Somos sequences in algebra, geometry & number theory

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Tuesday 2nd August 2016

Outline

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3. Discrete BKP (Miwa, cube) equation
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History

Somos ('80s) noticed that, choosing six initial 1s, the recurrence

$$\tau_{n+6}\tau_n = \alpha\,\tau_{n+5}\tau_{n+1} + \beta\,\tau_{n+4}\tau_{n+2} + \gamma\,\tau_{n+3}^2$$

with coefficients $\alpha = \beta = \gamma = 1$ produces the sequence

$$1, 1, 1, 1, 1, 1, 3, 5, 9, 23, 75, 421, 1103, 5047, 41783, 281527, \ldots$$

(A006722 in Sloane's). The key observation was the Laurent property, i.e.

$$\tau_n \in \mathbb{Z}[\tau_0^{\pm 1}, \tau_1^{\pm 1}, \ldots, \tau_5^{\pm 1}, \alpha, \beta, \gamma] \qquad \forall n \in \mathbb{Z}.$$

This property holds for order $k \leq 7$ only in

$$\mathrm{Somos-k:} \qquad \tau_{n+k}\tau_n = \sum_{j=1}^{\lfloor k/2 \rfloor} \alpha_j\,\tau_{n+k-j}\tau_{n+j},$$

if all coefficients α_j are non-zero.

Prehistory: elliptic divisibility sequences (EDS)

A special case of Somos-4:

$$\tau_{n+4}\tau_n = (\tau_2)^2\tau_{n+3}\tau_{n+1} - \tau_3\tau_1(\tau_{n+2})^2.$$

EDS (Ward): choose $\tau_1 = 1, \tau_2, \tau_3, \tau_4 \in \mathbb{Z}$, with $\tau_2|\tau_4$. Then

$$\tau_n \in \mathbb{Z} \quad \text{with} \quad \tau_n|\tau_m \quad \text{whenever} \quad n|m.$$

- Term $\tau_n \leftrightarrow n \cdot P \in E$, elliptic curve (cf. division polynomials)
- Generate large primes (Chudnovsky$\times$2) ...
- ... but only finitely many (Everest-Miller-Stephens)
- Hilbert's 10th problem undecidable over $\mathbb{Z}[S^{-1}]$ (Poonen)
- Cryptographic applications (Shipsey, Swart, Stange)

Analytic solution of Somos-4

For the general Somos-4 recurrence,

$$\tau_{n+4}\tau_n = \alpha\,\tau_{n+3}\tau_{n+1} + \beta\,(\tau_{n+2})^2,$$

the solution of the initial value problem has the form

$$\tau_n = AB^n\,\frac{\sigma(v_0 + nv)}{\sigma(v)^{n^2}},$$

for suitable $A, B \in \mathbb{C}^*$, $v_0, v \in \mathbb{C}$ mod Λ, where $\sigma(z;\Lambda)$ denotes the Weierstrass sigma-function associated with the elliptic curve

$$E: \qquad y^2 = 4x^3 - g_2x - g_3$$

that is birationally equivalent to the biquadratic defined by

$$H = u_0u_1 + \frac{\alpha}{u_0} + \frac{\alpha}{u_1} + \frac{\beta}{u_0u_1}, \qquad \text{with} \qquad u_n = \frac{\tau_{n+2}\,\tau_n}{(\tau_{n+1})^2}.$$

Analytic solution of Somos-4 (continued)

Main idea: 2D symplectic map for u_n, with first integral H.
Explicit formulae:

$$g_2 = \frac{H^4 - 8\beta H^2 - 24\alpha^2 H + 16\beta^2}{12\alpha^2},$$

$$g_3 = -\frac{H^6 - 12\beta H^4 - 36\alpha^2 H^3 + 48\beta^2 H^2 + 144\alpha^2\beta H + 216\alpha^4 - 64\beta^3}{216\alpha^3},$$

$$v \in \mathbb{C} \bmod \Lambda \quad \leftrightarrow \quad \left(\frac{H^2/4 - \beta}{3\alpha}, \sqrt{\alpha} \right) \in E.$$

Example: Sequence of points $P_0 + n \cdot P \in E$ corresponding to

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, \ldots \quad (A006720 \,\mathrm{in\ OEIS})$$

has $v_0 \leftrightarrow P_0 = (-1, 1)$, $v \leftrightarrow P = (1, 1)$ on $E: \ y^2 = 4x^3 - 4x + 1$.

Discrete KP equation

Partial difference equation in three independent variables (n_1, n_2, n_3):

$$T_1 T_{-1} = T_2 T_{-2} + T_3 T_{-3}.$$

Notation: $T = T(n_1, n_2, n_3)$ with $T_{\pm 1} = T(n_1 \pm 1, n_2, n_3)$ etc.

- Integrability (Lax pair, solitons, algebro-geometric solutions)
- Laurent property (Fomin & Zelevinsky)
- Cluster structure (Di Francesco & Kedem, Okubo)

Plane wave solutions

Consider plane wave reductions with a quadratic exponential gauge:

$$T(n_1, n_2, n_3) = a_1^{n_1^2} a_2^{n_2^2} a_3^{n_3^2}\, \tau(n),$$

where distinct δ_j (either all integer or all half-integer) are chosen such that

$$n = n_0 + \delta_1 n_1 + \delta_2 n_2 + \delta_3 n_3, \qquad \delta_1 > \max(\delta_2, \delta_3).$$

Set $\alpha = a_2^2/a_1^2$, $\beta = a_3^2/a_1^2$, and $\tau_n = \tau(n)$, to find

$$\tau_{n+\delta_1}\tau_{n-\delta_1} = \alpha\, \tau_{n+\delta_2}\tau_{n-\delta_2} + \beta\, \tau_{n+\delta_3}\tau_{n-\delta_3},$$

which is a 3-term Gale-Robinson/Somos type recurrence, with vanishing algebraic entropy (Mase).

Example: $(\delta_1, \delta_2, \delta_3) = (2, 1, 0)$ gives Somos-4:

$$\tau_{n+4}\tau_n = \alpha\, \tau_{n+3}\tau_{n+1} + \beta\, \tau_{n+2}^2.$$

Reductions of discrete KP Lax pair

Scalar linear system for $\Psi = \Psi(n_1, n_2, n_3)$:

$$\begin{array}{rcll} \Psi_{-1,2} + F\,\Psi_{2,-3} & = & \Psi, & \text{with} \quad F = T_{-1,3}\,T_{2,-3}/(TT_{-1,2}); \\ G\,\Psi_{1,2} + \Psi_{2,3} & = & \Psi, & \text{with} \quad G = T_{-1,3}\,T_{1,2}/(TT_{2,3}). \end{array}$$

Compatibility condition:

$$R_{1,-3} = R, \qquad R = (T_1\,T_{-1} - T_3\,T_{-3})/(T_2\,T_{-2}).$$

Now set $\Psi(n_1, n_2, n_3) = \lambda_1^{n_1}\lambda_2^{n_2}\lambda_3^{n_3}\phi(n)$, and apply the plane wave reduction, taking $\zeta = \lambda_2\lambda_1^{-1}$, $\xi = (a_1^2\lambda_1\lambda_2)^{-1}$, $\lambda_3 = a_3^2\lambda_2$ to find

$$\begin{array}{rcl} \phi_{n+\delta_1-\delta_2} - X_n\,\phi_{n+\delta_1-\delta_3} & = & \zeta\phi_n, \\ Y_n\,\phi_{n+\delta_1+\delta_2} + \beta\zeta\,\phi_{n+\delta_2+\delta_3} & = & \xi\phi_n, \end{array}$$

with $X_n = \tau_{n+2\delta_1-2\delta_3}\tau_{n+\delta_1-\delta_2}/(\tau_{n+2\delta_1-\delta_2-\delta_3}\tau_{n+\delta_1-\delta_3})$, and $Y_n = \tau_{n+2\delta_1+\delta_2-\delta_3}\tau_n/(\tau_{n+2\delta_1-\delta_2-\delta_3}\tau_{n+\delta_1-\delta_3})$, which is a scalar Lax pair with two spectral parameters ζ, ξ.

Extra freedom: The coefficient α does not appear in the scalar Lax pair

$$\begin{aligned} \phi_{n+\delta_1-\delta_2} - X_n\,\phi_{n+\delta_1-\delta_3} &= \zeta\phi_n, \\ Y_n\,\phi_{n+\delta_1+\delta_2} + \beta\zeta\,\phi_{n+\delta_2+\delta_3} &= \xi\phi_n. \end{aligned}$$

The general compatibility condition has a periodic coefficient:

$$\tau_{n+\delta_1}\tau_{n-\delta_1} = \alpha_n\,\tau_{n+\delta_2}\tau_{n-\delta_2} + \beta\,\tau_{n+\delta_3}\tau_{n-\delta_3}, \qquad \alpha_{n+\delta_1-\delta_3} = \alpha_n,$$

The scalar Lax pair is equivalent to a matrix linear system of size $K = \max(\delta_1 - \delta_2, \delta_1 - \delta_3)$, of the form

$$\mathbf{L}_n(\zeta)\,\mathbf{\Phi}_n = \xi\mathbf{\Phi}_n, \qquad \mathbf{\Phi}_{n+1} = \mathbf{M}_n(\zeta)\,\mathbf{\Phi}_n.$$

This yields the discrete Lax equation

$$\mathbf{L}_{n+1}\mathbf{M}_n = \mathbf{M}_n\mathbf{L}_n,$$

preserving the spectral curve

$$P(\zeta,\xi) \equiv \det(\mathbf{L}_n(\zeta) - \xi\mathbf{1}) = 0.$$

Cluster algebras

Somos/Gale-Robinson recurrences are a special case of T-systems, which arise from mutations in a cluster algebra, defined as follows:-
Quiver Q (no 1- or 2-cycles) $\leftrightarrow B = (b_{j\ell}) \in \mathrm{Mat}_r(\mathbb{Z})$, skew-symmetric.
Matrix mutation: $B \mapsto B' = \mu_k\, B = (b'_{j\ell})$, where

$$b'_{j\ell} = \begin{cases} -b_{j\ell}, & j = k \quad \text{or} \quad \ell = k, \\ b_{j\ell} + [-b_{jk}]_+ b_{k\ell} + b_{jk}[b_{k\ell}]_+ & \text{otherwise}, \end{cases}$$

with $[b]_+ = \max(b, 0)$. Cluster mutation: $\mathbf{x} = (x_j) \mapsto \mathbf{x}' = \mu_k(\mathbf{x}) = (x'_j)$, where $x'_j = x_j$ for $j \neq k$ and

$$x'_k\, x_k = \prod_{j=1}^{r} x_j^{[b_{jk}]_+} + \prod_{j=1}^{r} x_j^{[-b_{jk}]_+}.$$

The (coefficient-free) cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is the $\mathbb{Z}$-algebra generated by the cluster variables produced by all possible mutations of the initial $\mathbf{x}$.

Cluster algebras with periodicity

In special cases, the action of a sequence of matrix mutations is equivalent to a permutation. For $\rho : (1, 2, 3, \ldots, r) \mapsto (r, 1, 2, \ldots, r-1)$, the case $\mu_1 B = \rho B$ (period 1) was completely classified by Fordy & Marsh: the entries of B must satisfy $b_{j,r} = b_{1,j+1}$ and

$$b_{j+1,k+1} = b_{j,k} + b_{1,j+1}[-b_{1,k+1}]_+ - b_{1,k+1}[-b_{1,j+1}]_+, \qquad 1 \leq j, k \leq r-1.$$

Then the cluster map $\varphi = \rho^{-1} \cdot \mu_1$ is equivalent to iterating the recurrence

$$x_{n+r}x_n = \prod_j x_{n+j}^{[b_{1,j+1}]_+} + \prod_j x_{n+j}^{[-b_{1,j+1}]_+},$$

which has palindromic exponents, and preserves the presymplectic form

$$\omega = \sum_{j<k} b_{jk} \, \mathrm{d} \log x_j \wedge \mathrm{d} \log x_k .$$

Symplectic form and U-system

For $\mathbf{w} \in \ker B$, $\lambda \in \mathbb{C}^*$ consider the scaling action

$$\mathbf{x} \longrightarrow \lambda^{\mathbf{w}} \cdot \mathbf{x} = (\lambda^{w_j} x_j), \qquad \mathbf{x}^{\mathbf{v}} = \prod_j x_j^{v_j} \longrightarrow \lambda^{(\mathbf{v},\mathbf{w})} \mathbf{x}^{\mathbf{v}}$$

So $\mathbb{Q}^r = \mathrm{im} B \oplus \ker B \implies$ basis of $\mathrm{im} B$ provides full set of invariants

$$\mathbf{u} = (u_1, \ldots, u_{2m}), \qquad m = \frac{1}{2} \mathrm{rk} B,$$

and φ projects down to a symplectic map $\hat{\varphi}$ for $\mathbf{u}$, with

$$\hat{\omega} = \sum_{j<k} \hat{b}_{jk} \, \mathrm{d} \log u_j \wedge \mathrm{d} \log u_k , \quad \mathcal{M}^{-T} B \mathcal{M}^{-1} = \begin{pmatrix} \hat{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Furthermore, can always choose $\mathcal{M}$ with a "palindromic basis" for $\mathrm{im} B$ so the map $\hat{\phi}$ can be written as a recurrence, called the U-system.

Example: $u_{n+2} u_{n+1}^2 u_n = \alpha u_{n+1} + \beta$ is the U-system for Somos-4.

Liouville integrability of U-systems

Conjecture: *The U-systems obtained from the plane wave reductions of the discrete Hirota equation are integrable in the Liouville sense.*

This is easy to verify in low dimensions, by checking directly that the coefficients of the spectral curve are in involution with respect to the log-canonical Poisson bracket given by the Toeplitz matrix $C = \hat{B}^{-1}$.

Various examples: Lyness maps, discrete reductions of modified Bogoyavlensky lattices, DTKQ systems,...

Main example: Two-parameter family associated with reductions of the discrete KdV lattice, corresponding to

$$(\delta_1, \delta_2, \delta_3) = \left(N + \frac{1}{2}M, \frac{1}{2}M, \left|N - \frac{1}{2}M\right|\right) \text{ or } \left(M + \frac{1}{2}N, \frac{1}{2}N, \left|M - \frac{1}{2}N\right|\right).$$

Reductions of discrete KdV

The version of the discrete KdV equation due to Hirota can be written

$$v_{j+1,k+1} - v_{jk} = \alpha \left(\frac{1}{v_{j+1,k}} - \frac{1}{v_{j,k+1}} \right).$$

Taking $v_{jk} \to v_n$, $n = Nj + Mk$ yields the $(M, -N)$-reduction

$$v_{n+M+N} - v_n = \alpha \left(\frac{1}{v_{n+N}} - \frac{1}{v_{n+M}} \right).$$

Henceforth assume $N > M$, $\gcd(M, N) = 1$. Hirota's tau-function gives

$$v_n = \frac{\tau_n \tau_{n+N+M}}{\tau_{n+M} \tau_{n+N}},$$

leading to a trilinear form which integrates in two different ways to produce

$$\begin{aligned} \tau_{n+2N+M}\tau_n &= -\alpha\, \tau_{n+2N}\tau_{n+M} + \beta_n\, \tau_{n+N+M}\tau_{n+N}, \\ \tau_{n+2M+N}\tau_n &= \alpha\, \tau_{n+2M}\tau_{n+N} + \tilde{\beta}_n\, \tau_{n+N+M}\tau_{n+M}, \end{aligned}$$

where $\beta_{n+M} = \beta_n$, $\tilde{\beta}_{n+N} = \tilde{\beta}_n$.

Bi-Hamiltonian structures: example (2,3)

Example: Setting $(M, N) = (2, 3)$ and dropping n, the first bilinear equation is a Somos-8,

$$\tau_8\tau_0 = -\alpha\tau_6\tau_2 + \beta_0\tau_5\tau_3, \qquad \beta_{n+2} = \beta_n.$$

The exchange matrix is

$$B = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\ & & 0 & 1 & 1 & 0 & 0 & -1 \\ & & & 0 & 1 & 1 & -1 & 0 \\ & & & & 0 & 1 & 1 & -1 \\ & & * & & & 0 & 0 & 1 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{pmatrix},$$

and its rows are spanned by the palindromic basis given by $(1, -1, 0, -1, 1, 0, 0, 0)$ and its three shifts.

Bi-Hamiltonian structures: example (2,3) (continued)

The symplectic coordinates are $u_j = \frac{\tau_j \tau_{j+4}}{\tau_{j+1}\tau_{j+3}}$, $j = 0, 1, 2, 3$.

$$\text{1st} \quad \text{U} - \text{system}: \qquad u_4 u_3 u_2 u_1 u_0 = -\alpha u_2 + \beta_0, \quad \beta_{n+2} = \beta_n,$$

with the Poisson bracket in 4D being

$$\{u_j, u_{j+1}\}_1 = 0, \quad \{u_j, u_{j+2}\}_1 = u_j u_{j+2}, \quad \{u_j, u_{j+3}\}_1 = -u_j u_{j+3}.$$

For the dKdV map

$$v_5 - v_0 = \alpha(v_3^{-1} - v_2^{-1}), \quad \text{with} \quad v_0 = \frac{\tau_0 \tau_5}{\tau_2 \tau_3},$$

we have

$$v_0 = u_0 u_1, \; v_1 = u_1 u_2, \; v_2 = u_2 u_3, \; v_3 = \frac{\beta_0 - \alpha u_2}{u_0 u_1 u_2}, \; v_4 = \frac{\beta_1 - \alpha u_1}{u_1 u_2 u_3}.$$

Bi-Hamiltonian structures: example (2,3) (continued)

Thus the log-canonical bracket for u_j lifts to

$$\{v_0, v_1\}_1 = v_0 v_1, \{v_0, v_2\}_1 = v_0 v_2, \{v_0, v_3\}_1 = -v_0 v_3 - \alpha, \{v_0, v_4\}_1 = -v_0 v_4.$$

The second bilinear equation is a Somos-7, which for $\tilde{u}_j = \frac{\tau_j \tau_{j+3}}{\tau_{j+1}\tau_{j+2}}$ leads to

$$\text{2nd} \quad \text{U} - \text{system}: \qquad \tilde{u}_4 \tilde{u}_3 \tilde{u}_2^2 \tilde{u}_1 \tilde{u}_0 = \tilde{\beta}_0 \tilde{u}_2 + \alpha, \quad \tilde{\beta}_{n+3} = \tilde{\beta}_n,$$

and the bracket $\{,\}_2$ for $\tilde{u}_j$ in 4D has the same coefficients as that for u_j. From $v_j = \tilde{u}_j \tilde{u}_{j+1} \tilde{u}_{j+2}$ the lift of this bracket gives

$$\begin{aligned} \{v_0, v_1\}_2 &= v_0 v_1, & \{v_0, v_2\}_2 &= v_0 v_2 - \alpha, \\ \{v_0, v_3\}_2 &= -v_0 v_3 - \alpha, & \{v_0, v_4\}_2 &= -v_0 v_4 + \frac{\alpha^2}{v_2^2}. \end{aligned}$$

These two brackets for the dKdV reduction are compatible; the difference $\{,\}_1 - \{,\}_2$ comes from a Lagrangian structure (Tran).

Bi-Hamiltonian structures: general case

For general coprime (M, N), $\mathrm{rk} B = 2\left\lfloor \frac{N+M-1}{2} \right\rfloor$, and the two U-systems of corresponding dimension preserve the same log-canonical Poisson bracket. In particular, when $N + M$ is odd the bracket has the form

$$\{u_j, u_k\} = c_{k-j}\, u_j u_k, \qquad c_k = -c_{-k},$$

where the coeffients c_k for $k = 1, \ldots, N + M - 2$ satisfy

$$\begin{aligned} c_k &= -c_{N+M-k}, & 2 \leq k \leq (N + M - 1)/2, \\ c_k &= c_{k+N-M}, & 0 \leq k \leq M - 2, \\ c_k &= c_{k+2M}, & 1 \leq k \leq (N - M - 1)/2. \end{aligned}$$

The solution for c_k lies in a tableau built from $r = \frac{k}{2M} \mod N + M$.

Conclusion: Compatibility of the lifted brackets $\{,\}_{1,2}$ proves Liouville integrability of both the dKdV reductions and the pair of U-systems.

Analytic solution of Somos-6

Theorem [H] For arbitrary $\mathrm{A}, \mathrm{B}, \mathrm{C} \in \mathbb{C}^*$, $\mathbf{v}_0 \in \mathbb{C}^2 \bmod \Lambda$, the sequence

$$\tau_n = \mathrm{A}\mathrm{B}^n\mathrm{C}^{n^2-1}\frac{\sigma(\mathbf{v}_0 + n\mathbf{v})}{\sigma(\mathbf{v})^{n^2}}$$

associated with a genus 2 curve X satisfies Somos-6 with coefficients

$$\alpha = \frac{\sigma^2(3\mathbf{v})\mathrm{C}^{10}}{\sigma^2(2\mathbf{v})\sigma^{10}(\mathbf{v})}\,\hat{\alpha}, \qquad \beta = \frac{\sigma^2(3\mathbf{v})\mathrm{C}^{16}}{\sigma^{18}(\mathbf{v})}\,\hat{\beta},$$

$$\gamma = \frac{\sigma^2(3\mathbf{v})\mathrm{C}^{18}}{\sigma^{18}(\mathbf{v})}\left(\wp_{11}(3\mathbf{v}) - \hat{\alpha}\wp_{11}(2\mathbf{v}) - \hat{\beta}\wp_{11}(\mathbf{v})\right),$$

$$\hat{\alpha} = \frac{\wp_{22}(3\mathbf{v}) - \wp_{22}(\mathbf{v})}{\wp_{22}(2\mathbf{v}) - \wp_{22}(\mathbf{v})}, \quad \hat{\beta} = \frac{\wp_{22}(2\mathbf{v}) - \wp_{22}(3\mathbf{v})}{\wp_{22}(2\mathbf{v}) - \wp_{22}(\mathbf{v})} = 1 - \hat{\alpha},$$

provided that $\mathbf{v} \in \mathrm{Jac}(X)$ satisfies the constraint

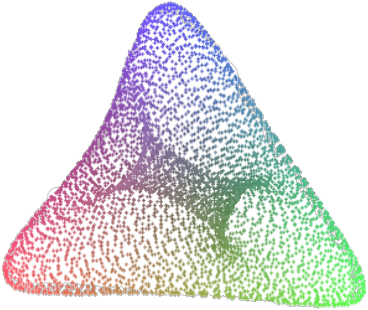
$$\det\begin{pmatrix} 1 & 1 & 1 \\ \wp_{12}(\mathbf{v}) & \wp_{12}(2\mathbf{v}) & \wp_{12}(3\mathbf{v}) \\ \wp_{22}(\mathbf{v}) & \wp_{22}(2\mathbf{v}) & \wp_{22}(3\mathbf{v}) \end{pmatrix} = 0.$$

Solution of the initial value problem for Somos-6

Question: Given initial data $\tau_0, \ldots, \tau_5$ and coefficients α, β, γ for Somos-6, how to reconstruct parameters $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and genus 2 curve

$$X : \qquad z^2 = 4s^5 + c_4 s^4 + c_3 s^3 + c_2 s^2 + c_1 s + c_0,$$

with vectors $\mathbf{v}_0, \mathbf{v} \in \mathrm{Jac}(X)$ such that $\mathbf{v}$ satisfies the constraint?
Idea: Consider map in 4D satisfied by $u_n = \tau_{n+2}\tau_n/(\tau_{n+1})^2$.



Reduction of discrete BKP

Bilinear form of discrete BKP:

$$T_{123}T - T_1T_{23} + T_2T_{31} - T_3T_{12} = 0.$$

Plane wave reduction with quadratic gauge:

$$T(n_1,n_2,n_3) = \delta_1^{n_2n_3}\delta_2^{n_3n_1}\delta_3^{n_1n_2}\,\tau_n, \qquad n = n_1+2n_2+3n_3, \quad \delta_1 = \sqrt{-\frac{\alpha}{\beta\gamma}}.$$

Reduction of the discrete BKP scalar Lax triad gives a Lax pair with

$$\mathbf{L}(x) = \begin{pmatrix} \frac{A_2x^2+A_1x}{x+\lambda} & \frac{A_2'x^2+A_1'x}{x+\lambda} & \frac{A_1''x+A_0''}{x+\lambda} \\ B_2\,x^2+B_1x & B_1'x & B_1''\,x+B_0'' \\ C_2x^2+C_1x & C_2'x^2+C_1'x & C_1''x+C_0'' \end{pmatrix},$$

where x is a spectral parameter and A_j, B_j etc. are rational functions of

$$\lambda = \frac{\delta_1\beta^2}{\alpha^2}, \quad \mu = -\frac{\delta_1\beta^3}{\gamma^2}, \quad P_n = -\frac{\delta_1\beta}{u_nu_{n+1}}, \quad R_n = \frac{\delta_1\gamma}{u_nu_{n+1}u_{n+2}}.$$

The second half of the Lax pair is

$$\mathbf{M}(x) = \frac{1}{R_0}\begin{pmatrix} -1 & 1 & 0 \\ -\frac{x}{\lambda} - 1 & 1 & \frac{1}{\lambda} \\ 0 & (\lambda P_0 R_1 R_2 + 1)x & -P_0 R_2 \end{pmatrix},$$

with $\det \mathbf{M}(x) = \dfrac{R_1 R_2}{R_0^2}\, x$, and the discrete Lax equation $\tilde{\mathbf{L}}\mathbf{M} = \mathbf{M}\mathbf{L}$ is equivalent to the reduced Somos-6 map in 4D:

$$\hat{\varphi}: \quad u_{n+4}\, u_{n+3}^2\, u_{n+2}^3\, u_{n+1}^2\, u_n = \alpha\, u_{n+3}\, u_{n+2}^2\, u_{n+1} + \beta\, u_{n+2} + \gamma,$$

The spectral curve

$$S: \qquad \det(\mathbf{L}(x) - y\mathbf{1}) = 0$$

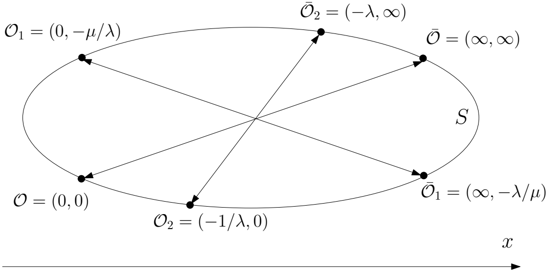
provides two independent first integrals of the map, denoted K_1, K_2.

Spectral curve: Jacobian and Prym variety

The curve S has genus 4 and is given by

$$(x+\lambda)\,y^3+(x\,K_1+\mu+x^2\,K_2)\,y^2-(\mu\,x^4+K_1\,x^3+x^2\,K_2)\,y-\lambda\,x^4-x^3=0.$$

It has the involution $\iota\,:\,(x,y)\to(1/x,1/y)$ with two fixed points $(\pm1,1)$.



The quotient $C=S/\iota$ has genus 2, and Jac(S) is isogenous to Jac(C) $\times$ Prym(S,ι), where Prym(S,ι) $=\mathrm{im}(1-\iota)=\ker(1+\iota)^0$.

Solving the initial value problem from the spectral data

Theorem: [H & Fedorov] A generic complex invariant manifold $\mathcal{I}_K = \{K_1 = const,\ K_2 = const\}$ for the reduced Somos-6 map $\hat{\varphi}$ is isomorphic to $\mathrm{Prym}(S, \iota) \cong \mathrm{Jac}(X)$ for a genus 2 curve X.

- $\mathrm{Prym}(S, \iota)$ is isomorphic to a 2D Jacobian (Mumford, Dalaljan).
- Effective description of 2-fold branched coverings of hyperelliptic curves (Levin).
- Isospectral manifold for Lax matrix $\mathbf{L}(x)$ is isomorphic to $\mathcal{I}_K$.
- The eigenvector bundle for $\mathbf{L}(x)\psi = y\psi$ defines a point in $\mathrm{Jac}(S)$.
- $\tilde{\psi} = \mathbf{M}(x)\psi$ shifts by the divisor $\mathcal{V} = \bar{\mathcal{O}} - \mathcal{O}$, and $\iota(\mathcal{V}) = -\mathcal{V}$.

Now $\mathcal{V}$ corresponds to a vector $\mathbf{v} \in \mathrm{Jac}(X)$, and from $\mathcal{O}_1 - \bar{\mathcal{O}}_1 \equiv 2\mathcal{V}$, $\mathcal{O}_2 - \bar{\mathcal{O}}_2 \equiv 3\mathcal{V}$ one sees that $\mathbf{v}$ satisfies the constraint. Hence $\mathbf{v}_0$ and the other parameters in the solution are recovered from the initial data.

Classical Somos-6 sequence

For the original Somos-6 sequence,

$$(\tau_0, \ldots, \tau_5) = (1, 1, 1, 3, 5, 9), \qquad \alpha = 1, \qquad \beta = 1, \qquad \gamma = 1,$$

one finds

$$\lambda = i, \qquad \mu = -i, \qquad K_1 = 19, \qquad K_2 = 14i.$$

This yields the genus 2 curve

$$X : \qquad z^2 = 4s^5 - 233s^4 + 1624s^3 - 422s^2 + 36s - 1,$$

and one has the constant $\mathrm{C} = i/\sqrt{20}$ and a pair of divisors on X,

$$\mathcal{D} = (s_1, z_1) + (s_2, z_2) - 2\infty, \qquad \mathcal{D}_0 = (s_1^{(0)}, z_1^{(0)}) + (s_2^{(0)}, z_2^{(0)}) - 2\infty$$

where

$$s_{1,2} = -8 \pm \sqrt{65}, \qquad z_{1,2} = 20i(129 \mp 16\sqrt{65}),$$
$$s_{1,2}^{(0)} = 5 \pm 2\sqrt{6}, \qquad z_{1,2}^{(0)} = 4i(71 \pm 29\sqrt{6}),$$

corresponding to $\mathbf{v}, \mathbf{v}_0$ respectively.