Integrable background geometries: review and outlook

David M. J. Calderbank

University of Bath

Durham, 2016

SIGMA **10** (2014), arXiv:1403.3471

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Apologia

- (Hitchin) Integrability is like jazz: if you have to ask what it is, you will never know. This makes me tone deaf.
- Attempt to collect and organise examples (taxonomy).
- Focus on integrable systems related to twistor theory (Ward).
- Mostly old ideas (original article on webpage in 2001).
- Very many contributors over the years... can only name a few.

Plan

Aim to address a key issue: what is the geometry of reductions of SDYM? Main contentions:

- It does not suffice to restrict to SDYM on flat $\mathbb{R}^4$
- Instead SDYM and reductions are defined over *background geometries* in dimension ≤ 4
- Background geometries are themselves solutions of (dispersionless) integrable systems

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Proceed by dimension (and history)

4. Dimension four: selfduality, twistor theory and integrability
3. Dimension three: Einstein–Weyl geometry and monopoles
2. Dimension two: spinor vortices and Higgs bundles
1. Dimension one: Riccati spaces and isomonodromy
0. Null reductions: projective surfaces and twisted flat pencils
-1. Higher dimensions: quaternionic geometries and reductions

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4. Selfdual Yang–Mills (SDYM)

G-connection D on a vector bundle V over M = affine 4-space

- $TM = M \times \mathbb{C}^4$ with coordinate vector fields $\partial_1, \partial_2, \partial_3, \partial_4$
- Trivialize $V \cong M \times \mathbb{C}^k$, $D_i = \partial_i + A_i$, for $A_i \in \mathfrak{g} \subseteq \mathrm{End}(\mathbb{C}^k)$
- Curvature $F_{ij} = -F_{ji} = [D_i, D_j] = \partial_i A_j - \partial_j A_i + [A_i, A_j]$

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SDYM equations: $F_{12} = 0 = F_{34}$, $F_{14} = F_{23}$

$\Leftrightarrow [L_1, L_2] = 0$ for *Lax pair* $L_1 = D_1 + \zeta D_3$, $L_2 = D_2 + \zeta D_4$

- $\partial_1 + \zeta\partial_3$, $\partial_2 + \zeta\partial_4$ commute and span null planes for (conformal class of) metric $dx_1\, dx_4 - dx_2\, dx_3$

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- $\partial_1 + \zeta\partial_3$, $\partial_2 + \zeta\partial_4$ commute and span null planes for (conformal class of) metric $dx_1\, dx_4 - dx_2\, dx_3$
- Can also view $\mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ with $\partial_1 + \zeta\partial_3 = (1, 0) \otimes (1, \zeta)$ and $\partial_2 + \zeta\partial_4 = (0, 1) \otimes (1, \zeta)$; null planes are $\mathbb{C}^2 \otimes (1, \zeta)$
- Take $\zeta \in \mathbb{C}P^1 = \mathbb{C} \cup \infty$: have rank 2 integrable distribution on $M \times \mathbb{C}P^1$; twistor space $\mathbb{T}$ is 3-diml space of leaves
- Have $M \xleftarrow{\pi} M \times \mathbb{C}P^1 \xrightarrow{\alpha} \mathbb{T}$ and $\pi^* V \cong \alpha^* W$ for vector bundle $W \to \mathbb{T}$ s.t. $\forall x \in M$, W is trivial on $\alpha(\pi^{-1}(x))$

4. Selfdual 4-manifolds and their twistor spaces

Generalize to M with $TM = E \otimes H$, for $E \to M, H \to M$ rank 2

- Locally $E \cong M \times \mathbb{C}^2$, $H \cong M \times \mathbb{C}^2$ and have vector fields $V_1 + \zeta V_3 \leftrightarrow (1,0) \otimes (1,\zeta)$ and $V_2 + \zeta V_4 \leftrightarrow (0,1) \otimes (1,\zeta)$
- Key requirement: there are lifts of these vector fields to $P(H) \cong M \times \mathbb{C}P^1$ which span an integrable distribution

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- Twistor space Z is space of leaves, so have double fibration

$$\begin{array}{ccc} & P(H) & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\alpha} \\ M^4 & & Z^3 \end{array}$$

- Key property: M is moduli space of "twistor lines"; for $x \in M$, $\alpha(\pi^{-1}(x)) \cong \mathbb{C}P^1$ in Z, with normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^2$

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- Can solve generalization of SDYM to $V \to M$ via $W \to Z$ with $W|_{\alpha(\pi^{-1}(x))}$ trivial on each twistor line
- If this works, say M (and E, H) is an *integrable background geometry* (IBG) for SDYM

4. Smörgåsbord of recipes

Heavenly hermeneutics

- Commuting independent vector fields $V_1 + \zeta V_3$ and $V_2 + \zeta V_4$ on M make it into an IBG (Mason–Newman, Joyce, Dunajski)
- If V_j are volume preserving (divergence-free), M carries a selfdual vacuum Einstein (SDVE) metric (Plebanski).

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- Gibbons–Hawking: can construct SDVE metrics from solutions of $U(1)$ monopole equations $*df = dA$ on $\mathbb{R}^3$
- Ward: can also use solutions of Hitchin equations on $\mathbb{R}^2$ or Nahm equations on $\mathbb{R}$, provided gauge group is contained in volume preserving diffeomorphisms of Σ^2 or Σ^3 respectively

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Bait and switch map (aka "lets twist again"). Suppose:

- M is an IBG
- G acts freely on M preserving structure of $TM = E \otimes H$
- P is a principal $\tilde{G}$-bundle, with $\dim \tilde{G} = \dim G$
- P admits a $\tilde{G}$-connection solving SDYM

Then P/G is an IBG (with a free action of $\tilde{G}$ preserving structure).

4. What is going on?

- Suppose G acts freely on M, an IBG (for SDYM)
- SDYM on M reduces to a gauge field equation on $Q = M/G$
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Such a coherent picture cannot be obtained without admitting the most general IBGs. In particular, for SDYM, we must admit that the relevant lifts of $V_1 + \zeta V_3$ and $V_2 + \zeta V_4$ differ from the coordinate lifts by multiples of ∂_ζ, i.e., derivatices with respect to the spectral parameter. The appearance of such derivatives is a hallmark of dispersionless integrable systems: IBGs belong here.

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Rest of the talk: illustrate this in each dimension.

3. Einstein–Weyl spaces and Jones–Tod constructions

- If an IBG M (for SDYM) odmits a free nondegenerate conformal $U(1)$ action then $B = M/U(1)$ is an Einstein–Weyl 3-manifold, i.e., $\mathrm{Ric}^{\nabla}_{o} = 0$ for a torsion-free conformal connection ∇ on B
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- Conversely if (A, ϕ) is a solution of the monopole equation on (B, ∇), where the gauge group is a subgroup of the diffeomorphisms of a 1-manifold, then the associated bundle of 1-manifolds is an IBG for SDYM
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- Constructions are mutually inverse when gauge group is $U(1)$
- Special cases: Gibbons–Hawking; LeBrun hyperbolic Ansatz
- When M is SVDE and the $U(1)$ action is isometric, B is given by a solution of $SU(\infty)$ Toda equation $u_{xx} + u_{yy} + (e^u)_{zz} = 0$, and the $U(1)$ monopole equation reduces to its linearization. However, only the solution u_z yields a SDVE metric.

3. Minitwistor theory of Einstein–Weyl spaces

- B, ∇ Einstein–Weyl implies that $TB \cong S^2 H$ for a rank 2 bundle $H \to B$, and $P(H) \cong B \times \mathbb{C}P^1$ has a rank 2 integrable distribution (Lax pair)
- Thus have a "mini" twistor correspondence (double fibration)

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- Thus B is moduli space of minitwistor lines $\alpha(\pi^{-1}(x)) \cong \mathbb{C}P^1$ in S, which have normal bundle $\mathcal{O}(2)$
- Solutions of the Bogomolny monopole equations correspond to holomorphic vector bundles on S which are trivial on minitwistor lines

2. Spinor vortices and generalized Hitchin equations

- If an IBG M (for SDYM) admits a free nondegenerate conformal action of a 2-dimensional Lie group G, then $\Sigma = M/G$ is a conformal surface carrying a solution (C, ψ, ∇) on a spinorial version of the vortex equations:

$$\overline{\partial}^{\nabla} C = 0 \qquad \overline{\partial}^{\nabla} \psi = -3C\overline{\psi} \qquad s^{\nabla} = \psi\overline{\psi} - 2C\overline{C},$$

- The symmetry reduction of SDYM equation to Σ is a background-coupled generalization of Hitchin's equations for Higgs pairs (A, Φ)

$$F^A - [\Phi, \overline{\Phi}] = \psi \wedge \overline{\Phi} + \overline{\psi} \wedge \Phi$$
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- Conversely can construct M from solutions of generalized Hitchin equations on Σ with gauge group a subgroup of diffeomorphisms of a 2-manifold
- Have a twistor correspondence but twistor space is a non-Hausdorff complex curve

1. Riccati spaces and generalized Nahm equations

- If an IBG M (for SDYM) admits a free nondegenerate conformal action of a 3-dimensional Lie group G, then $\Gamma = M/G$ is a curve carrying a solution B of the Riccati equation
$$\partial_t B = (B^2)_0$$
for symmetric traceless 3×3 matrices

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- If an IBG M (for SDYM) admits a free nondegenerate conformal action of a 3-dimensional Lie group G, then $\Gamma = M/G$ is a curve carrying a solution B of the Riccati equation

$$\partial_t B = (B^2)_0$$

  for symmetric traceless 3×3 matrices

- The symmetry reduction of SDYM equation to Γ is a background-coupled generalization of Nahm's equations

$$\partial_t \Phi_i - \frac{1}{2} \sum_{i,j,k=1}^{3} \varepsilon_{ijk} [\Phi_j, \Phi_k] = \sum_{j=1}^{3} B_{ij} \Phi_j$$

- Conversely can construct M from solutions of generalized Hitchin equations on Γ with gauge group a subgroup of diffeomorphisms of a 3-manifold

1. Geometry of Riccati equation

- Really B is a section of $\mathrm{End}(\mathcal{E})$ which is trace-free and symmetric with respect to an inner product on $E \cong \Gamma \times \mathbb{C}^3$
- For an orthonormal frame e_1, e_2, e_3 of E, let

$$e_\zeta = \frac{1}{2}(\zeta^2 + 1)e_1 + i\zeta e_2 + \frac{i}{2}(\zeta^2 - 1)e_3$$

  This is null with respect to inner product: $\langle e_\zeta, e_\zeta \rangle = 0$. Thus ζ parametrizes the conic $\langle v, v \rangle = 0$ in $P(E) \cong \Gamma \times \mathbb{C}P^2$

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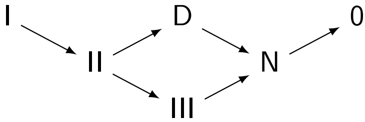
- $\langle v + \lambda Bv, v \rangle = 0$ defines a pencil (one parameter family) of conics in $P(E) \cong \Gamma \times \mathbb{C}P^2$. Base locus (intersection) consists of four points (counted with multiplicity), classified by



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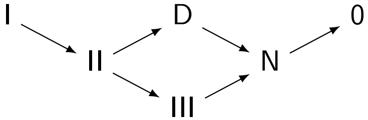

- Generalized Nahm equation has Lax pair

$$\langle B(e_\zeta), e_\zeta \rangle \partial_\zeta + \Phi(e_\zeta) \qquad \partial_t + \langle B(e_\zeta), e'_\zeta \rangle \partial_\zeta + \Phi(e'_\zeta)$$

  Interpretation: $\partial_\zeta + \Phi(e_\zeta)/\langle B(e_\zeta), e_\zeta \rangle$ is isomonodromic.

0. Null reductions

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- Can also consider null reductions; most interesting cases are reductions to 2 dimensions.

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$$\mathrm{d}^{\nabla}\psi = 0, \qquad \mathrm{d}^{\nabla}\chi = 0, \qquad F^{\nabla} = \chi \wedge \psi.$$

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For $\phi = \psi = 0$ this means $d^A + \lambda\Phi$ is flat.
- If G is two dimensional with non-twistorial null surfaces as orbits then M/G has a projective structure $[\nabla]$ (twistor space is dual surface, and twistor lines have normal bundle $\mathcal{O}(1)$). SDYM reduces to solutions (A, Φ) of
$$\nabla^A\Phi = \tfrac{1}{2}\mathrm{d}^{\nabla, A}\Phi.$$

-1. Higher dimensions

- Higher degree Lax pairs are obtained by generalizing M^4 to M^{2k}, where $TM = E \otimes H$ with H rank 2 and E rank k.
- Have a double fibration

$$\begin{array}{ccc} & P(H) & \\ {}^{\pi}\swarrow & & \searrow^{\alpha} \\ M^{2k} & & Z^{k+1} \end{array}$$

where twistor lines $\alpha(\pi^{-1}(x)) \cong \mathbb{C}P^1$ have normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^k$. When k is even, M is a quaternionic manifold.

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  where twistor lines $\alpha(\pi^{-1}(x)) \cong \mathbb{C}P^1$ have normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^k$. When k is even, M is a quaternionic manifold.
- Reductions are complicated, but may be classified in terms of sheaves on $\mathbb{C}P^1$.
- Example of reduction for $k = 2m$ even is B^{3m} with $TB = V \otimes S^2H$ where V has rank m, H has rank 2. Twistor lines have normal bundle $\mathcal{O}(2) \otimes \mathbb{C}^m$.
- Real point however is that all these geometries have Lax distributions with geometric interpretation.