Wild character varieties, meromorphic Hitchin systems and Dynkin diagrams

P. Boalch, CNRS Orsay

The Lax project

Try to classify integrable systems with nice properties

- finite dimensional complex algebraic completely integrable Hamiltonian system $(M, \mathcal{X})$
- admits a good Lax representation (any genus)

upto isomorphism (isogeny, deformation, ...)

Then look at different representations of each one

The Lax project

E.g. Look at isospectral deformations of rational matrix

$$A(z)$$

$$\Sigma = \det(A(z)-\lambda) \rightsquigarrow \text{spectral curve}$$

$$\mathcal{M}^* = \{ A \mid \text{orbits of polar parts fixed} \} / G \quad \text{symplectic}$$

- lots of examples of such integrable systems

  Jacobi, Garnier, ...

The Lax project

Hitchin systems $\left(\text{fix } G = GL_n(\mathbb{C}),\ \Sigma \text{ compact Riemann surface}\right)$

$$T^*\mathrm{Bun}_G = \left\{ (V, \Phi) \mid V \text{ stable}, \Phi \in H^0(\mathrm{End}\, V \otimes \Omega^1) \right\} / \text{iso.}$$

$$\cap$$

$$\mathcal{M}_{Dol} = \left\{ (V, \Phi) \mid \text{stable pair} \right\} / \text{iso.}$$

(Higgs bundles)

$$\downarrow \chi$$

$$\mathbb{H}$$

The Lax project

Hitchin systems (fix $G = GL_n(\mathbb{C})$, Σ compact Riemann surface)

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$$\begin{array}{c} \cap \\ \mathcal{M}_{Dol} \end{array} = \{(V,\Phi) \mid \text{stable pair}\}/\text{iso.}$$

(Higgs bundles)

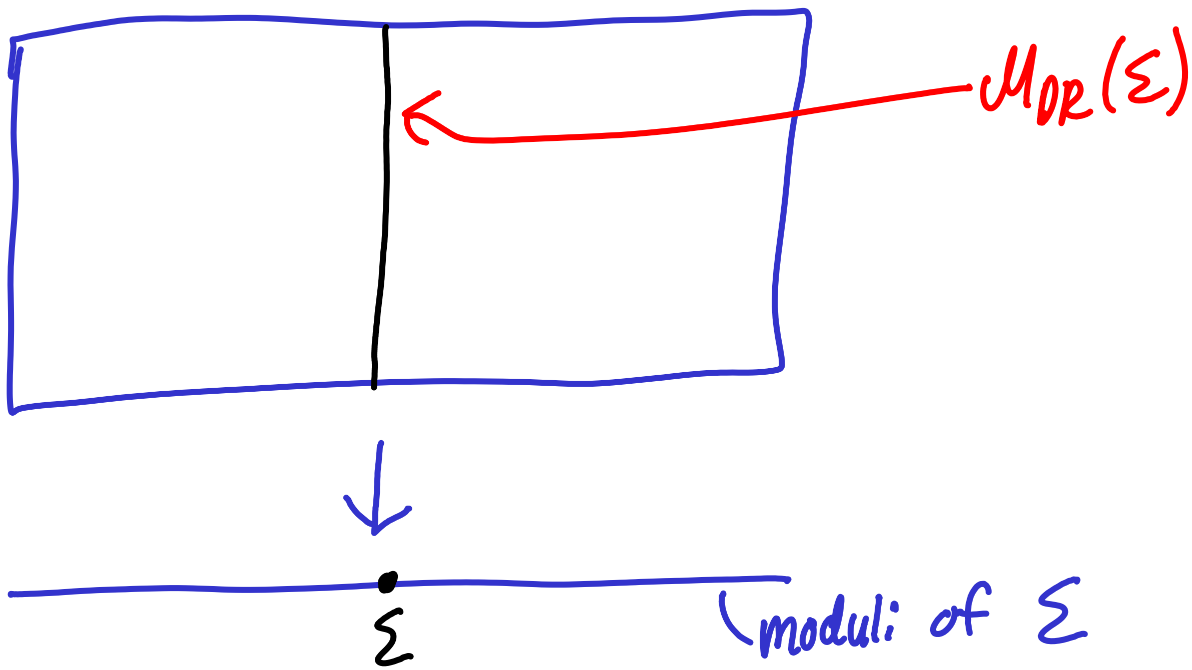
$$\mathcal{M}_{Dol} \xrightarrow{\chi} \mathbb{H}$$

2. Hyperkahler: $\mathcal{M}_{Dol} \overset{\text{nonabelian Hodge}}{\cong} \mathcal{M}_{DR} \overset{RH}{\cong} \mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G$

Higgs — Connections — character variety

The Lax project

Vary Σ $\rightsquigarrow$ isomonodromy connection on spaces of connections



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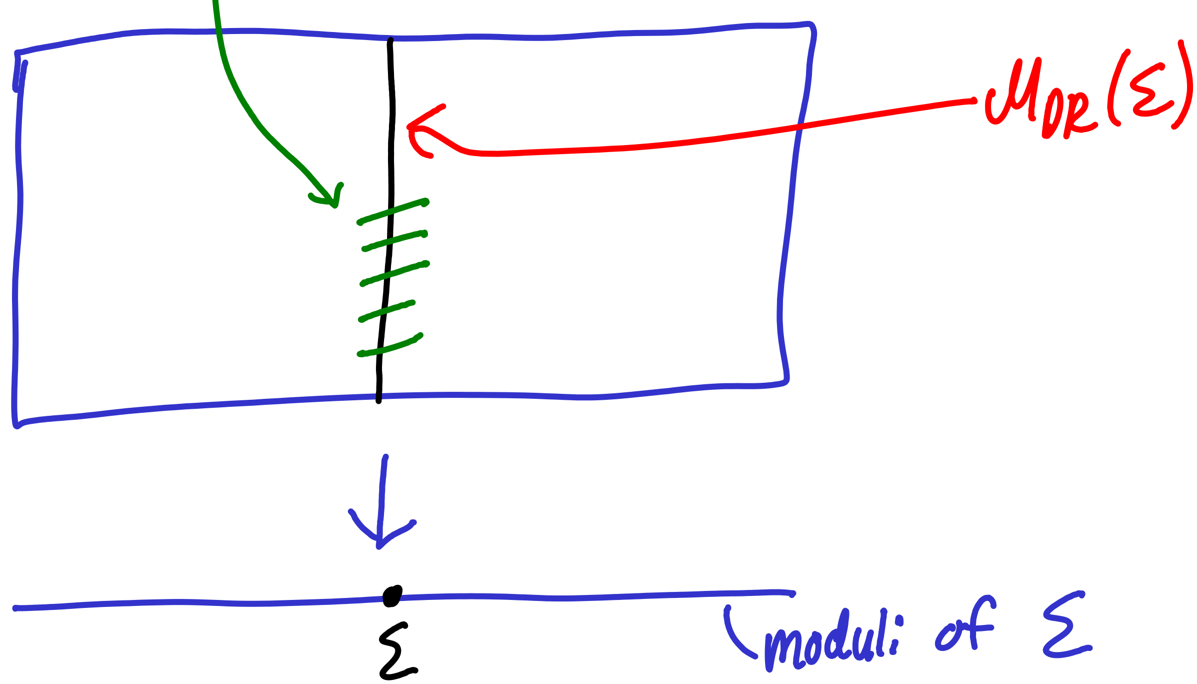

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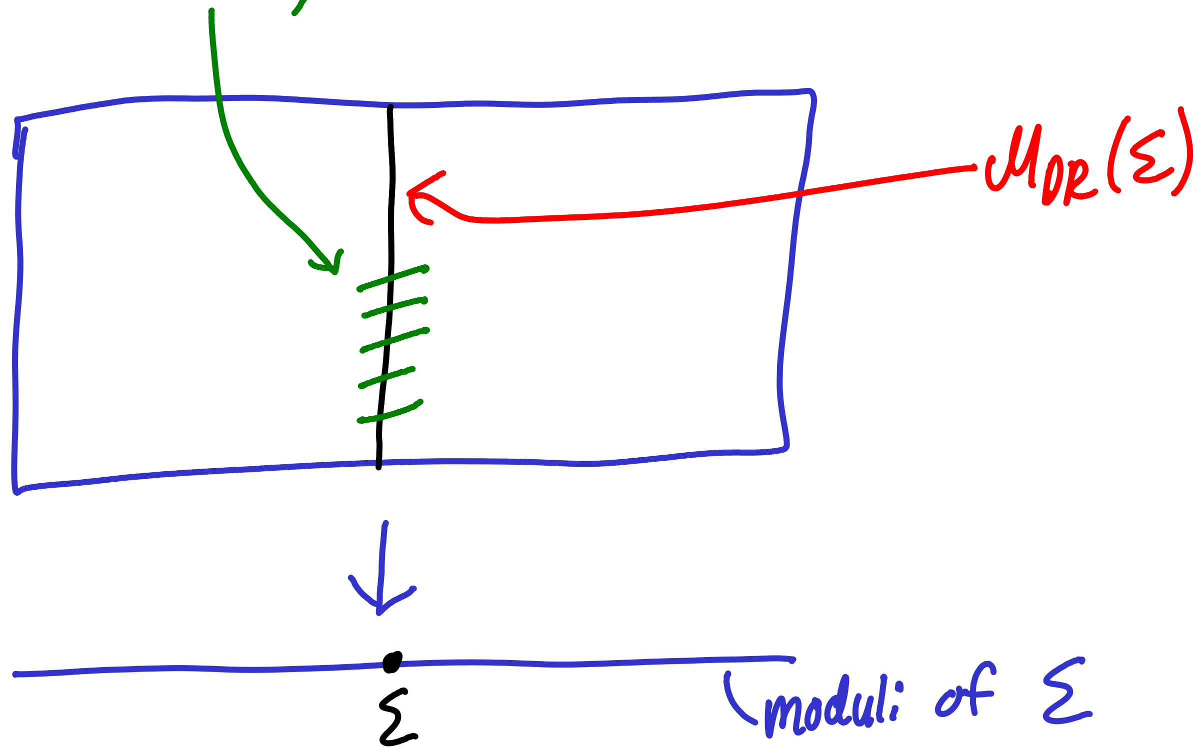

- Classify both ACIHS & isomonodromy systems at same time

(i.e. classify hyperkahler manifolds with such extra structure)

The Lax project

Back to rational matrices:

- $A(z)\,dz$ is a meromorphic Higgs field (V trivial)

- $d - A(z)\,dz$ is a meromorphic connection (V trivial)

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- Nitsure, Bottacin, Markman ~'95 ACIHS in Poisson sense
- PB. '99 Symplectic forms on $\mathcal{M}_{DR} \cong \mathcal{M}_B$ (mero. Atiyah-Bott / Goldman)
- Biquard-B. '01 Hyperkahler structure
- Algebraic approach to symplectic forms: Woodhouse '00, Krichever '01, B. '02, 09, 11, B.-Yamakawa '15

The Lax project

$$\underset{\text{mero. Higgs}}{\mathcal{M}_{Dol}} \overset{\substack{\text{wild}\\ \text{nonabelian Hodge}}}{\cong} \underset{\text{mero. Connections}}{\mathcal{M}_{DR}} \overset{\text{RHB}}{\cong} \underset{\text{wild character variety}}{\mathcal{M}_B} = \{\text{monodromy \& Stokes data}\}$$

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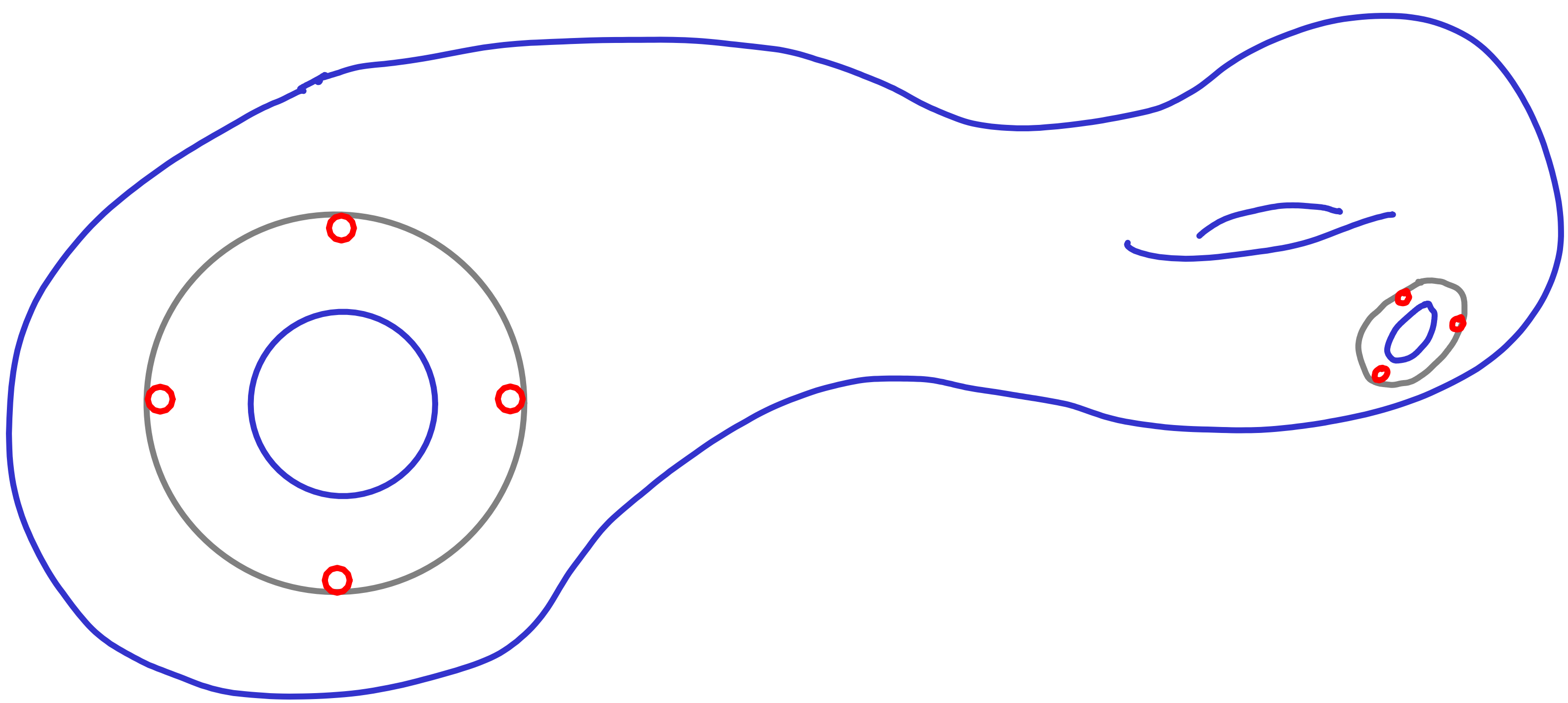

Example

Φ	Higgs Integrable system $\mathcal{M}_{Dol}$	Connections Isomonodromy system $\mathcal{M}_{DR}$	Monodromy/Stokes $\mathcal{M}_B$
$(A_1 + A_2 z)\frac{dz}{z}$	Manakov	Dual Schlesinger	G^*

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Duality:

$$A + P(z-B)^{-1}Q \longleftrightarrow B + Q(z-A)^{-1}P \qquad \text{(up to signs)}$$

AHH, Harnad
Fourier-Laplace

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$\mathfrak{sl}_3$

4 poles $\mathfrak{gl}_2$

Painlevé 6

$\mathcal{M}_B \cong$ Fricke-Klein-Vogt surface

$$xyz + x^2 + y^2 + z^2 + ax + by + cz = d$$

(Hyperkähler four manifold)

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$$xyz + x^2+y^2+z^2 + ax+by+cz = d$$

$$\cong \quad \mathcal{C}/\!/T, \quad \mathcal{C} \subset SL_3^*, \quad \dim \; 6-2\cdot 2 = 2$$

$$\cong \quad \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_4 /\!/ GL_2, \quad \dim \; 4\cdot 2 - 2\cdot 3 = 2$$

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$$\cong \quad \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C}_\infty /\!/ G_2 \qquad \dim\ 3\cdot 6 + 12 - 2\cdot 14 = 2 \quad (a=b=c)$$

G_2 representation of Painlevé VI (B.–Paluba, JAG '16)

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$\mathcal{M}_B \cong$ Flaschka-Newell surface

$$xyz + x+y+z = b-b^{-1} \qquad b\in\mathbb{C}^*$$

(New hyperkahler 4manifold, via Biquard-B. '01)

Example

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⋮			

Dynkin diagrams

Okamoto ('80s):

P_6 has D_4 affine Weyl group symmetry

P_2 – A_1 ————————————

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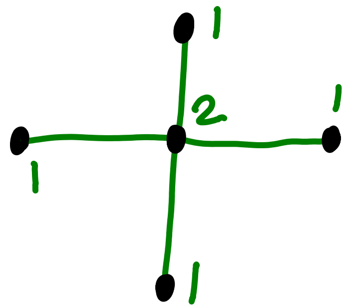

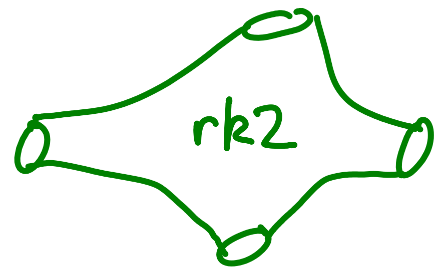

$\mathcal{M}^* \cong D_4$ ALE space / quiver variety $\hookrightarrow \mathcal{M}_{DR} \cong \mathcal{M}_B$

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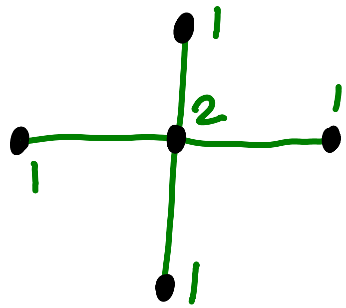

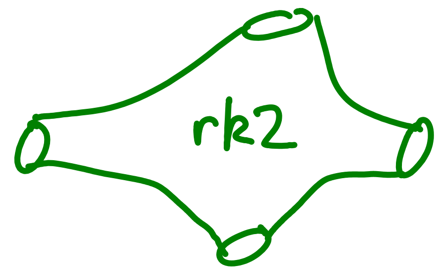

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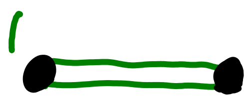
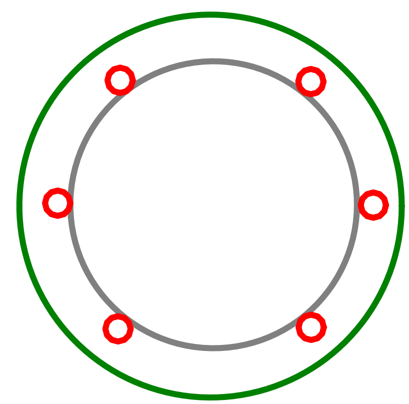
P_2





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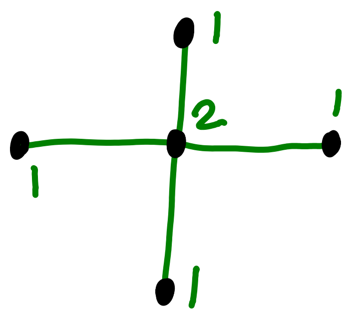

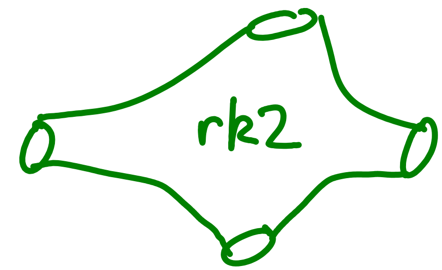

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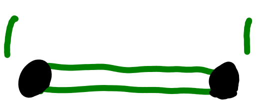

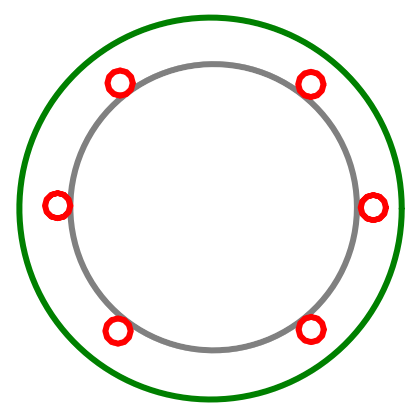

$\mathcal{M}^* \cong A_1$ ALE space / Eguchi-Hanson $\hookrightarrow \mathcal{M}_{DR} \cong \mathcal{M}_B$

(Ex.3, 0706.2634)

Spaces from graphs/quivers

$\Gamma = \circ\!\!-\!\!\!-\!\!\!-\!\!\circ \qquad I = \{\text{nodes}(\Gamma)\}$

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$$V = V_1 \oplus V_2 \qquad\qquad (I \text{ graded complex vector space})$$

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$$\Gamma = \begin{tikzcd} V_1 \arrow[r, dash] \arrow[r, bend left, "a"] & V_2 \arrow[l, bend left, "b"] \end{tikzcd} \qquad I = \{\text{nodes}(\Gamma)\}$$

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with moment map $\mu(a,b) = (ab, -ba)$

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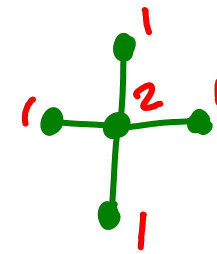
Additive/Nakajima quiver variety:

$$\mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda}\, H = \mu^{-1}(\lambda)/H \qquad (\lambda \in \mathbb{C}^I \subset \mathrm{Lie}(H)^*)$$

Spaces from graphs/quivers

Kronheimer '89: If Γ an affine ADE Dynkin graph,



$\dim V_i \sim$ minimal null root then

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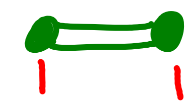

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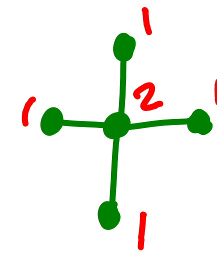

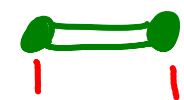

Multiplicative version

$$\Gamma = \begin{tikzcd} \overset{V_1}{\circ} \arrow[r, bend left, "a"] & \overset{V_2}{\circ} \arrow[l, bend left, "b"] \end{tikzcd}$$

$$\mathrm{Rep}^*(\Gamma, V) = \{ (a,b) \mid 1+ab \text{ invertible} \}$$

$$\cap$$

$$\mathrm{Rep}(\Gamma, V)$$

"invertible representations"

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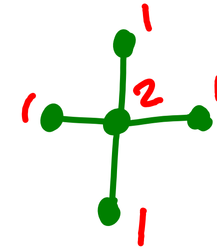

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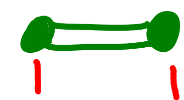

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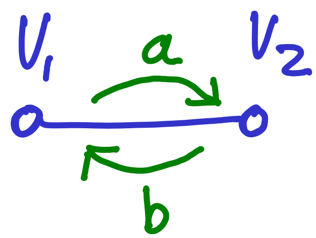
$\Gamma =$ 

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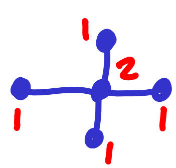
$$\cap$$

$$\mathrm{Rep}(\Gamma, V)$$

"invertible representations"

Thm (VandenBergh '04) $\mathrm{Rep}^*(\Gamma, V)$ is a "multiplicative" (or "quasi") Hamiltonian H-space with group valued moment map $\mu(a,b) = \left(1+ab, (1+ba)^{-1}\right) \in H$

E.g. Mult. Quiver Var. () $\cong \{ xyz + x^2 + y^2 + z^2 = ax + by + cz + d \}$

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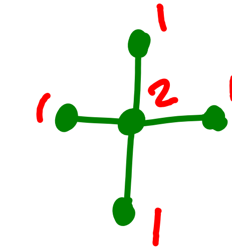
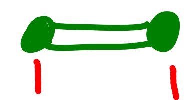


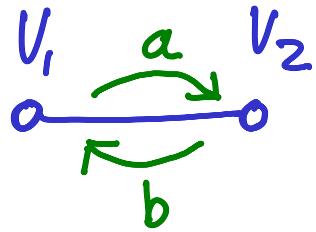
$\dim V_i \sim$ minimal null root then

$\mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} H$ is $\mathbb{C}$ dim^{n} 2



Multiplicative version

$\mathcal{B}(V_1, V_2) :=$

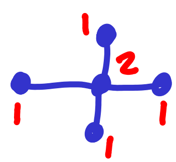
$\Gamma =$ 

$\mathrm{Rep}^*(\Gamma, V) = \{ (a,b) \mid 1+ab \text{ invertible} \}$

$\cap$ "invertible representations"

$\mathrm{Rep}(\Gamma, V)$

Thm (Van den Bergh '04) $\mathrm{Rep}^*(\Gamma, V)$ is a "multiplicative" (or "quasi") Hamiltonian H-space with group valued moment map $\mu(a,b) = (1+ab, (1+ba)^{-1}) \in H$

E.g. Mult. Quiver Var. () $\cong \{ xyz + x^2 + y^2 + z^2 = ax + by + cz + d \}$

Qn Suppose $\Gamma =$  or  etc

then what is $\operatorname{Rep}^*(\Gamma, V)$?

Multiplicative version

$$\Gamma = \begin{tikzcd} V_1 \arrow[r, bend left, "a"] & V_2 \arrow[l, bend left, "b"] \end{tikzcd}$$

$\mathcal{B}(V_1, V_2) :=$

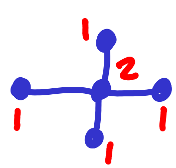
$$\operatorname{Rep}^*(\Gamma, V) = \{ (a,b) \mid 1+ab \text{ invertible} \}$$

$$\cap$$

$$\operatorname{Rep}(\Gamma, V)$$

"invertible representations"

Thm (Van den Bergh '04) $\operatorname{Rep}^*(\Gamma, V)$ is a "multiplicative" (or "quasi") Hamiltonian H-space with group valued moment map $\mu(a,b) = \left(1+ab, (1+ba)^{-1}\right) \in H$

E.g. Mult. Quiver Var. () $\cong \{ xyz + x^2 + y^2 + z^2 = ax + by + cz + d \}$

SPECIMEN ALGORITHMI SINGVLARIS.

Auctore

L. EVLERO.

1.

Consideratio fractionum continuarum, quarum vsum vberrimum per totam Analysin iam aliquoties ostendi, deduxit me ad quantitates certo quodam modo ex indicibus formatas, quarum natura ita est comparata, vt singularem algorithmum requirat. Cum igitur summa Analyseos inuenta maximam partem algorithmo ad certas quasdam quantitates accommodato

6. Haec ergo teneatur definitio ſignorum ( ), inter quae indices ordine a ſiniſtra ad dextram ſcribere conſtitui; atque indices hoc modo clauſulis incluſi inpoſterum denotabunt numerum ex iſtis indicibus formatum. Ita a ſimpliciſſimis caſibus inchoando, habebimus:

$$(a) = a$$
$$(a,b) = ab + 1$$
$$(a,b,c) = abc + c + a$$
$$(a,b,c,d) = abcd + cd + ad + ab + 1$$
$$(a,b,c,d,e) = abcde + cde + ade + abe + abc + e + c + a$$

etc.

ex

"Euler's continuant polynomials"

G. G. Stokes 1857

VI. *On the Discontinuity of Arbitrary Constants which appear in Divergent Developments.* *By* G. G. STOKES, M.A., D.C.L., Sec. R.S., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

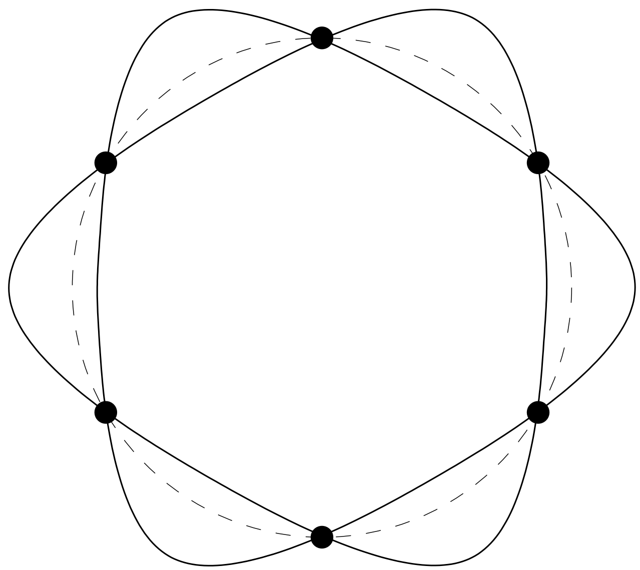
[Read *May* 11, 1857.]

IN a paper "On the Numerical Calculation of a class of Definite Integrals and Infinite Series," printed in the ninth volume of the *Transactions* of this Society, I succeeded in developing the integral $\int_0^\infty \cos\frac{\pi}{2}(w^3 - mw)\,dw$ in a form which admits of extremely easy numerical calculation when m is large, whether positive or negative, or even moderately large. The method there followed is of very general application to a class of functions which frequently occur in physical problems. Some other examples of its use are given in the same paper; and I was enabled by the application of it to solve the problem of the motion of the fluid surrounding a pendulum of the form of a long cylinder, when the internal friction of the fluid is taken into account *.

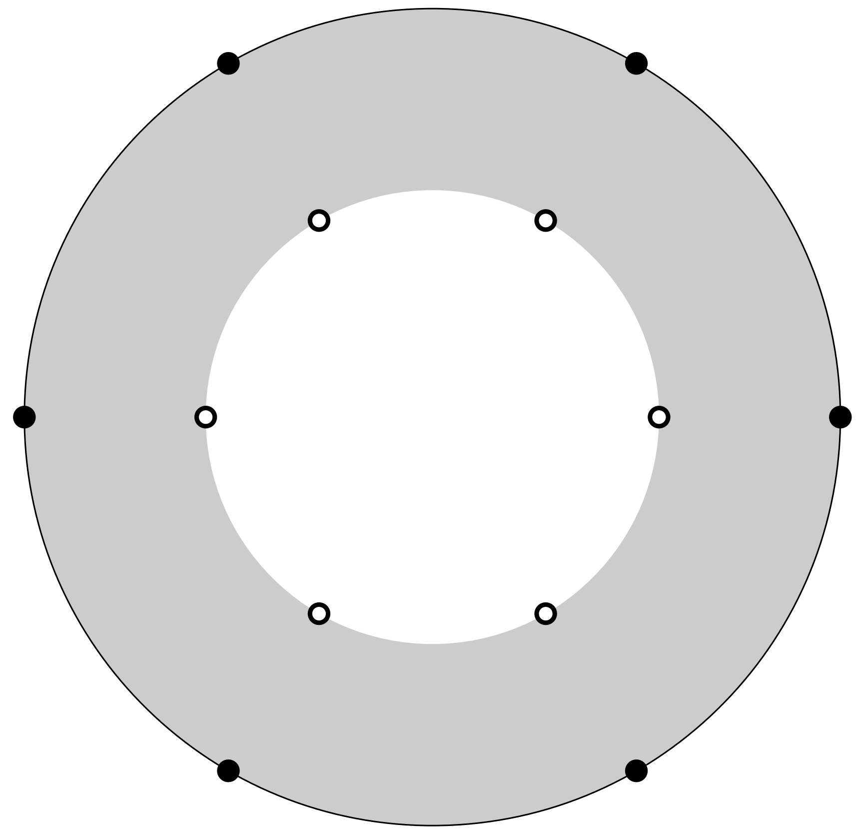
These functions admit of expansion, according to ascending powers of the variables, in series which are always convergent, and which may be regarded as defining the functions for all values of the variable real or imaginary, though the actual numerical calculation would involve a labour increasing indefinitely with the magnitude of the variable. They satisfy certain linear differential equations, which indeed frequently are what present themselves in the first instance, the series, multiplied by arbitrary constants, being merely their integrals. In my former paper, to which the present may be regarded as a supplement, I have employed these equations to obtain integrals in the form of descending series multiplied by exponentials. These integrals, when once the arbitrary constants are determined, are exceedingly convenient

Stokes structures

(Sibuya 1975, Deligne 1978, Malgrange 1980 ...)



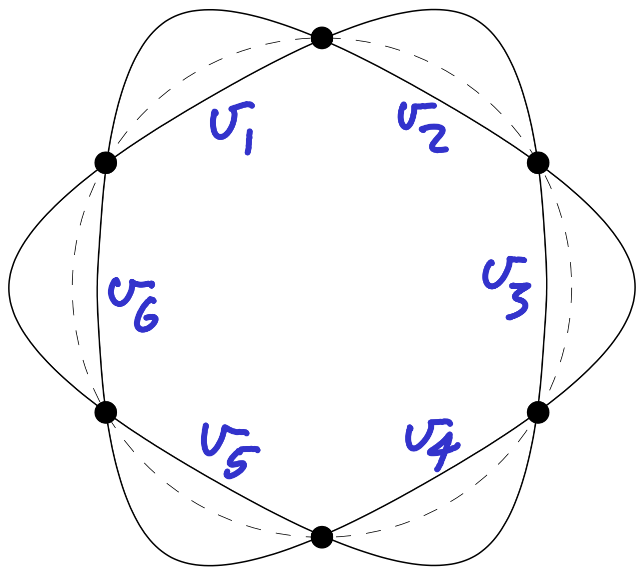
Stokes diagram with Stokes directions



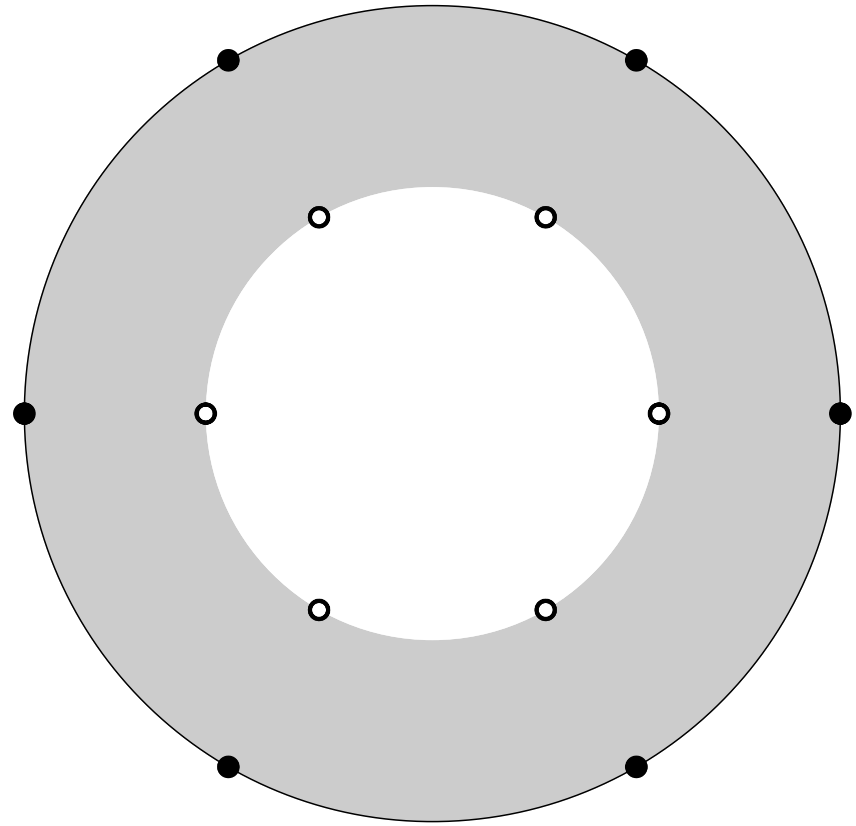
Halo at ∞ with singular directions

Stokes structures

(Sibuya 1975, Deligne 1978, Malgrange 1980 ...)



Stokes diagram with Stokes directions

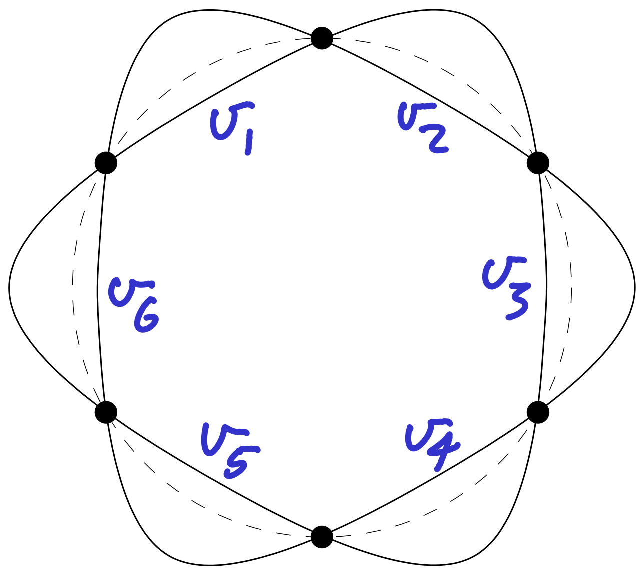


Halo at ∞ with singular directions

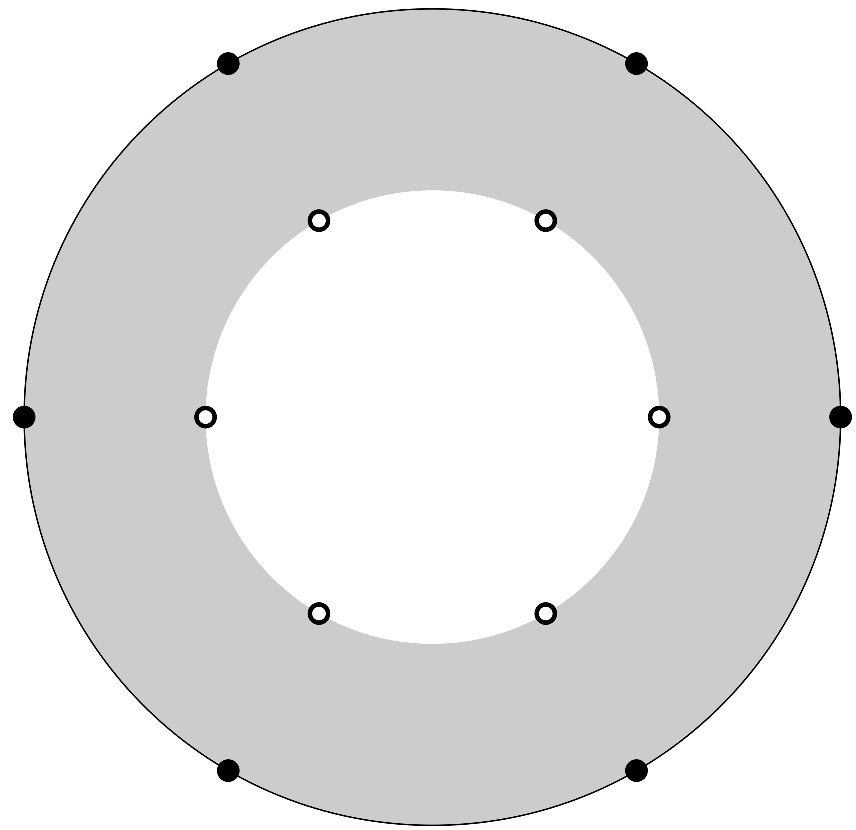
Subdominant solutions $\quad v_i \nparallel v_{i+1}$

Stokes structures

(Sibuya 1975, Deligne 1978, Malgrange 1980 ...)



Stokes diagram with Stokes directions



Halo at ∞ with singular directions

Subdominant solutions $v_i \nparallel v_{i+1}$

$$\mathcal{M}_B \cong \{ xyz + x + y + z = b - b^{-1} \}$$

$$\cong \left\{ (p_1, \dots, p_6) \in (\mathbb{P}^1)^6 \;\middle|\; \begin{array}{c} p_i \neq p_{i+1} \bmod 6 \\ \dfrac{(p_1-p_2)(p_3-p_4)(p_5-p_6)}{(p_2-p_3)(p_4-p_5)(p_6-p_1)} = b^2 \end{array} \right\} \Big/ PSL_2(\mathbb{C})$$

Cartoon

∞-d Ham^n geometry

e.g. connections on C^∞ bundles / Riemann surfaces

$\cup$ $\qquad\qquad\qquad\qquad$ $/\!/\mathcal{G}_1$

Hamiltonian geometry

$\mathcal{O} \subset \mathfrak{g}^*, \quad T^*G$

quasi-Hamiltonian geometry

$\mathcal{C} \subset G, \quad D = G \times G$

$\mu^{-1}(0)/G$

mult. sp. quotient $\quad \mu^{-1}(1)/G$

Additive symplectic geometry

$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

Multiplicative symplectic geometry

Betti spaces, character varieties

Cartoon

∞-d Ham^n geometry

e.g. connections on C^∞ bundles / Riemann surfaces

$\cup$ $\qquad\qquad$ $/\!/\mathcal{G}_1$

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Multiplicative symplectic geometry

Betti spaces, character varieties

$$\left\{d-\sum\frac{A_i}{z-a_i}dz \;\middle|\; A_i\in\mathcal{O}_i,\ \sum A_i=0\right\}/G$$

Cartoon

∞-d Ham^n geometry
e.g. connections on C^∞ bundles / Riemann surfaces

$\cup$

$/\!/ \mathcal{G}_1$

Hamiltonian geometry
$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

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mult. sp. quotient $\mu^{-1}(1)/G$

Additive symplectic geometry
$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

$\mathcal{M}^*$

$\xrightarrow{RH}$

Multiplicative symplectic geometry
Betti spaces, character varieties

$\mathcal{M}_B$

Cartoon

∞-d Ham^n geometry

e.g. connections on C^∞ bundles / Riemann surfaces

$\cup$ $\qquad\qquad\qquad$ $\downarrow /\!/ \mathcal{G}_1$

Hamiltonian geometry

$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

$\downarrow \mu^{-1}(0)/G$

Additive symplectic geometry

$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

$\mathcal{M}^*$

quasi-Hamiltonian geometry

$\mathcal{C} \subset G$, $D = G \times G$

mult. sp. quotient $\downarrow \mu^{-1}(1)/G$

Multiplicative symplectic geometry

Betti spaces, wild character varieties

$\mathcal{M}_B$

$\xRightarrow{\text{RHB}}$

Wild Character Varieties

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

symplectic variety

Σ compact Riemann Surface $\Longrightarrow$ $\mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

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$$\Big|\Big|\!\wr \ RH$$

$\mathcal{M}_{DR} = \{\text{Alg. connections on } G\text{-bundles on } \Sigma\}/\text{isom}$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ $\Longrightarrow$

symplectic variety

$$\mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

$$\Big|\Big|\Big| \int \text{RH}$$

$$\mathcal{M}_{DR} = \{\text{Alg. connections on } G\text{-bundles on } \Sigma\}/\text{isom}$$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

Poisson variety

$\Longrightarrow \quad \mathcal{M}_B^{tame} = \mathrm{Hom}(\pi_1(\Sigma^\circ), G)/G$

$\wr\!\wr$ RH

$\mathcal{M}_{DR} = \{$ Alg. connections on G-bundles on Σ° with reg. sing.s $\} /$ isom

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Poisson scheme (∞-type)

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \ldots, a_m)$ $\Longrightarrow$ $\mathcal{M}_B$

$$\mathcal{M}_B \overset{RHB}{\cong} \mathcal{M}_{DR}$$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

$$\mathcal{M}_{DR} = \{\text{Alg. connections on } G\text{-bundles on } \Sigma^\circ\} / \text{isom}$$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Poisson variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \dots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

$\cong$ RHB

$\mathcal{M}_{DR} = \left\{ \begin{array}{l} \text{Alg. connections on } G\text{-bundles on } \Sigma^0 \\ \text{with irreg. types } \underset{\sim}{Q} \end{array} \right\} \Big/ \text{isom}$

$\Sigma^0 = \Sigma \setminus \underset{\sim}{a}$

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

Poisson variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \dots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

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$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

$Q_i \in \underbrace{\mathfrak{t}_i}_{\text{Cartan subalg.}} \subset \mathfrak{g}((z_i))$

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

Poisson variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \dots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

$$\mathcal{M}_B \overset{RHB}{\cong}$$

$$\mathcal{M}_{DR} = \{\text{Alg. connections on } G\text{-bundles on } \Sigma^\circ \text{ with irreg. types } \underset{\sim}{Q}\} / \text{isom}$$

$$\nabla \cong dQ_i + \Lambda_i \frac{dz_i}{z_i} + \text{holom.}$$

Cartan subalg.

e.g. $Q_i \in \mathfrak{t}((z_i)) \subset \mathfrak{g}((z_i))$ $\qquad \mathfrak{t} \subset \mathfrak{g}$

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

Wild Riemann surface $(\Sigma, \underset{\sim}{a}, \underset{\sim}{Q}) \Rightarrow$ wild character variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \dots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

$\|\wr$ RHB

$\mathcal{M}_{DR} = \{$ Alg. connections on G-bundles on Σ° with irreg. types $\underset{\sim}{Q}\} / \text{isom}$

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Cartan subalg.

e.g. $\quad Q_i \in \mathfrak{t}((z_i)) \subset \mathfrak{g}((z_i)) \qquad \mathfrak{t} \subset \mathfrak{g}$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\qquad G = GL_2(\mathbb{C})$

$Q = A/z^k$, $A = \begin{pmatrix} a & \\ & b \end{pmatrix}$ $a \neq b$

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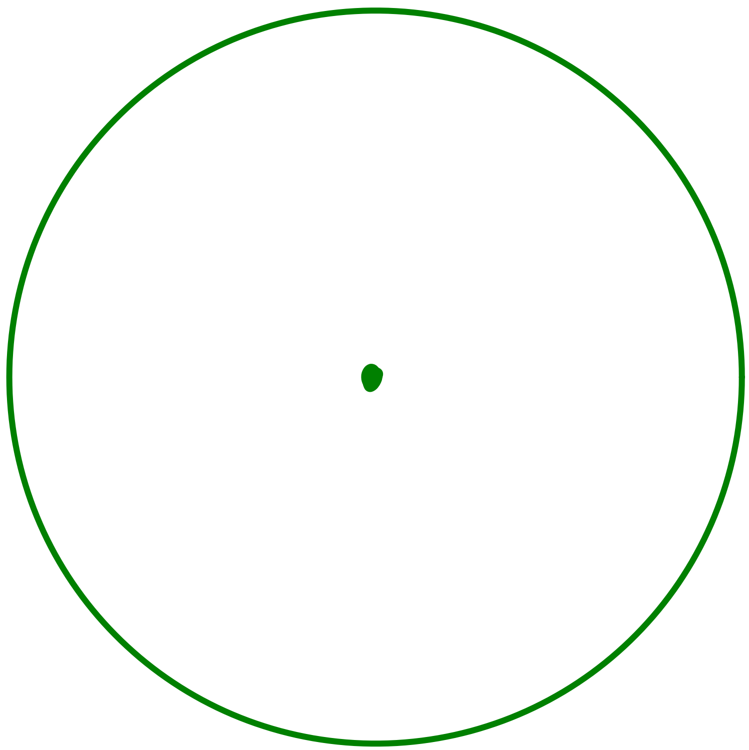

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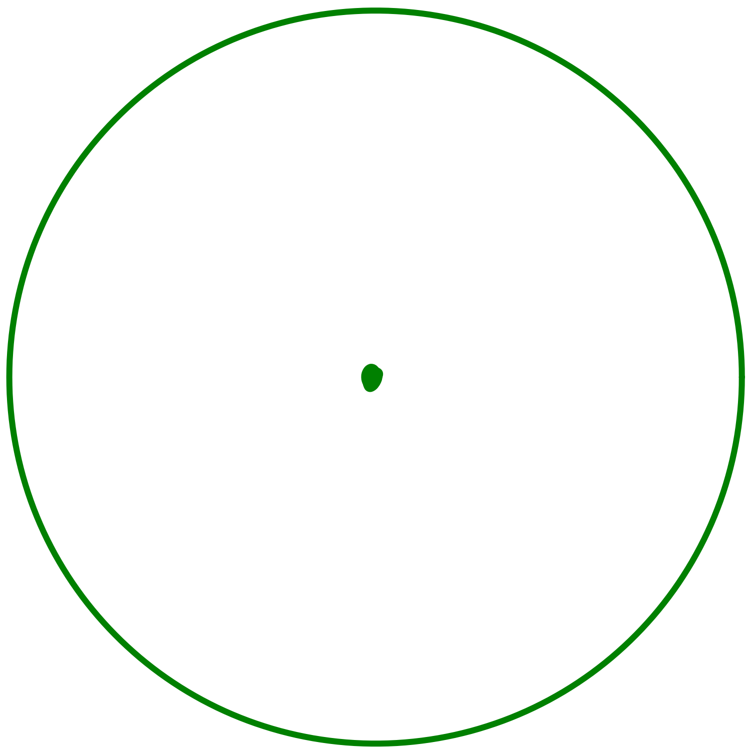

$Q \Rightarrow$

- centraliser group $H = T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset G$
  $C_G(Q)$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

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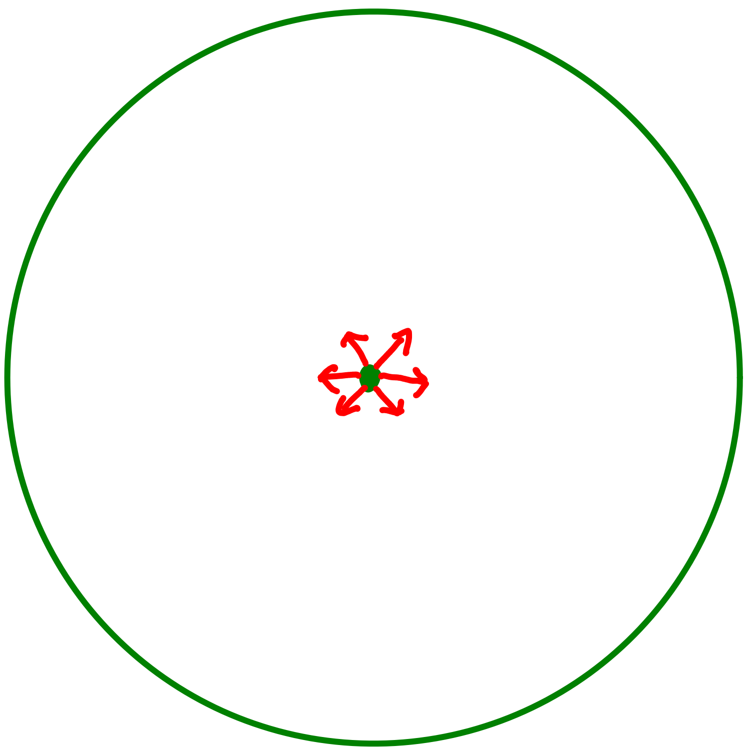

$Q \Rightarrow$

- centraliser group $C_G(Q)$ $H = T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset G$
- singular directions $\mathbb{A}$

Wild Character Varieties

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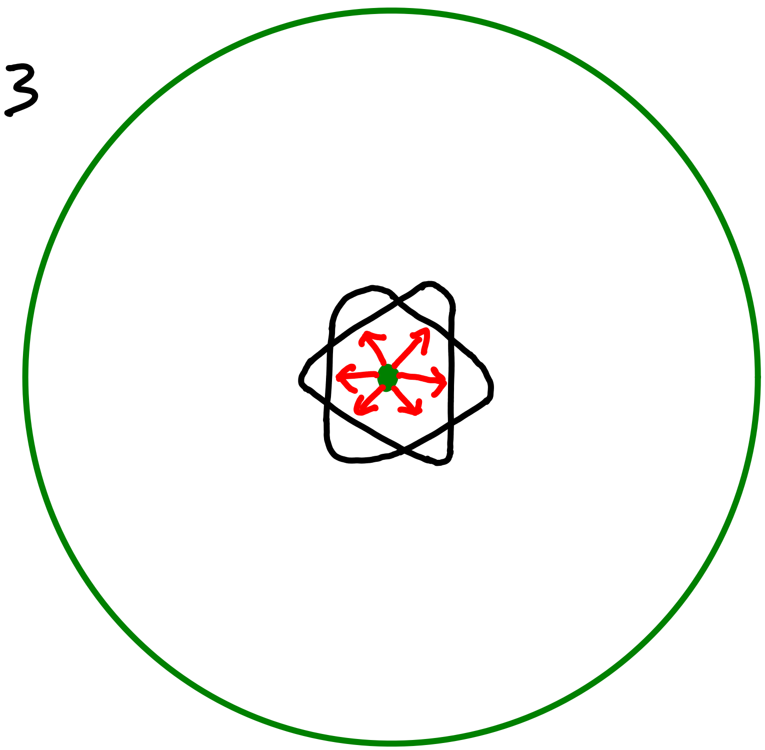
$k=3$



$Q \Rightarrow$

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- singular directions $\mathbb{A}$

Solutions involve $\exp(Q)$

$Q = \text{diag}(q_1, q_2)$

Stokes diagram: plot growth of $\exp(q_1)$, $\exp(q_2)$

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

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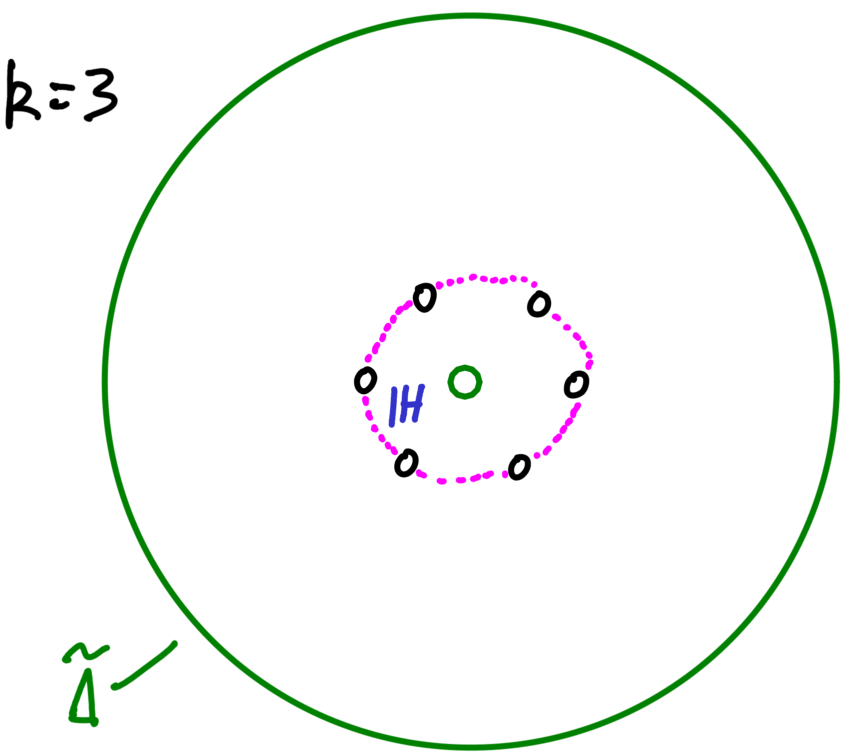

o $e(d)$ extra punctures

IH halo/annulus

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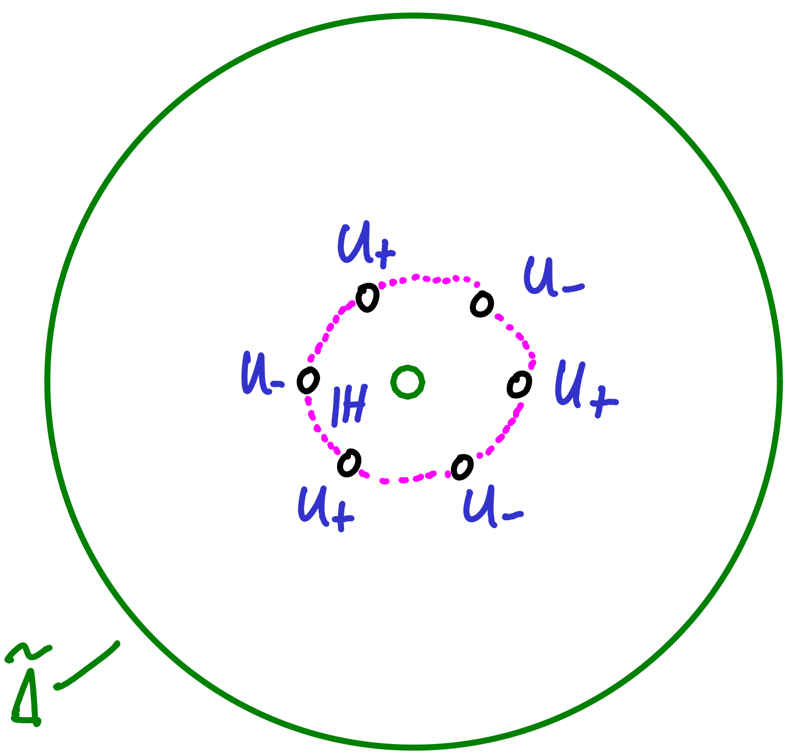

$Q \Rightarrow$

- centraliser group $C_G(Q)$ $\quad H = T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset G$
- singular directions $\mathbb{A}$
- Stokes groups $\mathbb{S}to_d \subset G \quad \forall d \in \mathbb{A}$

  $\cong U_+$ or U_- here

  $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}$

$\circ$ $\quad e(d)$ extra punctures

$\mathbb{H}$ halo/annulus

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

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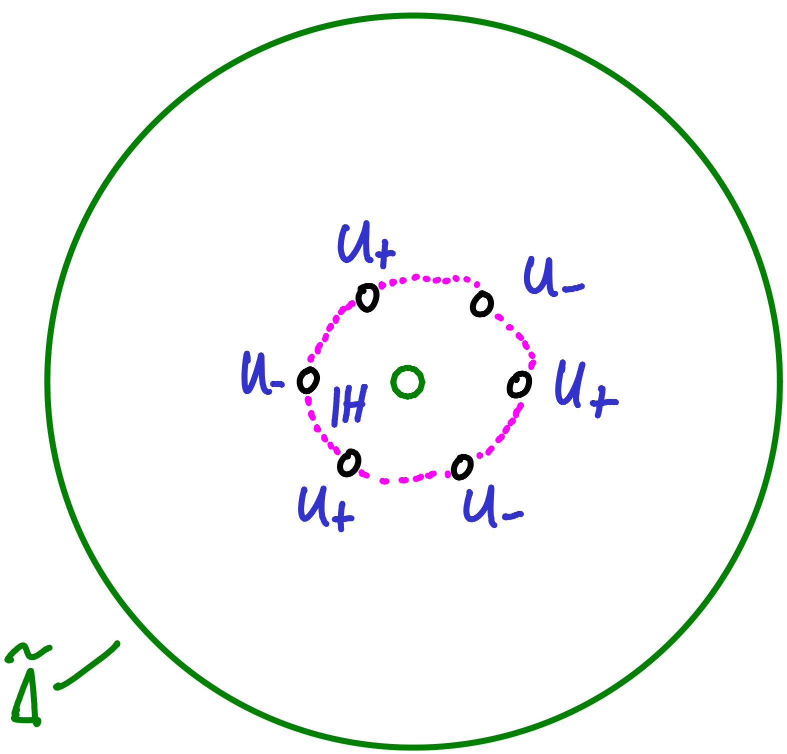

Stokes local system:

- G local system on $\tilde{\Delta}$
- flat reduction to H in $\mathbb{H}$
- monodromy around $e(d)$ in $\mathbb{S}to_d$

$\circ$ $e(d)$ extra punctures

$\mathbb{H}$ halo/annulus

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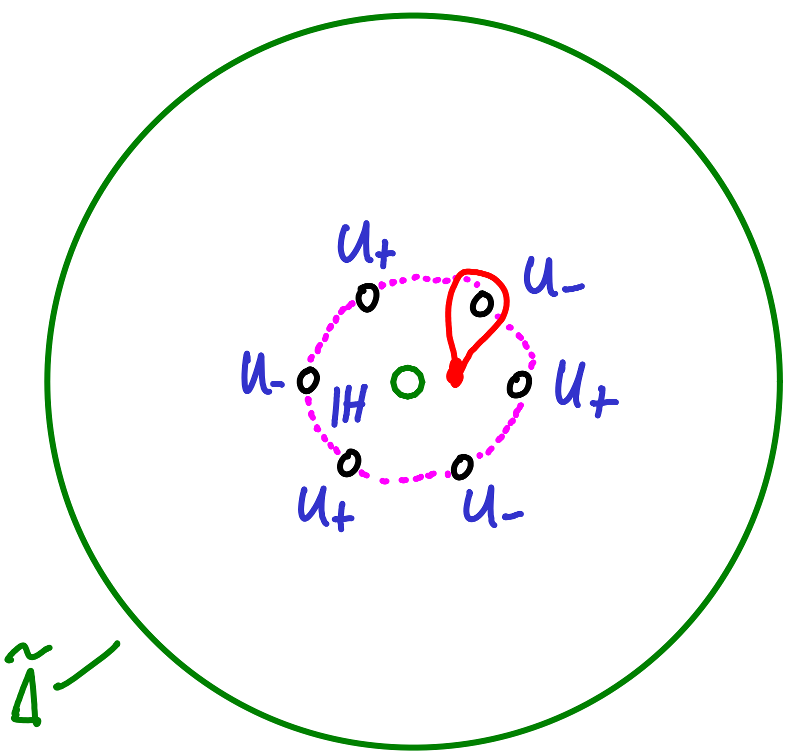

o $e(d)$ extra punctures

$\mathbb{H}$ halo/annulus

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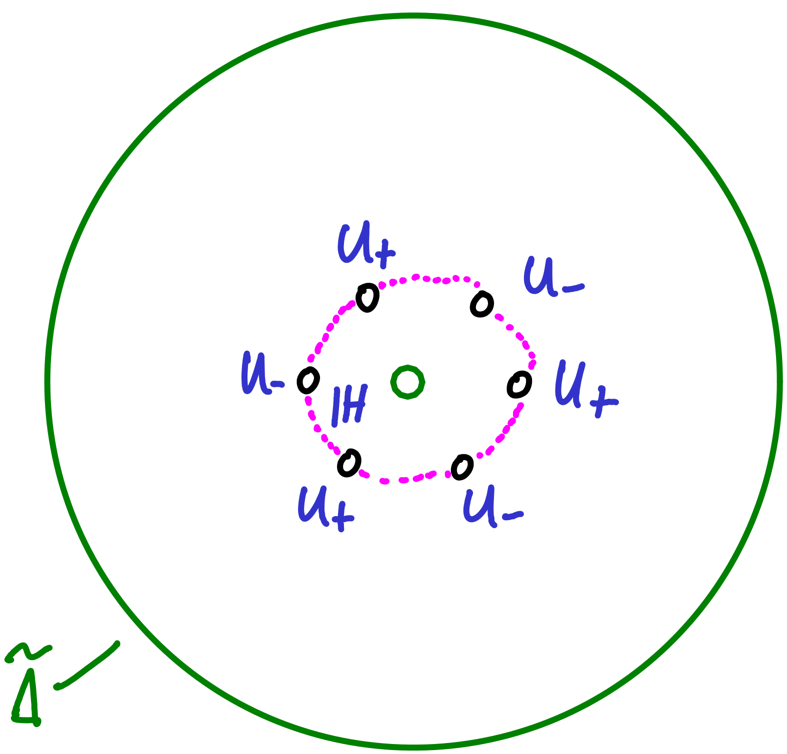

o $e(d)$ extra punctures

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Stokes local system:

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- flat reduction to H in $\mathbb{H}$
- monodromy around $e(d)$ in $\mathbb{S}to_d$

— Topological data that the multisummation approach to Stokes data gives

$$\left\{ \begin{array}{c} \text{Connections with} \\ \text{irreg. type } Q \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \text{Stokes local} \\ \text{systems} \end{array} \right\}$$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\qquad G = GL_2(\mathbb{C})$

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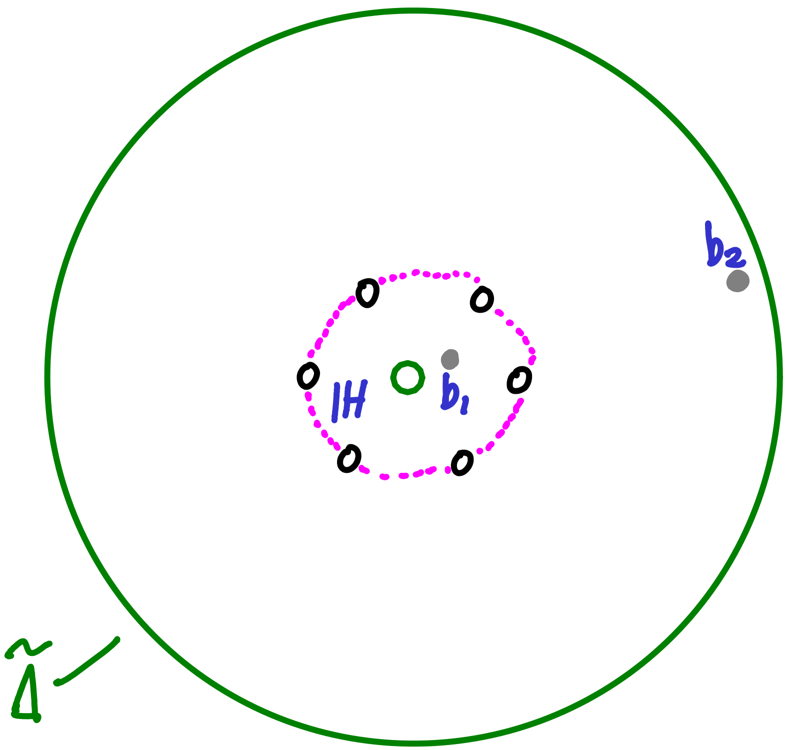

basepoints b_1, b_2

o $e(d)$ extra punctures

$\mathbb{H}$ halo/annulus

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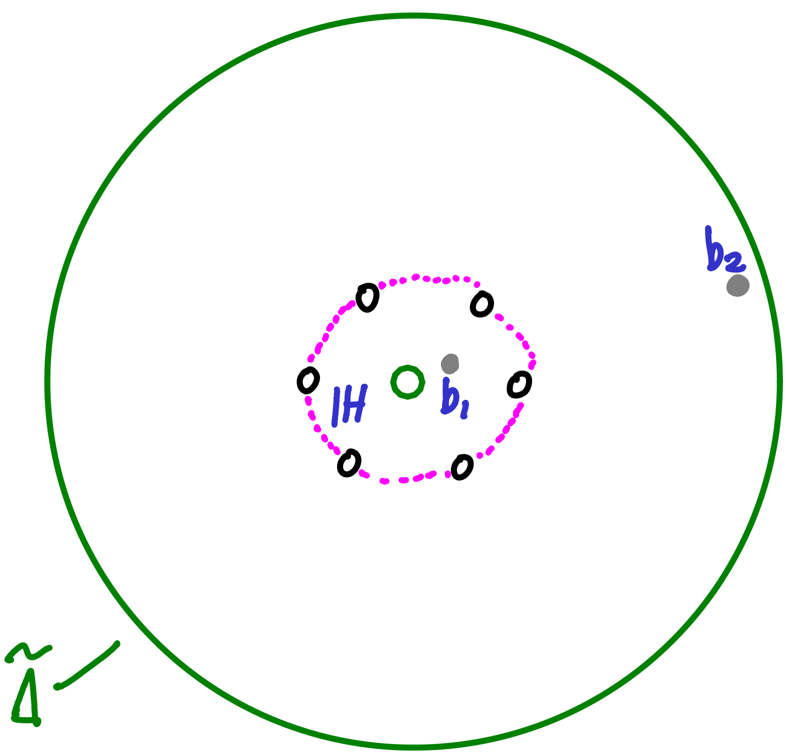

basepoints b_1, b_2

$$\Pi = \Pi_1(\tilde{\Delta}, \{b_1, b_2\})$$

$\circ$ $e(d)$ extra punctures

$\mathbb{H}$ halo/annulus

Wild Character Varieties

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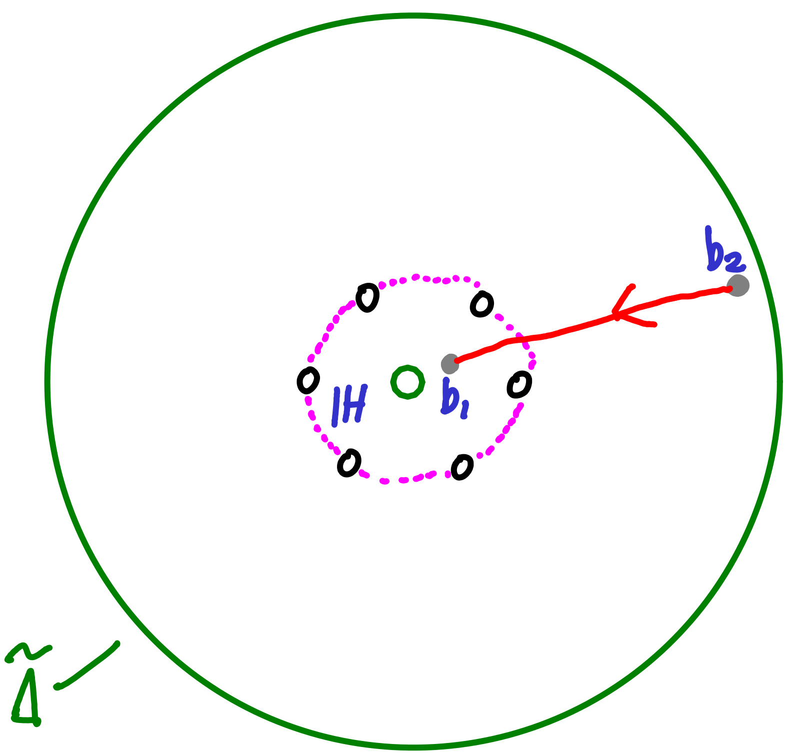

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$\circ$ $\quad e(d)$ extra punctures

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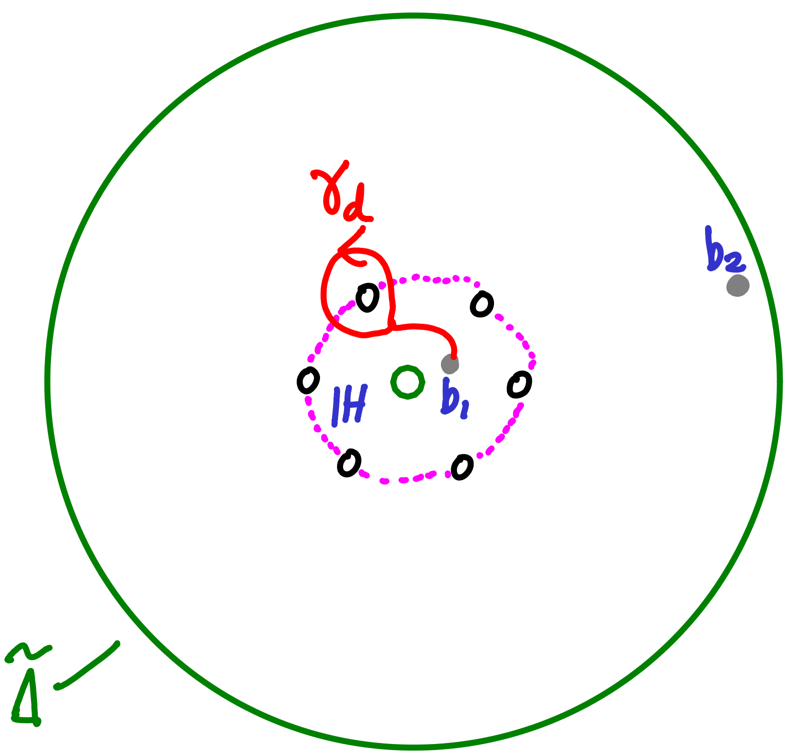

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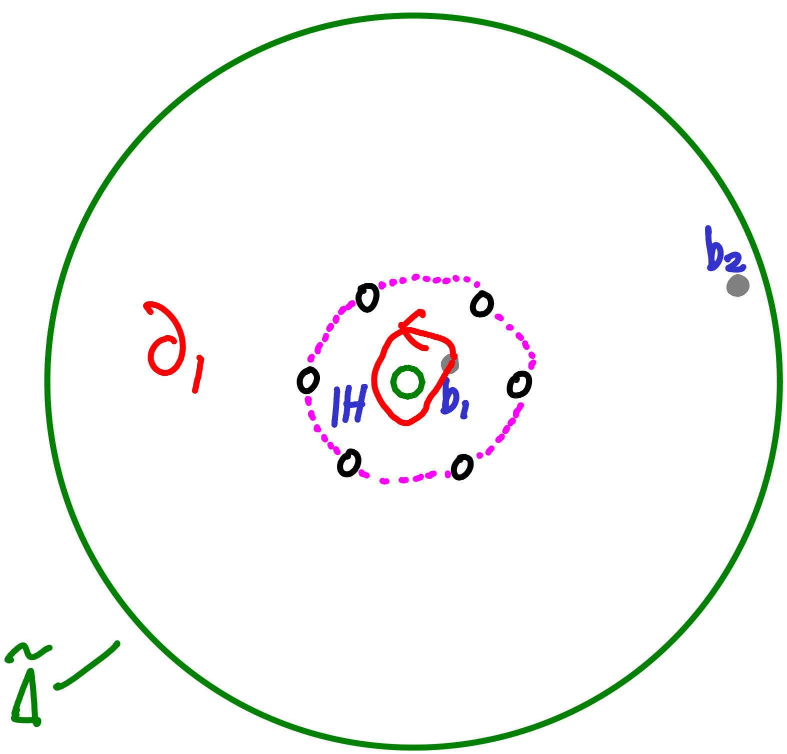

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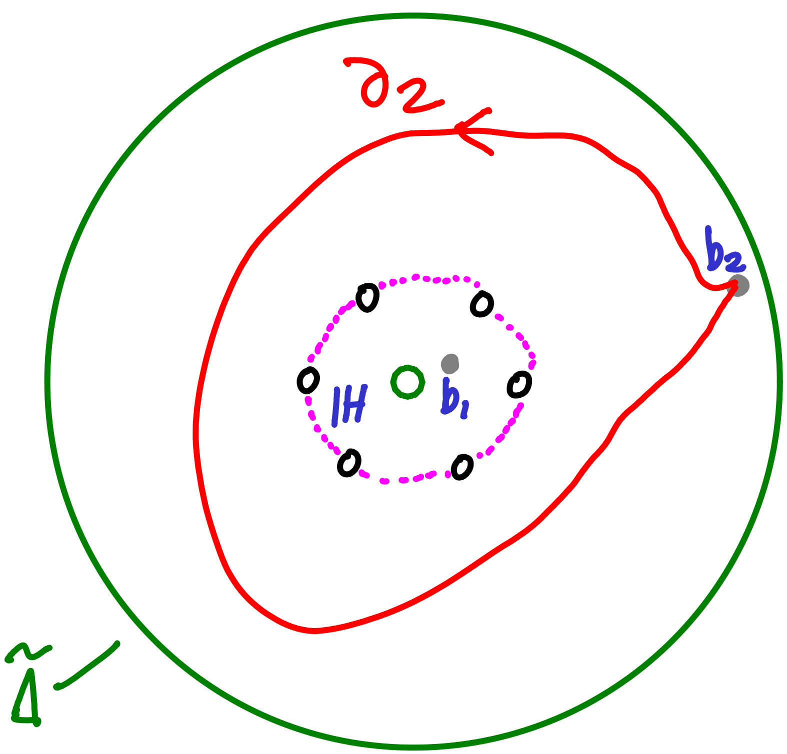

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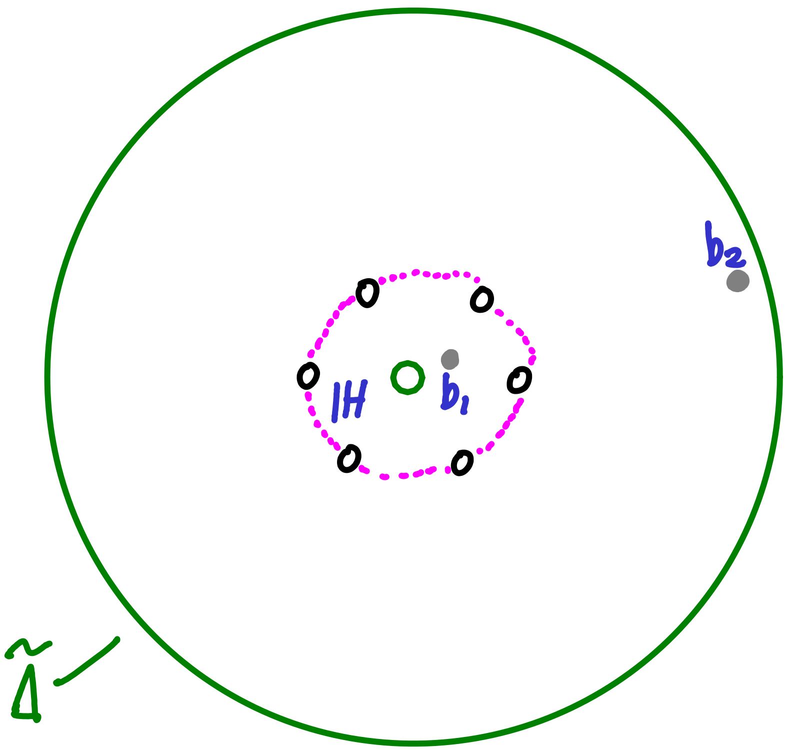

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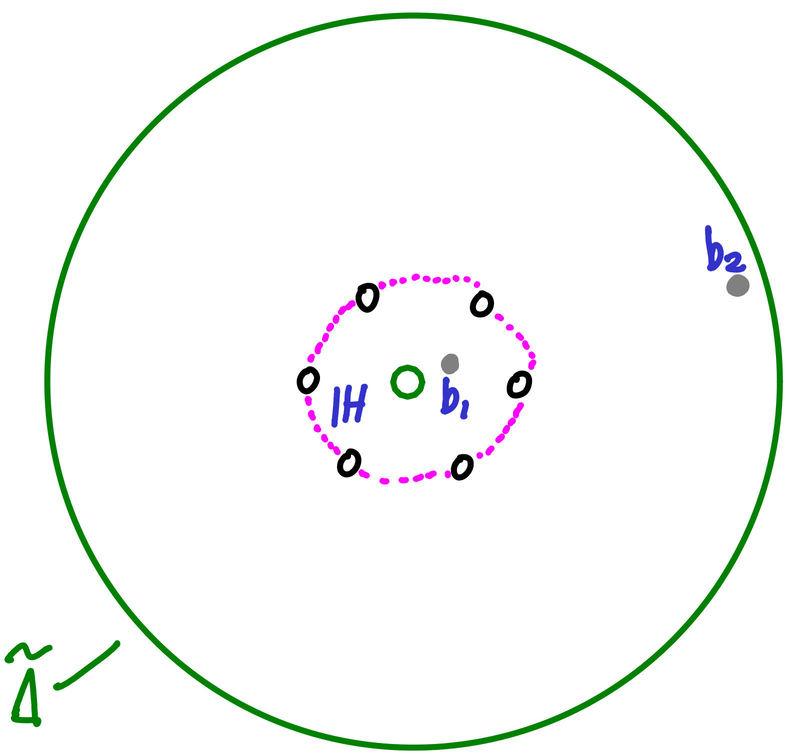

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$$\tilde{\mathcal{M}}_B = \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$$

$$= \left\{ \rho : \Pi \to G \;\middle|\; \begin{array}{l} \rho(\partial_1) \in H \\ \rho(\gamma_d) \in \mathrm{Sto}_d \quad \forall d \in \mathcal{A} \end{array} \right\}$$

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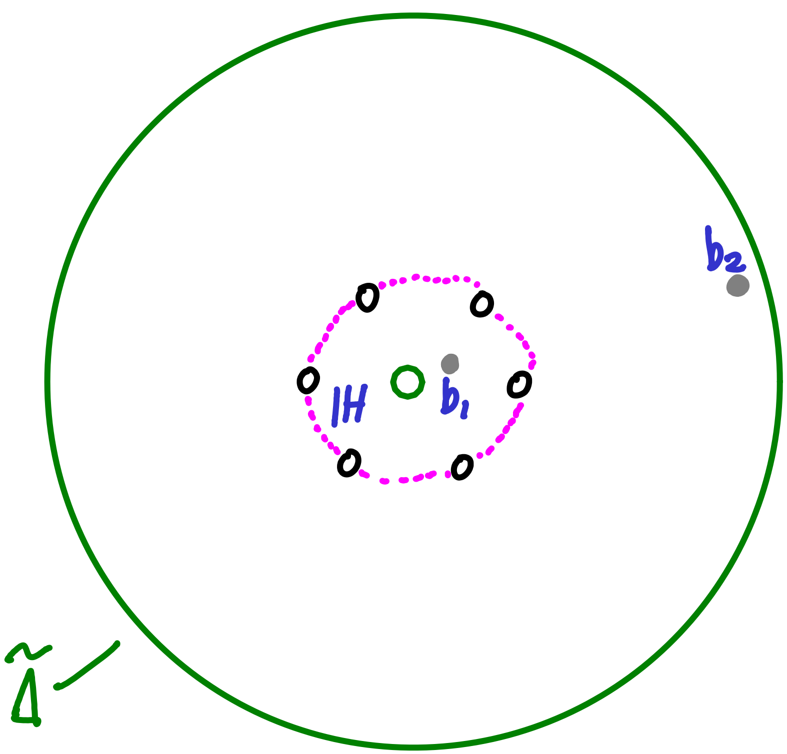

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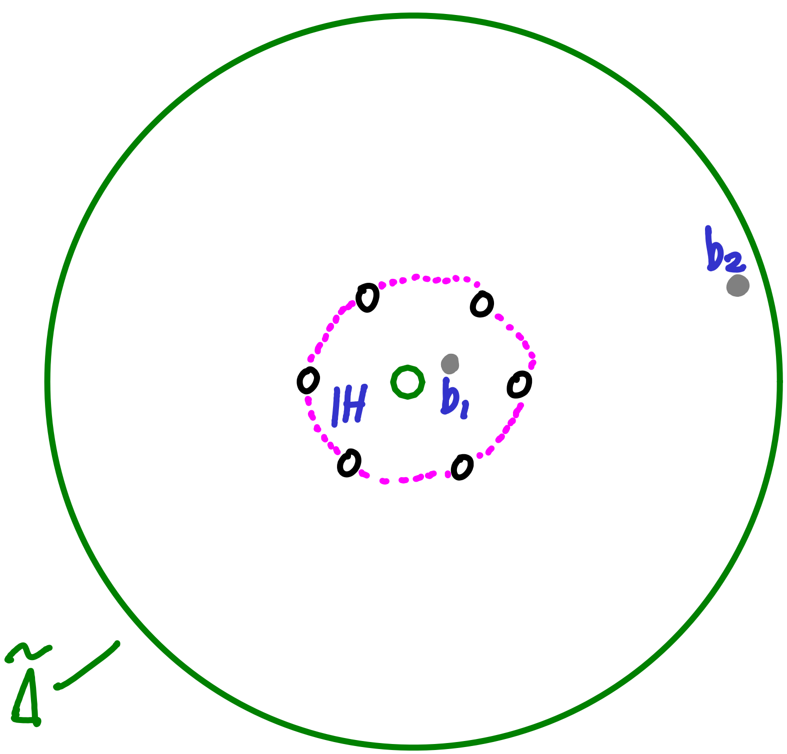

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$\underset{\sim}{S} = (S_1, \dots, S_{2k})$ $S_{odd/even} \in U_{+/-}$

Moment map $\mu(C, \underset{\sim}{S}, h) = (C^{-1} h S_{2k} \cdots S_2 S_1 C, h^{-1}) \in G \times H$

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— Euler's continuants are group valued moment maps

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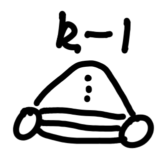
$\Gamma =$ 

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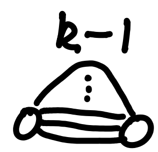
$$=: \operatorname{Rep}^*(\Gamma, V) \qquad \Gamma =$$ 

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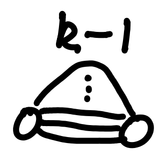
$=: \mathrm{Rep}^*(\Gamma, V)$ $\qquad \Gamma =$ 

$, \qquad V = \mathbb{C} \oplus \mathbb{C}$

[Similarly for $V = V_1 \oplus V_2$ any dimension
(2009-2015) Γ any "fission graph"]

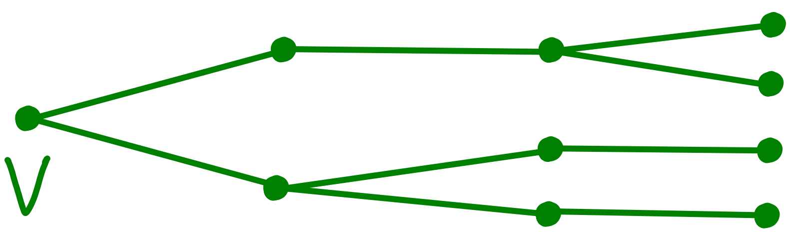
$$\mu(a_1, \dots, b_{k-1}) = \left((a_1, b_1, \dots, a_{k-1}, b_{k-1}), (b_{k-1}, \dots, b_1, a_1)^{-1} \right)$$

Fission graphs (arXiv 0806, apx C) $G = GL(V)$

$$Q = A_r/z^r + \cdots + A_1/z \qquad (A_i \in \mathfrak{t})$$

$$= A_r w^r + \cdots + A_1 w \qquad w = 1/z$$

$r = 3$:



"fission tree"

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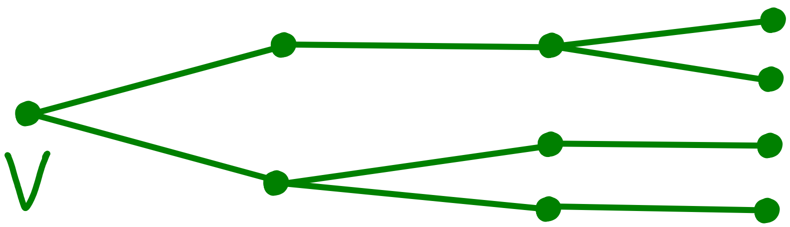
$r=3$: 

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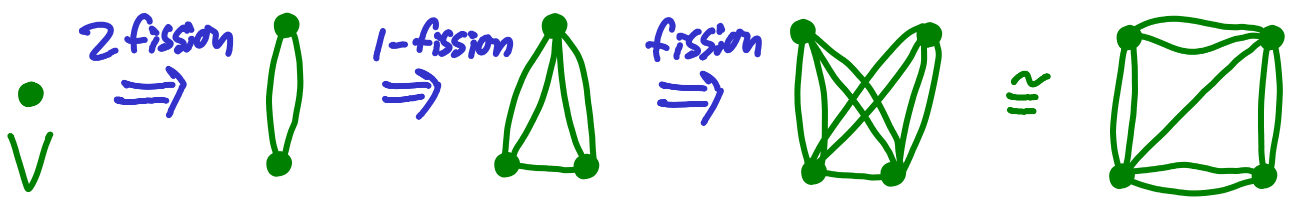
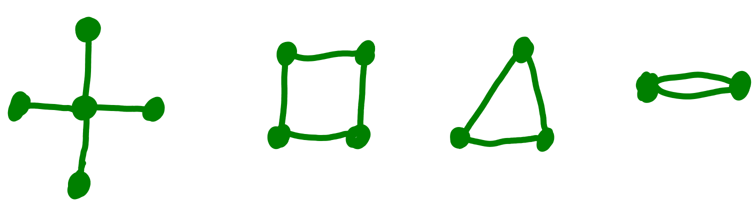
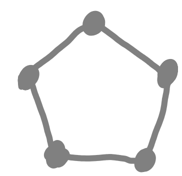


"fission graph" $\Gamma(Q)$

- $r=2$ get all complete k-partite graphs
- e.g.  but not 

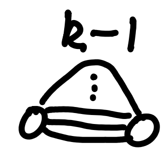
$Q = \mathrm{diag}(q_1, \dots, q_n) \Rightarrow$ nodes $= \{1, \dots, n\}$, # edges $i \leftrightarrow j$ $= \deg_w(q_i - q_j) - 1$

Wild Character Varieties

In this example $\left((\mathbb{P}^1, 0, Q) \quad Q = A/z^k, \quad GL_2(\mathbb{C}) \right)$

$$\mathcal{M}_B = \widetilde{\mathcal{M}}_B \,/\!\!/_{(q_1,q_2)}\, H$$

$$= \mathrm{Rep}^*(\Gamma, V) \,/\!\!/_{(q_1,q_2)}\, H$$

$\Gamma =$ 

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"multiplicative quiver variety"

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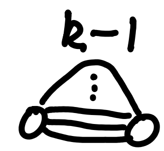
$\Gamma =$ 

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E.g. $k=3$ (Painlevé 2 Betti space)

$$\mathcal{M}_B \cong \left\{ xyz + x + y + z = b - b^{-1} \right\} \quad b \in \mathbb{C}^* \text{ constant}$$

(Flaschka-Newell surface)

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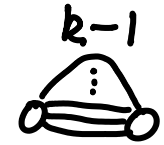
$\Gamma =$ 

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"multiplicative quiver variety"

Also $\mathcal{M}^* \cong \mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} H$ "Nakajima/additive quiver variety"

(P.B 2008, Hiroe-Yamakawa 2013)

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$$\boxed{\begin{array}{ccc} \mathcal{M}^* & \xRightarrow{\text{RHB}} & \mathcal{M}_B \\ \cong & & \cong \\ \mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} H & & \mathrm{Rep}^*(\Gamma, V) /\!\!/_{q} H \end{array}}$$

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607) "K2 surfaces"

affine Weyl group	E_8	E_7	E_6	D_4	$A_3 = D_3$	D_2	D_1	D_0	A_2	A_1	A_0
minimal rank of bundles	6	4	3	2	2	2	2	2	2	2	2
pole orders	1+1+1	1+1+1	1+1+1	1+1+1+1	2+1+1	2+2	2+2	2+2	3+1	4	4

affine Weyl group

minimal rank of bundles

pole orders

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Tame ← | → Wild

	Tame				Wild						
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minimal rank of bundles	6	4	3	2	2	2	2	2	2	2	2
pole orders	1+1+1	1+1+1	1+1+1	1+1+1+1	2+1+1	2+2	2+2	2+2	3+1	4	4

affine Weyl group

minimal rank of bundles

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(Nonabelian Hodge surfaces) (1203.6607) "K2 surfaces"

E_8 E_7 E_6

P_3 P_3' P_3''

D_2 D_1 D_0

D_4 $A_3 = D_3$

P_6 P_5

A_2 A_1 A_0

P_4 P_2 P_1

Phase spaces for Painlevé differential equations

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607) "K2 surfaces"

$\mathcal{M}^* \cong$ ALE

$\mathcal{M}^* \cong$ ALF

Atiyah-Hitchin

E_8 E_7 E_6

D_4 $A_3 = D_3$

D_2 D_1 D_0

A_2 A_1 A_0

$T^*\mathbb{P}^1$ $\mathbb{C}^2$

[$\mathcal{M}^* \subset \mathcal{M}$ open piece where bundle holom. trivial]