deg_u u^{i,s} = 1, \quad i = 1, \ldots, n,\ s \geqslant 1$$

  and zero otherwise.
- The first filtration on $\hat{\mathcal{A}}[\lambda]$ is given by

$$F^r \hat{\mathcal{A}}^p[\lambda] = \{ f \in \hat{\mathcal{A}}^p[\lambda], p + \deg_u f \geqslant r \}.$$

- Denote Δ_k the homogeneous components of D_λ on $\hat{\mathcal{A}}[\lambda]$:

$$D_\lambda = \Delta_{-1} + \Delta_0 + \ldots, \qquad \deg_u \Delta_k = k.$$

- The page E_0 of the associated spectral sequence is:

$$(E_0, d_0) = (\hat{\mathcal{A}}[\lambda], \Delta_{-1}),$$

$$\Delta_{-1} = \sum_{s \geqslant 1} (u^i - \lambda) f^i \theta_i^{1+s} \frac{\partial}{\partial u^{i,s}}.$$

Proposition

The first page is given by

$$E_1 = H(\hat{\mathcal{A}}[\lambda], \Delta_{-1}) \cong \hat{\mathcal{C}}[\lambda] \oplus \bigoplus_{i=1}^{n} \mathrm{Im}\left(\hat{d}_i : \hat{\mathcal{C}}_i \to \hat{\mathcal{C}}_i\right)$$

where

$$\hat{\mathcal{C}} := C^\infty(U)[[\theta_1^0, \dots, \theta_n^0, \theta_1^1, \dots, \theta_n^1]],$$

$$\hat{\mathcal{C}}_i := \hat{\mathcal{C}}[[\{u^{i,s}, \theta_i^{s+1}, s \geqslant 1\}]],$$

$$\hat{d}_i = \sum_{s \geqslant 1} \theta_i^{s+1} \frac{\partial}{\partial u^{i,s}} \quad \textit{(de Rham)}.$$

Proof

To prove the Poincaré lemma

$$H(\hat{\mathcal{C}}_i, \hat{d}_i) = \hat{\mathcal{C}}$$

we can define an homotopy map, $i = 1, \ldots, n$, $s \geqslant 1$

$$h_{i,s} = \frac{\partial}{\partial \theta_i^{s+1}} \int du^{i,s},$$

with zero integration constant, then we have

$$h_{i,s}\hat{d}_i + \hat{d}_i h_{i,s} = 1 - \pi_{u^{i,s}} \pi_{\theta_i^{s+1}}.$$

Proof

Similarly, to prove the Proposition we use two homotopy maps. The first is

$$h_{i,s} := \sigma_i \frac{1}{u^i - \lambda} \frac{1}{f^i} \frac{\partial}{\partial \theta_i^{s+1}} \int du^{i,s}$$

which satisfies

$$h_{i,s}\Delta_{-1} + \Delta_{-1} h_{i,s} = 1 - p_{i,s},$$

$$\begin{aligned} p_{i,s} :=& \pi_{u^{i,s}} \pi_{\theta_i^{s+1}} + \Big(1 - \pi_{u^{i,s}} \pi_{\theta_i^{s+1}} + \\ & - \sum_{\substack{t \geqslant 1 \\ j}} \frac{f^j}{f^i} \frac{\partial}{\partial \theta_i^{s+1}} \theta_j^{t+1} \int du^{i,s} \frac{\partial}{\partial u^{j,t}} \Big) \pi_{\lambda - u^i}. \end{aligned}$$

It follows that we can kill the dependence on all the variables $u^{i,s}$, θ_i^{s+1} with $i = 1, \ldots, n$, $s \geqslant 1$, in the λ-dependent part of any cocyle.

Proof

The second homotopy map is, for $i \neq j$

$$h_{i,s;j,t} = \frac{1}{u^i - u^j} \frac{1}{f^i f^j} \frac{\partial}{\partial \theta_i^{s+1}} \frac{\partial}{\partial \theta_j^{t+1}} \int du^{i,s} \int du^{j,t}$$

and we have for $\Delta_{-1} = d'' - \lambda d'$

$$[h_{i,s;j,t}, d'' d'] = 1 - p_{i,s;j,t} + (...)d' + (...)d'',$$

where we did not specify the last two terms since they vanish when applied on elements in $\mathrm{Ker}\, d' \cap \mathrm{Ker}\, d''$, and

$$p_{i,s;j,t} := \pi_{u^{i,s}} \pi_{\theta_i^{s+1}} + \pi_{u^{j,t}} \pi_{\theta_j^{t+1}} - \pi_{u^{i,s}} \pi_{\theta_i^{s+1}} \pi_{u^{j,t}} \pi_{\theta_j^{t+1}}.$$

This allows to kill mixed terms in the λ independent part of a cocycle.

Second page

- The second page E_2 is given by

$$E_2 = H(E_1, d_1) = H\left(\hat{\mathcal{B}}, \Delta_0\right),$$

$$\hat{\mathcal{B}} := \hat{\mathcal{C}}[\lambda] \oplus \bigoplus_{i=1}^{n} \operatorname{Im}\left(\hat{d}_i : \hat{\mathcal{C}}_i \to \hat{\mathcal{C}}_i\right).$$

$$
\begin{aligned}
\Delta_0 &= (-\lambda + u^i) f^i \theta_i^1 \frac{\partial}{\partial u^i} \\
&+ \sum_{\substack{s=a+b\\ s,a\geqslant 1; b\geqslant 0}} (-\lambda + u^i)\binom{s}{b} \partial_j f^i u^{j,a} \theta_i^{1+b} \frac{\partial}{\partial u^{i,s}} + \sum_{\substack{s=a+b\\ s,a\geqslant 1; b\geqslant 0}} \binom{s}{b} f^i u^{i,a} \theta_i^{1+b} \frac{\partial}{\partial u^{i,s}} \\
&+ \frac{1}{2} \sum_{\substack{s=a+b\\ s\geqslant 1; a,b\geqslant 0}} (-\lambda + u^i)\binom{s}{b} \partial_j f^i u^{j,1+a} \theta_i^{b} \frac{\partial}{\partial u^{i,s}} + \frac{1}{2} \sum_{\substack{s=a+b\\ s\geqslant 1; a,b\geqslant 0}} \binom{s}{b} f^i u^{i,1+a} \theta_i^{b} \frac{\partial}{\partial u^{i,s}} \\
&+ \frac{1}{2} \sum_{\substack{s=a+b\\ s\geqslant 1; a,b\geqslant 0}} (-\lambda + u^i)\binom{s}{b} f^i \frac{\partial_i f^j}{f^j} u^{j,1+a} \theta_j^{b} \frac{\partial}{\partial u^{i,s}} + \frac{1}{2} \sum_{\substack{s=a+b\\ s\geqslant 1; a,b\geqslant 0}} \binom{s}{b} f^i u^{i,1+a} \theta_i^{b} \frac{\partial}{\partial u^{i,s}} \\
&- \frac{1}{2} \sum_{\substack{s=a+b\\ s\geqslant 1; a,b\geqslant 0}} (-\lambda + u^j)\binom{s}{b} f^j \frac{\partial_j f^i}{f^i} u^{i,1+a} \theta_j^{b} \frac{\partial}{\partial u^{i,s}} - \frac{1}{2} \sum_{\substack{s=a+b\\ s\geqslant 1; a,b\geqslant 0}} \binom{s}{b} f^i u^{i,1+a} \theta_i^{b} \frac{\partial}{\partial u^{i,s}} \\
&+ \frac{1}{2} \sum_{\substack{s=a+b\\ s,a,b\geqslant 0}} (-\lambda + u^j)\binom{s}{b} \partial_i f^j \theta_j^{a} \theta_j^{1+b} \frac{\partial}{\partial \theta_i^s} + \frac{1}{2} \sum_{\substack{s=a+b\\ s,a,b\geqslant 0}} \binom{s}{b} f^i \theta_i^{a} \theta_i^{1+b} \frac{\partial}{\partial \theta_i^s} \\
&+ \frac{1}{2} \sum_{\substack{s=a+b\\ s,a,b\geqslant 0}} (-\lambda + u^j)\binom{s}{b} f^j \frac{\partial_j f^i}{f^i} \theta_i^{a} \theta_j^{1+b} \frac{\partial}{\partial \theta_i^s} + \frac{1}{2} \sum_{\substack{s=a+b\\ s,a,b\geqslant 0}} \binom{s}{b} f^i \theta_i^{a} \theta_i^{1+b} \frac{\partial}{\partial \theta_i^s} \\
&- \frac{1}{2} \sum_{\substack{s=a+b\\ s,a,b\geqslant 0}} (-\lambda + u^j)\binom{s}{b} f^j \frac{\partial_j f^i}{f^i} \theta_j^{a} \theta_i^{1+b} \frac{\partial}{\partial \theta_i^s} - \frac{1}{2} \sum_{\substack{s=a+b\\ s,a,b\geqslant 0}} \binom{s}{b} f^i \theta_i^{a} \theta_i^{1+b} \frac{\partial}{\partial \theta_i^s} \,.
\end{aligned}
$$

Second filtration

- The second page E_2 is given by

$$E_2 = H\left(\hat{\mathcal{B}}, \Delta_0\right),$$

$$\hat{\mathcal{B}} := \hat{\mathcal{C}}[\lambda] \oplus \bigoplus_{i=1}^{n} \mathrm{Im}\left(\hat{d}_i : \hat{\mathcal{C}}_i \to \hat{\mathcal{C}}_i\right).$$

- To compute E_2 we introduce a filtration on $\hat{\mathcal{B}}$:

$$F^r\hat{\mathcal{B}} = \{f \in \hat{\mathcal{B}}, \deg_{\theta^1} f - \deg_\theta f \leqslant -r\}.$$

- The differential splits in $\Delta_0 = \Delta_{01} + \Delta_{00} + \Delta_{0,-1}$, where Δ_{01} is the part that increases the number of θ_i^1 by one.

Explicitly:

$$
\begin{aligned}
\Delta_{01} &= (-\lambda + u^i) f^i \theta_i^1 \frac{\partial}{\partial u^i} \\
&+ \sum_{s\geqslant 1} \frac{s+2}{2} f^i u^{i,s} \theta_i^1 \frac{\partial}{\partial u^{i,s}} \\
&- \frac{1}{2} \sum_{s\geqslant 1} (-\lambda + u^j) s f^j \frac{\partial_j f^i}{f^i} u^{i,s} \theta_j^1 \frac{\partial}{\partial u^{i,s}} \\
&- \frac{1}{2} (-\lambda + u^j) \partial_i f^j \theta_j^1 \theta_j^0 \frac{\partial}{\partial \theta_i^0} + \frac{1}{2} \sum_{s\geqslant 0} f^i (s-1) \theta_i^1 \theta_i^s \frac{\partial}{\partial \theta_i^s} \\
&- \frac{1}{2} \sum_{s\geqslant 0} (-\lambda + u^j) f^j \frac{\partial_j f^i}{f^i} (s+1) \theta_j^1 \theta_i^s \frac{\partial}{\partial \theta_i^s} \\
&+ \frac{1}{2} (-\lambda + u^j) f^j \frac{\partial_j f^i}{f^i} \theta_i^1 \theta_j^0 \frac{\partial}{\partial \theta_i^0}
\end{aligned}
$$

- The first page E_1' of the spectral sequence associated with the second filtration $F\hat{\mathcal{B}}$ is obtained by computing the cohomology:

$$E_1' = H(\hat{\mathcal{B}}, \Delta_{01}),$$

where

$$\hat{\mathcal{B}} := \hat{\mathcal{C}}[\lambda] \oplus \bigoplus_{i=1}^{n} \mathrm{Im}\left(\hat{d}_i : \hat{\mathcal{C}}_i \to \hat{\mathcal{C}}_i\right).$$

- The differential Δ_{01} leaves each summand invariant, hence we can compute the cohomology of each summand independently.

Vanishing of $H(\hat{\mathcal{C}}[\lambda], \Delta_{01})$

- The possible monomials in $\hat{\mathcal{C}}$ are

$$\theta^0_{i_1} \cdots \theta^0_{i_k} \theta^1_{j_1} \cdots \theta^1_{j_l}.$$

- Hence the cohomology $H^p_d(\hat{\mathcal{C}}[\lambda], \Delta_{01})$ vanishes, unless

$$d = 0, \ldots, n, \quad p = d, \ldots, d + n.$$

$\Rightarrow$ (case 1)

Third filtration

- Finally we need to compute, for fixed $i = 1, \ldots, n$:

$$H\left(\hat{\mathcal{B}}_i, \Delta_{01}\right),$$

where

$$\hat{\mathcal{B}}_i := \operatorname{Im}\left(\hat{d}_i : \hat{\mathcal{C}}_i \to \hat{\mathcal{C}}_i\right).$$

- We introduce a filtration on $\hat{\mathcal{B}}_i$ by:

$$F^r\hat{\mathcal{B}}_i = \{f \in \hat{\mathcal{B}}_i, \deg_{\theta_i^1} f - \deg_\theta f \leqslant -r\}$$

- Denote by $\theta_i^1 \mathcal{D}_i$ the part of Δ_{01} that increases the degree in θ_i^1.

$$\mathcal{D}_i := \sum_{s\geqslant 1} \frac{s+2}{2} f^i u^{i,s} \frac{\partial}{\partial u^{i,s}} + \sum_{s\geqslant 2} \frac{s-1}{2} f^i \theta_i^s \frac{\partial}{\partial \theta_i^s} - \frac{1}{2} f^i \theta_i^0 \frac{\partial}{\partial \theta_i^0} + \frac{1}{2} \sum_{j=1}^{n} (u^j - u^i) f^j \frac{\partial_j f^i}{f^i} \theta_j^0 \frac{\partial}{\partial \theta_i^0}$$

- The first page E_1'' of the spectral associated with the third filtration is obtained by computing the cohomology:

$$H(\hat{\mathcal{B}}_i, \theta_i^1 \mathcal{D}_i).$$

- Finally we can obtain the vanishing of the cohomology that implies the main theorem:

Proposition

The cohomology $H_d^p(\hat{\mathcal{B}}_i, \theta_i^1 \mathcal{D}_i)$ vanishes for all bi-degrees (p, d) unless

$$d = 2, \ldots, n+2, \quad q = d, \ldots, d+n-1.$$

$\Rightarrow$ (case 2)

Conclusions and open problems

For the semisimple N dimensional case

1. We show that most of the bihamiltonian cohomology in the general semisimple case vanishes, thus proving existence of deformations.
2. How to compute the remaining bihamiltonian cohomology groups, including the ones associated to the central invariants ?

Outline

D independent variables

Poisson bracket of Dubrovin-Novikov type with $x = (x^1, \dots, x^D)$, $u = (u^1, \dots, u^N)$:

$$\{u^i(x), u^j(y)\} = \sum_{\alpha=1}^{D} \big(g^{ij\alpha}(u(x))\partial_{x^\alpha}\delta(x-y) + \\ + b_k^{ij\alpha}(u(x))\partial_{x^\alpha}u^k(x)\delta(x-y)\big)$$

[Mokhov '88-'08, Ferapontov-Lorenzoni-Savoldi '15]

What can we say about the deformation theory of such Poisson brackets ?

- Differential polynomials

$$\mathcal{A} = C^\infty(U)[[\{\partial_{x^1}^{k_1}\cdots\partial_{x^D}^{k_D} u^i \text{ with } k_1,\dots,k_D \geqslant 0,\ (k_1,\dots,k_D)\neq 0\}]]$$

- Standard degree deg on $\mathcal{A}$:

$$\deg(\partial_{x^1}^{k_1}\cdots\partial_{x^D}^{k_D} u^i) = k_1+\cdots+k_D$$

- We consider dispersive deformations of multidimensional DN brackets of the form

$$\{u^i(x),u^j(y)\}^\epsilon = \{u^i(x),u^j(y)\}+ + \sum_{k>0}\epsilon^k \sum_{\substack{k_1,\dots,k_D\geqslant 0\\ k_1+\cdots+k_D\leqslant k+1}} A^{ij}_{k;k_1,\dots,k_D}(u(x))\partial_{x^1}^{k_1}\cdots\partial_{x^D}^{k_D}\delta(x-y)$$

where $A^{ij}_{k;k_1,\dots,k_D}\in\mathcal{A}$ and $\deg A^{ij}_{k;k_1,\dots,k_D} = k-k_1\cdots-k_D+1$.

- Miura-type transformations

$$v^i = u^i + \sum_{k\geqslant 1}\epsilon^k F^i_k$$

where $F^i_k\in\mathcal{A}$ and $\deg F^i_k = k$.

We consider the the scalar $N = 1$ case

$$\{u(x), u(y)\} = g(u(x))c^\alpha \frac{\partial}{\partial x^\alpha}\delta(x-y) + \frac{1}{2}g'(u(x))c^\alpha \frac{\partial u}{\partial x^\alpha}(x)\delta(x-y)$$

which in flat coordinates reduces to

$$\{u(x), u(y)\} = \sum_{\alpha=1}^{D} c^\alpha \frac{\partial}{\partial x^\alpha}\delta(x-y).$$

- Deformation theory is governed by Poisson cohomology groups $H^p(\hat{\mathcal{F}})$ associated with the Poisson bracket $\{u(x), u(y)\}$.
- Infinitesimal deformations $\longrightarrow H^2(\hat{\mathcal{F}})$
- Obstructions $\longrightarrow H^3(\hat{\mathcal{F}})$

Our main result

Define the ring of polynomials in the anticommuting variables θ^S

$$\Theta = \mathbb{R}[\{\theta^{(s_1,\dots,s_{D-1})}, s_i \geqslant 0\}]$$

and the auxiliary space:

$$H(D) = \frac{\Theta}{\partial_{x_1}\Theta + \cdots + \partial_{x_{D-1}}\Theta}.$$

Theorem

The Poisson cohomology of the Poisson bracket in bi-degree (p, d) is isomorphic to

$$H^p_d(D) \oplus H^{p+1}_d(D).$$

[C, Casati, Shadrin '15]

$D = 2$ independent variables

For $D = 1$ we recover scalar case of Getzler's theorem.

For $D = 2$ we have a closed formula for the dimension of $H^p_d(2)$:

d	0	1	2	3	4	5	6	7	8
$\dim H^2_d(\hat{\mathcal{F}})$	0	1	0	2	0	2	1	2	1
$\dim H^3_d(\hat{\mathcal{F}})$	0	0	0	1	0	1	2	1	2

Higher D

For $D \geqslant 2$ we expect the Poisson cohomology in $p = 2, 3$ to be highly non-trivial.

$D = 3$:

d	0	1	2	3	4	5	6	7	8
$\dim H^2_d(\hat{\mathcal{F}})$	0	2	1	8	3	16	13	26	26
$\dim H^3_d(\hat{\mathcal{F}})$	0	0	1	4	6	14	29	36	72

$D = 4$:

d	0	1	2	3	4	5	6
$\dim H^2_d(\hat{\mathcal{F}})$	0	3	3	20	15	66	73
$\dim H^3_d(\hat{\mathcal{F}})$	0	0	3	11	30	75	183

Remarks

- The situation in $D > 1$ looks much more complicated:
  - No Getzler's theorem on triviality
  - Many infinitesimal deformations, also non-homogeneous
  - A priori non-vanishing obstructions
- Deformation theory is non-empty: we find examples of nontrivial deformations of degree 2 for each $D > 2$

Remarks on the proof

1. The Poisson cohomology groups are invariant (up to isomorphism) under linear changes of the independent variables.
2. We can put the Poisson bracket in the special form

$$\{u(x), u(y)\} = \partial_{x^D} \delta(x - y).$$

3. We show that the following sequences are exact:

$$\begin{array}{ccccccccc}
0 & \to & \hat{\mathcal{A}}/\mathbb{R} & \xrightarrow{\partial_{x^1}} & \hat{\mathcal{A}} & \xrightarrow{\int dx^1} & \hat{\mathcal{F}}_1 & \to & 0 \\
0 & \to & \hat{\mathcal{F}}_1/\mathbb{R} & \xrightarrow{\partial_{x^2}} & \hat{\mathcal{F}}_1 & \xrightarrow{\int dx^2} & \hat{\mathcal{F}}_2 & \to & 0 \\
0 & \to & \hat{\mathcal{F}}_2/\mathbb{R} & \xrightarrow{\partial_{x^3}} & \hat{\mathcal{F}}_2 & \xrightarrow{\int dx^3} & \hat{\mathcal{F}}_3 & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \to & \hat{\mathcal{F}}_{D-1}/\mathbb{R} & \xrightarrow{\partial_{x^D}} & \hat{\mathcal{F}}_{D-1} & \xrightarrow{\int dx^D} & \hat{\mathcal{F}}_D & \to & 0
\end{array}$$

where

$$\hat{\mathcal{F}}_i = \frac{\hat{\mathcal{A}}}{\partial_{x^1}\hat{\mathcal{A}} + \cdots + \partial_{x^i}\hat{\mathcal{A}}}.$$

4. The differential associated to the Poisson bracket in special form

$$\Delta = \sum_S \theta^{S+\xi_D} \frac{\partial}{\partial u^S},$$

commutes with all the maps, therefore induces exact sequences of complexes.

5. The corresponding long exact sequences in cohomology allow us to compute inductively:

$$H(\hat{\mathcal{F}}_i) = \frac{\Theta}{\partial_{x^1}\Theta + \cdots + \partial_{x^i}\Theta},$$

for $i = 1, \ldots, D-1$.

6. The long exact sequence associated to the last line allows us to conclude.

Conclusion

For the D independent variables case

- $D > 1$ deformation theory highly nontrivial (unlike $D = 1$).
- Can we classify nontrivial (homogeneus) deformations ?