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THE BIHAMILTONIAN FORMALISM

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My task today is to present an overview of the bihamiltonian formalism from its birth, at the end of 70's, to the present application to soliton equations and beyond. It is a wide subject, with a long history full of detours. For this reason, I decided not to follow the historical pattern, and to adopt a more straight-forward and modern "axiomatic approach".

It has the advantage to clearly point out that the whole formalism rests on two primitive concepts and two basic principles, permitting in this way a great economy of thinking. Furthermore it allows to control more directly the far reaching consequences of the formalism.

My plan is to start from the primitive concepts of Poisson pencil and bihamiltonian system and to arrive to the so-called WDVV equations of Topological Field theories, passing through the theory of soliton equations, of Gelfand-Dikii type, thus providing a rather broad picture of the content and of the applica-tions of the bihamiltonian formalism.

Henceforth, my presentation will be splitted in three parts, dealing with:

1. the theoretical background
2. the applications to soliton equations
3. the far reaching and remote consequences

In the first part, I will present the basic principles supporting the whole theory. They are:

1. The Marsden-Ratiu reduction theorem

2. The Gelfand-Zakharevich "moment map"

In the second part I will use these principles to derive the soliton equations, in the specific setting of the Gelfand-Dikii scheme.

In the third part, finally, I will derive from the above analysis of the Gelfand-Dikii equations the concept of Lenard's square of exact 1-forms, which I will use to deduce the WDVV equations.

In discussing these topics I will be rather informal and selective. My unique purpose is to build a coherent picture of the "conceptual bridges" interrelating these different areas, and to explain where they come from according to the bihamiltonian perspective. Therefore I will be guilty both of skipping the proofs, and of omitting the proper references to other works describing the same landscape from different viewpoints.

PART 1 : The theoretical background

As said before, there are two primitive concepts at the beginning of the bihamiltonian formalism: the concepts of Poisson pencil and of bihamiltonian vector field.

They are rather simple.

A Poisson pencil is a pair of Poisson bivectors $P: T^*M \to TM$ and $Q: T^*M \to TM$ on a manifold M, such that any their linear combination

$$P(\lambda) = P + \lambda Q \qquad \lambda \in \mathbb{R}$$

is still a Poisson bivector.

A vector field $X : M \to TM$ on a bihamilto- nian manifold M is bihamiltonian if it can be written in the form

$$X = P\,dh = Q\,dk$$

for a suitable choice of the functions h and k.

A first remarkable property of Poisson pencils is shown by the MR reduction theorem.

The main ingredient of this theorem are the Casimir functions of one of the two Poisson bivector, let us say P. The level surfaces of these functions are the "symplectic leaves" of P. They define a first foliation of the manifold M in symplectic leaves.

The presence of the second Poisson bivector Q allows to define a second foliation inside the symplectic leaves of P. This foliation is spanned by the vector fields associated by Q to the Casimir functions of P:

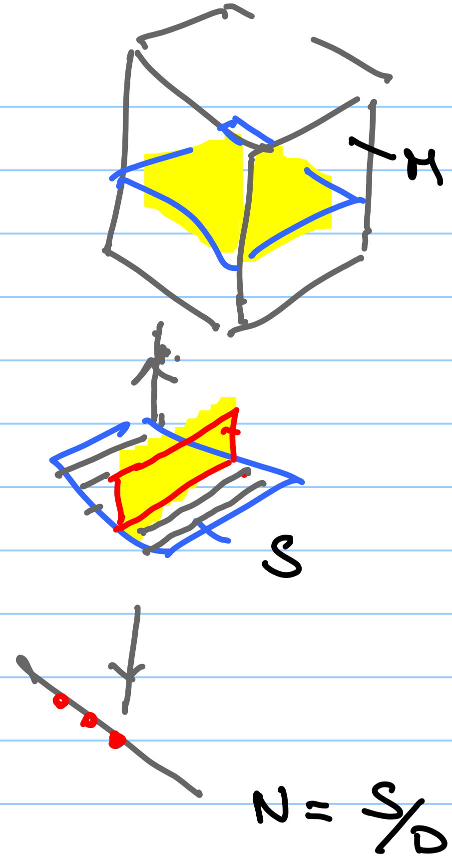
$$E = \langle Q\,dC \mid P\,dC = 0 \rangle$$

The geometry is shown by the following picture:

S is a leaf of P in M

D is a leaf of E in S

N is the space of leaves of E on S

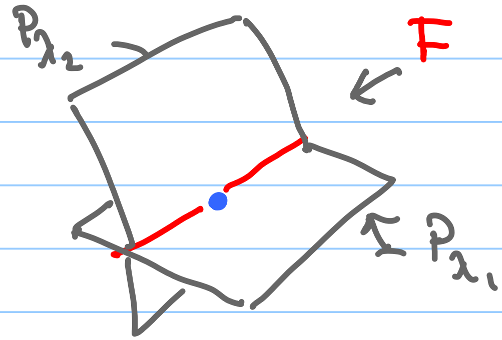


The theorem of Marsden-Ratiu tells us that the quotient space $N = S/_D$ is again a bihamiltonian manifold, and it gives a procedure to compute the Poisson pencil on this reduced space.

A second important element of the geometry of a bihamiltonian manifold has been identified by Gelfand and Zakharevich in their paper

"On the local geometry of a bihamiltonian structure".

It is a new foliation F determined by the intersection of the symplectic leaves of P_λ :



The study of this foliation is a hard problem, but here it will be enough to consider the case of Poisson pencils of Kronecker type. Roughly speaking, they are pencils of Poisson tensors of maximal rank $2n$ on a manifold of dimension $(2n+1)$. In this case GZ have shown that:

1. The Casimir function of P_λ depends polynomially on λ, and it can be written in the form

$$C(\lambda) = C_0 + C_1\lambda + \dots + C_n\lambda^n$$

2. The coefficients $(c_0, c_1, \ldots, c_n)$ are a set of $(n+1)$ (generically) independent functions that are in involution

3. Their level sets are the leaves of the foliation F

4. Their Hamiltonian vector fields span the leaves of F

Therefore the foliation F is Lagrangian, and the

Hamiltonian vector fields $(X_{c_1}, X_{c_2}, \dots, X_{c_n})$ form admitting $\mathcal{F}$ as its Lagrangian foliation.

The MR reduction theorem and the GZ result on the local structure of a bihamiltonian manifold are all we need to pursue our study of the applications of the bihamiltonian formalism. The two ideas to keep in mind are:

1. The MR theorem is a source of interesting Poisson pencils

2. The GZ theorem is a mechanism to extract hierarchies of integrable biha-miltonian vector fields out of a suitable class of Poisson pencils.

PART II : Applications to soliton equations

The link with the theory of soliton is provided by the remark that "loop algebras" are examples of bihamiltonian manifolds, when endowed with the canonical central extension of the Lie-Kirillov-Poisson structure. This means that "loop algebras" are a suitable place where to put the MR and GZ theorems at work.

I shall now describe the outcomes of this approach in the simplest case

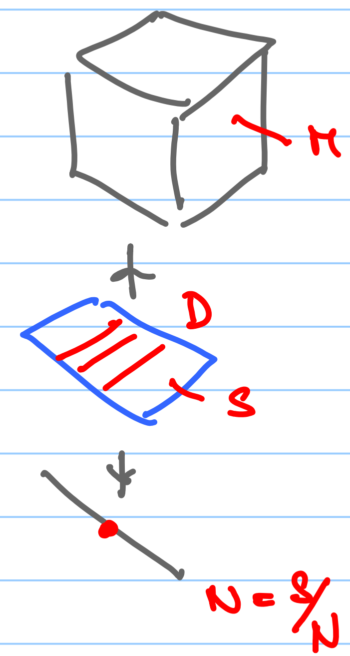
$$\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$$

referring to Lectures Notes in Physics 495 (1997), 256-296 for the proofs and further explanations which are omitted here for brevity.

MR reduction

The original phase space is the space of 2x2 traceless matrices

$$S = \begin{pmatrix} p(x) & r(x) \\ q(x) & -p(x) \end{pmatrix}$$



endowed with the Poisson pencil

$$\dot{p} = \left(\tfrac{1}{2}\alpha_x + q\beta - r\gamma\right) - \lambda\beta$$

$$\dot{q} = \left(\gamma_x + 2q\gamma - q\alpha\right) + \lambda\alpha$$

$$\dot{r} = \left(\beta_x - 2p\beta + r\alpha\right)$$

where (p,q,r) are the coordinates of the point S, $(\dot{p},\dot{q},\dot{r})$ are the components of a tangent vector at S, and (α,β,γ) are components of a covector.

The intermediate symplectic leaf is defined by the constraint

$\mathcal{S}$:

$r=1$.

The leaves of the auxiliary foliation $\mathcal{D}$ on $\mathcal{S}$ are formed by the pairs of functions (p,q) which satisfy the differential equation

$$\mathcal{D}: \qquad p_x + p^2 + q = u$$

where u is any given function, and the quotient space $N = \mathcal{S}/\mathcal{D}$ is therefore the space of the function $u(x)$.

By applying the HR reduction technique one obtains on N the Poisson pencil

$$\dot{u} = -\frac{1}{2} v_{xxx} + 2(u+\lambda)v_x + u_x v$$

of the KdV theory.

The GZ theorem

The next step is to apply the GZ type of analysis to find the Casimir function of the above Poisson pencil. It turns out that, in this infinite-dimensional setting, the Casimir function should be represented as a Laurent series

$$h(z) = z + \frac{h_1}{z} + \frac{h_2}{z^2} + \cdots$$

and not longer as a polynomial. As shown in the reference given above, this series verifies the

Riccati equation

$$h_x + h^2 + u = z^2$$

(where $z = \sqrt{\lambda}$), and the integrable bihamiltonian hierarchy defined by Gelfand and Zakharevich coincides with the KdV.

Of course there are many other possible descriptions of the KdV hierarchy. That presented above is not the more effective from a computational point of view, but it has the advantage to clearly point out that a great part of the theory of soliton equations rests on two simple ideas:

1. The MR reduction of a Poisson pencil
2. The GZ analysis of the Casimir functions of this pencil.

In the next part I want to point out another remarkable feature of the equations obtained by the above procedure.

PART III : Far reaching consequences

Nothing changes if one replaces $\mathfrak{sl}(2,\mathbb{R})$ by $\mathfrak{sl}(3,\mathbb{R})$ or by $\mathfrak{sl}(4,\mathbb{R})$, and so forth. The two steps procedure (first MR reduction, and then GZ analysis) continues to hold, practically unchanged. I am now interested in looking to one of these generalization to bring to light a peculiar property of the equations of motion constructed by this procedure, leading to the WDVV equations.

I consider the $sl(4,\mathbb{R})$ case. Skipping over all the intermediate steps, I go directly to the final result: the bihamiltonian equations associated with the Casimir function of the Poisson pencil. The only thing one has to know is that the reduced phase space may be parametrized by the first three components of the Casimir function, which has still the form

$$h(z) = z + \frac{h_1}{z} + \frac{h_2}{z^2} + \frac{h_3}{z^3} + \cdots$$

It turns out that, in these coordinates, the first three equations have the following form:

$$\frac{\partial}{\partial t_1}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

$$\frac{\partial}{\partial t_2}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = 2\begin{pmatrix} 0 & 0 & 1 \\ h_1 & 0 & 1 \\ h_2 & h_1 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} + \cdots\cdots$$

$$\frac{\partial}{\partial t_3}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = 3\begin{pmatrix} 0 & 0 & 1 \\ h_2 & h_1 & 0 \\ h_1^2 & h_2 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} + \cdots$$

up to terms containing higher-order x-derivatives of the field functions. So the above equations represent the dispersionless limit of the GD3 equations.

These equations are completely characterized by the matrices of the coefficients. More intrinsically they can be viewed as the matrices of the components of three tensor fields of type (1,1) on $\mathbb{R}^3$ parametrized by the coordinates (h_1, h_2, h_3). Let we call them the "primary fields" (K_1, K_2, K_3). The first tensor field K_1 is the identity. The other two are defined by

$$K_2\, dh_1 = dh_2 \qquad K_3\, dh_1 = dh_3$$

$$K_2\, dh_2 = dh_3 + h_1\, dh_1 \qquad K_3\, dh_2 = h_1\, dh_2 + h_2\, dh_1$$

$$K_2\, dh_3 = h_1\, dh_2 + h_2\, dh_1 \qquad K_3\, dh_3 = h_1^2\, dh_1 + h_2\, dh_2$$

One may check that they are commuting tensor fields with vanishing Haantjes torsion, and that, more importantly, they enjoy the property that all the second iterated 1-forms

$\kappa_a \kappa_b \, dh_1$, , for $a,b=1,2,3$, are exact. So they define a symmetric square of exact 1-forms. The table of the corresponding potentials is

h_1	h_2	h_3
h_2	$h_3 + \frac{1}{2} h_1^2$	$h_1 h_2$
h_3	$h_1 h_2$	$\frac{1}{2} h_2^2 + \frac{1}{3} h_1^3$

It is a symmetric matrix. I call it the Leonard's square of the potentials associated with the primary fields (K_1, K_2, K_3). The relation of this square with the WDVV equations is quite simple.

Take any solution $[\,h_1(\tau_1\tau_2\tau_3),\ h_2(\tau_1\tau_2\tau_3),$ $h_3(\tau_1\tau_2\tau_3)\,]$ of the equations

$$\frac{\partial}{\partial \tau_2}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ h_1 & 0 & 1 \\ h_2 & h_1 & 0 \end{pmatrix} \frac{\partial}{\partial \tau_1}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

$$\frac{\partial}{\partial \tau_3}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ h_2 & h_1 & 0 \\ h_1^2 & h_2 & 0 \end{pmatrix} \frac{\partial}{\partial \tau_1}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

that is of the dispersionless GD3 equations written in terms of the normalized times $\tau_1 = t_1$, $\tau_2 = \frac{1}{2} t_2$, $\tau_3 = \frac{1}{3} t_3$.

These equations are compatible due to the properties.

of the tensor fields K_1, K_2, K_3. Hence there exists a common solution, as desired. Look at the solution as a change of coordinates on our manifold, from the old coordinates (h_1, h_2, h_3) to the new coordinates (z_1, z_2, z_3). Write the entries of the Lenard's square in the new coordinates.

Claim: For any solution of the renormalized dispersionless GD 3 equations, the matrix of the potentials of the Lenard's square is the Hessian matrix of a function $F(\tau_1, \tau_2, \tau_3)$. This function satisfies the WDVV equations automatically.

Example. A particularly simple solution of the above equations is given by

$$h_1 = \tau_3 \qquad h_2 = \tau_2 \qquad h_3 = \tau_1$$

as one may readily check. To this solution corresponds a very simple polynomial function $F(\tau_1, \tau_2, \tau_3)$. It is the "prepotential" associated with the space of orbits of Coxeter group of type A_3. It is one of the simplest solutions of the WDVV equations.

What has been shown here, in the simple case of the Lie algebra A_3, it is true in general. It can be shown conversely that any solution of the WDVV equations is associated to a Lenard's square of 1-forms, in the way illustrated before $\blacksquare$