Dispersionless (3+1)-dimensional integrable hierarchies

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Plan of presentation

1. The general R-matrix construction of integrable hierarchies
2. The contact bracket
3. Integrable (3+1)-dimensional infinite-component hierarchies and their reductions
4. Examples with finite number of fields

R-matrix construction

Let $\mathfrak{g}$ be an infinite-dimensional Lie algebra. The Lie bracket $[\cdot, \cdot]$ defines the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$: $\operatorname{ad}_a b = [a, b]$.
Recall that an $R \in \mathrm{End}(\mathfrak{g})$ is called a (classical) R-matrix if the R-bracket

$$[a, b]_R := [Ra, b] + [a, Rb]$$

is a new Lie bracket on $\mathfrak{g}$. The Jacobi identity is satisfied if R satisfies the so-called classical modified Yang–Baxter equation

$$[Ra, Rb] - R[a, b]_R - \alpha[a, b] = 0, \qquad \alpha \in \mathbb{R}.$$

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$$[Ra, Rb] - R[a, b]_R - \alpha[a, b] = 0, \qquad \alpha \in \mathbb{R}.$$

Let $L_i \in \mathfrak{g}$, $i \in \mathbb{N}$. Consider the associated hierarchies of flows

$$(L_n)_{t_r} = [RL_r, L_n], \qquad r, n \in \mathbb{N}.$$

R-matrix construction

Theorem I

Suppose that R is an R-matrix on $\mathfrak{g}$ which commutes with all derivatives ∂_{t_n}, i.e.,

$$(RL)_{t_n} = RL_{t_n}, \quad n \in \mathbb{N},$$

and obeys the classical modified Yang–Baxter equation for $\alpha \neq 0$. Let $L_i \in \mathfrak{g}$, $i \in \mathbb{N}$ satisfy considered hierarchies of flows.
Then the following conditions are equivalent:

i) the zero-curvature equations

$$(RL_r)_{t_s} - (RL_s)_{t_r} + [RL_r, RL_s] = 0, \quad r, s \in \mathbb{N}$$

hold;

ii) all L_i commute in $\mathfrak{g}$:

$$[L_i, L_j] = 0, \qquad i, j \in \mathbb{N}.$$

R-matrix construction

Moreover, if one (and hence both) of the above equivalent conditions holds, then considered flows commute, i.e.,

$$((L_n)_{t_r})_{t_s} - ((L_n)_{t_s})_{t_r} = 0, \quad n, r, s \in \mathbb{N};$$

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Moreover, if one (and hence both) of the above equivalent conditions holds, then considered flows commute, i.e.,

$$((L_n)_{t_r})_{t_s} - ((L_n)_{t_s})_{t_r} = 0, \quad n, r, s \in \mathbb{N};$$

Proof

$$\begin{aligned}
&(RL_r)_{t_s} - (RL_s)_{t_r} + [RL_r, RL_s] \\
&= R[RL_s, L_r] - R[RL_r, L_s] + [RL_r, RL_s] \\
&= [RL_r, RL_s] - R[L_r, L_s]_R = -\alpha[L_r, L_s]
\end{aligned}$$

$$\begin{aligned}
((L_n)_{t_r})_{t_s} - ((L_n)_{t_s})_{t_r} &= [RL_r, L_n]_{t_s} - [RL_s, L_n]_{t_r} \\
&= [(RL_r)_{t_s} - (RL_s)_{t_r}, L_n] + [RL_r, [RL_s, L_n]] \\
&\quad - [RL_s, [RL_r, L_n]] \\
&= [(RL_r)_{t_s} - (RL_s)_{t_r} + [RL_r, RL_s], L_n] \\
&= 0.
\end{aligned}$$

R-matrix construction

Now we present a procedure of extending the systems under study by adding an extra independent variable. Namely, we assume that all elements of $\mathfrak{g}$ depend on an additional independent variable y not involved in the Lie bracket, so all of the above results remain valid.
Consider an $\mathcal{L} \in \mathfrak{g}$ and the associated hierarchy of flows defined by

$$\mathcal{L}_{t_r} = [RL_r, \mathcal{L}] + (RL_r)_y, \qquad r \in \mathbb{N}.$$

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Theorem II

Suppose that $\mathcal{L} \in \mathfrak{g}$ and $L_i \in \mathfrak{g}$, $i \in \mathbb{N}$ are such that the zero-curvature equations hold for all $r, s \in \mathbb{N}$. Then the flows commute, i.e.,

$$(\mathcal{L}_{t_r})_{t_s} - (\mathcal{L}_{t_s})_{t_r} = 0, \quad r, s \in \mathbb{N}.$$

R-matrix construction

Proof

Using the evolution equations and the Jacobi identity for the Lie bracket we obtain

$$\begin{aligned}(\mathcal{L}_{t_r})_{t_s} - (\mathcal{L}_{t_s})_{t_r} &= [(RL_r)_{t_s} - (RL_s)_{t_r} + [RL_r, RL_s], \mathcal{L}] \\ &\quad + ((RL_r)_{t_s} - (RL_s)_{t_r} + [RL_r, RL_s])_y \\ &= 0.\end{aligned}$$

The right-hand side of the above equation vanishes by virtue of the zero curvature equations.

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It is well known that whenever $\mathfrak{g}$ admits a decomposition into two Lie subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ such that

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \qquad [\mathfrak{g}_\pm, \mathfrak{g}_\pm] \subset \mathfrak{g}_\pm, \qquad \mathfrak{g}_+ \cap \mathfrak{g}_- = \emptyset,$$

the operator

$$R = \frac{1}{2}(P_+ - P_-) = P_+ - \frac{1}{2}$$

where $P_\pm$ are projectors onto $\mathfrak{g}_\pm$, satisfies the classical modified Yang–Baxter equation with $\alpha = \frac{1}{4}$, i.e., R is a classical R-matrix.

R-matrix construction

Next, let us specify the dependence of L_j on y via the so-called Lax–Novikov equations

$$[L_j, \mathcal{L}] + (L_j)_y = 0, \qquad j \in \mathbb{N}.$$

Then, our previously considered equations take the following form:

$$(L_s)_{t_r} = [B_r, L_s], \qquad r, s \in \mathbb{N},$$

$$(B_r)_{t_s} - (B_s)_{t_r} + [B_r, B_s] = 0,$$

$$\mathcal{L}_{t_r} = [B_r, \mathcal{L}] + (B_r)_y, \qquad n, r \in \mathbb{N}$$

where $B_i = P_+ L_i$.

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where $B_i = P_+ L_i$.

For Lie algebras which admit an additional associative multiplication $\circ$ which obeys the Leibniz rule

$$\mathrm{ad}_a(b \circ c) = \mathrm{ad}_a(b) \circ c + b \circ \mathrm{ad}_a(c) \Leftrightarrow [a, b \circ c] = [a, b] \circ c + b \circ [a, c],$$

the commutative subalgebra in question is generated by rational powers of a given element $L \in \mathfrak{g}$. Here we relax that assumption.

The contact bracket

Consider a commutative and associative algebra A of formal series in p

$$A \ni f = \sum_i u_i p^i$$

with the standard multiplication

$$f_1 \cdot f_2 \equiv f_1 f_2, \qquad f_1, f_2 \in A.$$

The coefficients u_i of these series are assumed to be smooth functions of x, y, z and infinitely many times $t_1, t_2, \ldots$.

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The contact bracket on A will be denoted by $\{\cdot,\cdot\}_C$ and is defined by

$$\{f_1, f_2\}_C = \frac{\partial f_1}{\partial p}\frac{\partial f_2}{\partial x} - p\frac{\partial f_1}{\partial p}\frac{\partial f_2}{\partial z} + f_1\frac{\partial f_2}{\partial z} - (f_1 \leftrightarrow f_2).$$

This bracket is independent of y.

The contact bracket

Note that A is not a Poisson algebra as the contact bracket (9) does not obey the Leibniz rule:

$$\{f_1 f_2, f_3\}_C = \{f_1, f_3\}_C f_2 + f_1\{f_2, f_3\}_C - f_1 f_2\{1, f_3\}_C.$$

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To make contact with the R-matrix approach, we identify $\mathfrak{g}$ with A and the commutator $[\cdot,\cdot]$ in $\mathfrak{g}$ with the contact bracket. As for the choice of the splitting of $\mathfrak{g}$ into Lie subalgebras $\mathfrak{g}_\pm$ with $P_\pm$ being projections onto the respective subalgebras, so $\mathfrak{g}_\pm = P_\pm(\mathfrak{g})$, we have two natural choices when $R = P_+ - \frac{1}{2}$ satisfies the classical modified Yang–Baxter equation. These two choices are

$$P_+ = P_{\geqslant k}, \quad k = 0, 1$$

$$P_{\geqslant k}\left(\sum_{j=-\infty}^{\infty} u_j p^j\right) = \sum_{j=k}^{\infty} u_j p^j.$$

(3+1)-dimensional hierarchies

Consider first the case of $k = 0$ and the nth order Lax function from A

$$\mathcal{L} = u_n p^n + u_{n-1} p^{n-1} + \cdots + u_0 + u_{-1} p^{-1} + \cdots , \quad n > 0$$

and let

$$B_m \equiv P_+ L_m = v_{m,m} p^m + v_{m,m-1} p^{m-1} + \cdots + v_{m,0}, \qquad m > 0$$

where $u_i = u_i(\vec{t}, x, y, z)$, $v_{m,j} = v_{m,j}(\vec{t}, x, y, z)$, and $\vec{t} = (t_1, t_2, \ldots)$.

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where $u_i = u_i(\vec{t}, x, y, z)$, $v_{m,j} = v_{m,j}(\vec{t}, x, y, z)$, and $\vec{t} = (t_1, t_2, \ldots)$.

Substituting $\mathcal{L}$ and B_m into the equations

$$\mathcal{L}_{t_m} = \{B_m, \mathcal{L}\}_C + (B_m)_y$$

we obtain a hierarchy of infinite-component systems of the form

$$(u_r)_{t_m} = X_r^m[u, v_m], \qquad r \leq n + m, \quad r \neq 0, \ldots, m,$$
$$(u_r)_{t_m} = X_r^m[u, v_m] + (v_{m,r})_y, \qquad r = 0, \ldots, m.$$

where one has to put $u_r \equiv 0$ for $r > n$ and

(3+1)-dimensional hierarchies

$$X_r^m[u, v_m] \quad = \quad + \sum_{s=0}^{m} [s v_{m,s} (u_{r-s+1})_x - (r - s + 1) u_{r-s+1} (v_{m,s})_x$$
$$- (s-1) v_{m,s} (u_{r-s})_z + (r - s - 1) u_{r-s} (v_{m,s})_z],$$

for $r \leq m + n$. The fields u_r for $r \leq n$ are dynamical variables while equations for $n + m \geq r > n$ can be seen as nonlocal constraints on u_r defining the fields $v_{m,s}$.

(3+1)-dimensional hierarchies

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$$-(s-1)v_{m,s}(u_{r-s})_z + (r-s-1)u_{r-s}(v_{m,s})_z],$$

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The reader has to bear in mind that in addition, dependent variables $v_{m,s}$ are by construction related to each other through zero-curvature relations.

(3+1)-dimensional hierarchies

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Admissible constraints

- $$v_{m,m} = (u_n)^{\frac{m-1}{n-1}}, \quad n>1, \ m>1$$
- $$u_n = const, \ v_{m,m} = const, \ v_{m,m-1} = \begin{cases} \frac{m-1}{n-1}u_{n-1}, & n>1 \\ u_0 = const, & n=1 \end{cases}$$

(3+1)-dimensional hierarchies

Let us look on the case $n = 1$ more carefully. Taking $u_0 = 0$, Lax equation for

$$\mathcal{L} = p + u_{-1}p^{-1} + u_{-2}p^{-2} \cdots ,$$

and $m = 2$

$$B_2 = p^2 + v_1 p + v_0,$$

generates the following infinite field system:

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generates the following infinite field system:

$$\begin{aligned}
(v_1)_y &= (v_1)_x + (u_{-1})_z, \\
(v_0)_y &= (v_0)_x + (u_{-2})_z - 2(u_{-1})_x + 2u_{-1}(v_1)_z, \\
(u_r)_{t_2} &= 2(u_{r-1})_x - (u_{r-2})_z - (r+1)u_{r+1}(v_0)_x + v_0(u_r)_z \\
&\quad + (r-1)u_r(v_0)_z + v_1(u_r)_x - ru_r(v_1)_x + (r-2)u_{r-1}(v_1)_z,
\end{aligned}$$

where $r < 0$ and $v_{2,r} \equiv v_r$.

Natural $(2+1)$-dimensional reductions:

The reduction

$$\begin{aligned} 0 &= (v_1)_x + (u_{-1})_z, \\ 0 &= (v_0)_x + (u_{-2})_z - 2(u_{-1})_x + 2u_{-1}(v_1)_z, \\ (u_r)_{t_2} &= 2(u_{r-1})_x - (u_{r-2})_z - (r+1)u_{r+1}(v_0)_x + v_0(u_r)_z \\ &\quad +(r-1)u_r(v_0)_z + v_1(u_r)_x - ru_r(v_1)_x + (r-2)u_{r-1}(v_1)_z, \end{aligned}$$

when u_j, v_0 and v_1 are independent of y.

(3+1)-dimensional hierarchies

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&\quad +(r-1)u_r(v_0)_z + v_1(u_r)_x - ru_r(v_1)_x + (r-2)u_{r-1}(v_1)_z,
\end{aligned}
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when u_j, v_0 and v_1 are independent of y.

The reduction

$$
\begin{aligned}
(v_1)_y &= (u_{-1})_z, \\
(v_0)_y &= (u_{-2})_z + 2u_{-1}(v_1)_z, \\
(u_r)_{t_2} &= -(u_{r-2})_z + v_0(u_r)_z + (r-1)u_r(v_0)_z + (r-2)u_{r-1}(v_1)_z,
\end{aligned}
$$

when u_j, v_0 and v_1 are independent of x.

(3+1)-dimensional hierarchies

The reduction

$$(v_1)_y = (v_1)_x,$$
$$(v_0)_y = (v_0)_x - 2(u_{-1})_x,$$
$$(u_r)_{t_2} = 2(u_{r-1})_x - (r+1)u_{r+1}(v_0)_x + v_1(u_r)_x - ru_r(v_1)_x,$$

when u_j, v_0 and v_1 are independent of z.

(3+1)-dimensional hierarchies

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when u_j, v_0 and v_1 are independent of z.

The last system admits further reduction $v_1 = 0$ to the form

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It reduces to $(1+1)$-dimensional Benney system

$$(u_r)_{t_2} = 2(u_{r-1})_x - 2(r+1)u_{r+1}(u_{-1})_x, \quad r < 0,$$

when u_i are independent of *both* y and z and $v_0 = 2u_{-1}$.

(3+1)-dimensional hierarchies

The case $k = 1$

Similar considerations can be performed for $k = 1$. Let us look on that simplest case of

$$\mathcal{L} = p + u_0 + u_{-1}p^{-1} + \cdots$$

and

$$B_m \equiv P_+ L_m = v_{m,m-1}p^m + v_{m,m-2}p^{m-1} + \cdots + v_{m,1}p, \qquad m > 1$$

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The first flow for $m = 2$, where we put $v_{2,r} \equiv v_r$ to simplify writing, takes the form

$$\begin{aligned}
(v_2)_y &= (v_2)_x + u_0(v_2)_z + v_2(u_0)_z, \\
(v_1)_y &= (v_1)_x + u_0(v_1)_z + v_2(u_{-1})_z + 2u_{-1}(v_2)_z - 2v_2(u_0)_x, \\
(u_r)_{t_2} &= v_1(u_r)_x - ru_r(v_1)_x + (r-2)u_{r-1}(v_1)_z + 2v_2(u_{r-1})_x \\
&\quad - (r-1)u_{r-1}(v_2)_x - v_2(u_{r-2})_z + (r-3)u_{r-2}(v_2)_z, \quad r \leq 0.
\end{aligned}$$

$(2+1)$-dimensional and $(1+1)$-dimensional reductions are available as well.

(3+1)-dimensional reductions with finite number of fields

The case $k = 0$

We have a natural reduction to finite-component systems

$$\begin{aligned} \mathcal{L} &= u_n p^n + u_{n-1} p^{n-1} + \cdots + u_r p^r, \quad r = 0, 1, \\ B_m &= (u_n)^{\frac{m-1}{n-1}} p^m + v_{m,m-1} p^{m-1} + \cdots + v_{m,0} \end{aligned}$$

and

$$\begin{aligned} \mathcal{L} &= p^n + u_{n-1} p^{n-1} + \cdots + u_r p^r, \quad r = 0, 1, \\ B_m &= p^m + \frac{(m-1)}{(n-1)} u_{n-1} p^{m-1} + \cdots + v_{m,0}. \end{aligned}$$

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The case $k = 1$

We have a natural reduction to finite-component systems

$$\begin{aligned} \mathcal{L} &= u_n p^n + u_{n-1} p^{n-1} + \cdots + u_r p^r, \quad r = 1, 0, -1, \cdots \\ B_m &= (u_n)^{\frac{m-1}{n-1}} p^m + v_{m,m-1} p^{m-1} + \cdots + v_{m,1} p \end{aligned}$$

and

$$\mathcal{L} = p + u_0 + u_{-1} p^{-1} + \cdots + u_r p^r, \quad r = 0, 1, -1, \ldots$$

$$B_m = v_{m,m} p^m + v_{m,m-1} p^{m-1} + \cdots + v_{m,1} p, \qquad m > 1.$$

Examples

Example 1

Let

$$\mathcal{L} = p + u_0 + u_{-1}p^{-1}, \quad B_2 = v_2 p^2 + v_1 p,$$

then

$$\begin{aligned}
(u_{-1})_{t_2} &= u_{-1}(v_1)_x + v_1(u_{-1})_x,\\
(u_0)_{t_2} &= -2u_{-1}(v_1)_z + v_1(u_0)_x + u_{-1}(v_2)_x + 2v_2(u_{-1})_x,\\
(v_1)_y &= (v_1)_x + 2u_{-1}(v_2)_z + v_2(u_{-1})_z + u_0(v_1)_z - 2v_2(u_0)_x,\\
(v_2)_y &= (v_2)_x + u_0(v_2)_z + v_2(u_0)_z,
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(u_{-1})_{t_2} &= u_{-1}(v_1)_x + v_1(u_{-1})_x,\\
(u_0)_{t_2} &= -2u_{-1}(v_1)_z + v_1(u_0)_x + u_{-1}(v_2)_x + 2v_2(u_{-1})_x,\\
(v_1)_y &= (v_1)_x + 2u_{-1}(v_2)_z + v_2(u_{-1})_z + u_0(v_1)_z - 2v_2(u_0)_x,\\
(v_2)_y &= (v_2)_x + u_0(v_2)_z + v_2(u_0)_z,
\end{aligned}$$

(2+1)-dimensional reductions

For $z = 0$ and $v_2 = const = 1$ we get

$$\begin{aligned}
(u_{-1})_{t_2} &= u_{-1}(v_1)_x + v_1(u_{-1})_x,\\
(u_0)_{t_2} &= v_1(u_0)_x + 2(u_{-1})_x,\\
(v_1)_y &= (v_1)_x - 2(u_0)_x,
\end{aligned} \tag{1}$$

Examples

For $x = 0$ and $u_1 = const = 1$ we get

$$\begin{aligned}(u_0)_{t_2} &= -2(v_1)_z,\\ (v_1)_y &= 2(v_2)_z + u_0(v_1)_z,\\ (v_2)_y &= (u_0 v_2)_z\end{aligned}$$

Examples

For $x = 0$ and $u_1 = const = 1$ we get

$$(u_0)_{t_2} = -2(v_1)_z,$$
$$(v_1)_y = 2(v_2)_z + u_0(v_1)_z,$$
$$(v_2)_y = (u_0 v_2)_z$$

(1+1)-dimensional reductions

Further reduction by $y = 0$, leads to $(1+1)$-dimensional (t, x)-system

$$(u_{-1})_{t_2} = 2(u_{-1}u_0)_x,$$
$$(u_0)_{t_2} = 2(u_{-1} + u_0^2)_x,$$

where $v_1 = 2u_0$ and $(1+1)$-dimensional (t, z)-system

$$(u_0)_{t_2} = 2(u_0^{-2})_z,$$

where

$$v_2 = u_0^{-1}, \quad v_1 = -u_0^{-2}.$$

Examples

Example 2

Let

$$\mathcal{L} = u_3 p^3 + u_2 p^2 + u_1 p, \quad B_2 = v_2 p^2 + v_1 p,$$

then

$$\begin{aligned}
0 &= 2u_3(v_2)_z - v_2(u_3)_z, \\
0 &= u_2(v_2)_z - v_2(u_2)_z + 2u_3(v_1)_z + 2v_2(u_3)_x - 3u_3(v_2)_x \\
(u_3)_{t_2} &= v_1(u_3)_x + 2v_2(u_2)_x - 2u_2(v_2)_x - 3u_3(v_1)_x - v_2(u_1)_z + u_2(v_1)_z, \\
(u_2)_{t_2} &= (v_2)_y + v_1(u_2)_x + 2v_2(u_1)_x - 2u_2(v_1)_x - u_1(v_2)_x, \\
(u_1)_{t_2} &= (v_1)_y + v_1(u_1)_x - u_1(v_1)_x,
\end{aligned}$$

Example 2

Let

$$\mathcal{L} = u_3 p^3 + u_2 p^2 + u_1 p, \quad B_2 = v_2 p^2 + v_1 p,$$

then

$$\begin{aligned}
0 &= 2u_3 (v_2)_z - v_2 (u_3)_z, \\
0 &= u_2 (v_2)_z - v_2 (u_2)_z + 2u_3 (v_1)_z + 2v_2 (u_3)_x - 3u_3 (v_2)_x \\
(u_3)_{t_2} &= v_1 (u_3)_x + 2v_2 (u_2)_x - 2u_2 (v_2)_x - 3u_3 (v_1)_x - v_2 (u_1)_z + u_2 (v_1)_z, \\
(u_2)_{t_2} &= (v_2)_y + v_1 (u_2)_x + 2v_2 (u_1)_x - 2u_2 (v_1)_x - u_1 (v_2)_x, \\
(u_1)_{t_2} &= (v_1)_y + v_1 (u_1)_x - u_1 (v_1)_x,
\end{aligned}$$

with constraints

$$(v_1)_z = \left[\frac{1}{2} u_2 (u_3)^{-\frac{1}{2}}\right]_z - \left[\frac{1}{2} (u_3)^{\frac{1}{2}}\right]_x, \quad v_2 = (u_3)^{\frac{1}{2}}.$$

Examples

(2+1)-dimensional reductions

The reduction $z = 0$ leads to

$$v_2 = const = 1, \quad u_3 = const = 1, \quad v_1 = \frac{2}{3}u_2$$

and hence

$$(u_2)_{t_2} = 2(u_1)_x - \frac{2}{3}u_2(u_2)_x,$$
$$(u_1)_{t_2} = \frac{2}{3}[(u_2)_y + u_2(u_1)_x - u_1(u_2)_x].$$

Examples

(2+1)-dimensional reductions

The reduction $z = 0$ leads to

$$v_2 = const = 1, \quad u_3 = const = 1, \quad v_1 = \frac{2}{3}u_2$$

and hence

$$(u_2)_{t_2} = 2(u_1)_x - \frac{2}{3}u_2(u_2)_x,$$
$$(u_1)_{t_2} = \frac{2}{3}[(u_2)_y + u_2(u_1)_x - u_1(u_2)_x].$$

On the other hand, the reduction $x = 0$ leads to

$$(u_3)_{t_2} = u_2(v_1)_z - v_2(u_1)_z,$$
$$(u_2)_{t_2} = (v_2)_y,$$
$$(u_1)_{t_2} = (v_1)_y,$$

where

$$v_2 = (u_3)^{\frac{1}{2}}, \quad v_1 = \frac{1}{2}u_2(u_3)^{-\frac{1}{2}}.$$

Examples

(1+1)-dimensional reductions

The further reduction by $y = 0$, leads to $(1+1)$-dimensional system

$$(u_2)_{t_2} = 2(u_1)_x - \frac{2}{3} u_2 (u_2)_x,$$
$$(u_1)_{t_2} = \frac{2}{3}[u_2(u_1)_x - u_1(u_2)_x],$$

and

$$(u_3)_{t_2} = \frac{1}{2}\left[(u_3)^{-\frac{1}{2}}\right]_z .$$

with constraint

$$u_1 = const = 0, \quad u_2 = const = 1$$

Thank you for the attention