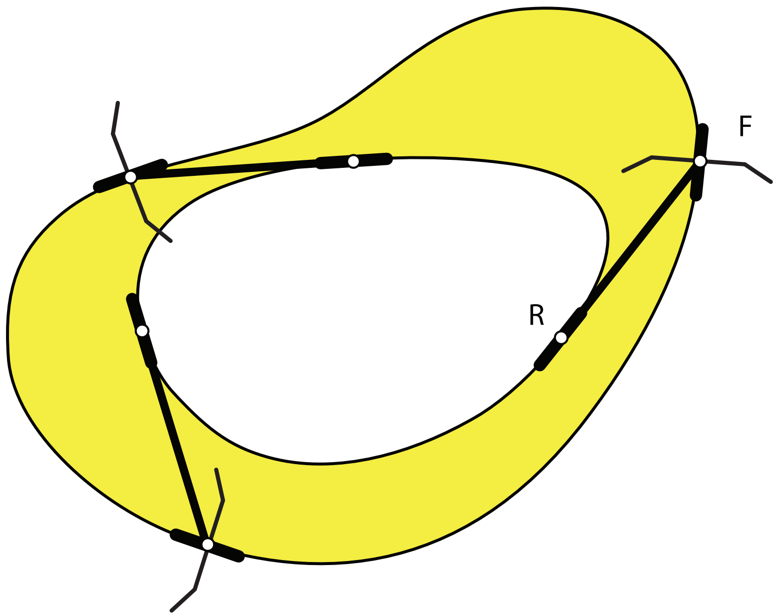
Tire track geometry and the filament equation: results and conjectures

Geometric and Algebraic Aspects of Integrability, July 2016

The model



Segment RF moves so that the trajectory of R is tangent to the segment. *Notation*: γ is the rear track, Γ is the front track. When the steering angle is 90°, γ has a cusp. One can do it in $\mathbf{R}^n$, and on a Riemannian manifold. (animation)

The bicycle constraint defines an n-dimensional non-integrable distribution on $S(T\mathbf{R}^n)$; the trajectories are horizontal curves.

The rear track γ has a coorientation determined by the direction of the motion. The map $\gamma \mapsto \Gamma$ is well defined (it depends on the length ℓ). Conversely, Γ defines the bicycle monodromy $M_{\Gamma,\ell} : S^{n-1} \to S^{n-1}$.

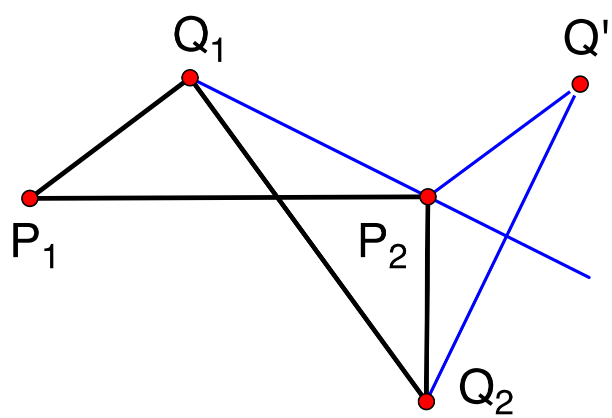
Theorem. *M is a Möbius transformation.*

Here S^{n-1} is the sphere at infinity of the hyperbolic space H^n, considered in the hyperboloid model in $\mathbf{R}^{n,1}$; the Möbius group is $O(n,1)$.

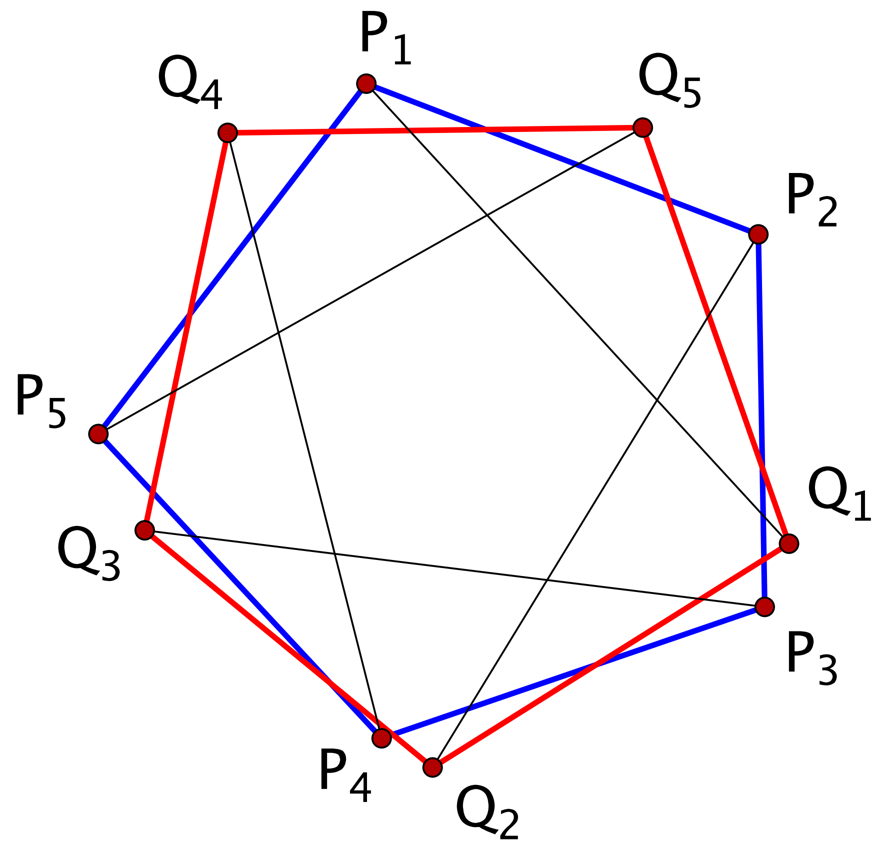
Bicycle (Darboux, Bäcklund) correspondence

Two front tracks that share the rear track, with opposite coorientations. Write: $\mathcal{B}_{2\ell}(\Gamma_1, \Gamma_2)$. Equivalently, two points, x_1 and x_2, traverse the curves Γ_1 and Γ_2 in such a way that the distance x_1x_2 is equal to 2ℓ, and the velocity of the midpoint of the segment x_1x_2 is aligned with the segment. (animation)

Discrete version (in $\mathbf{R}^n$):



Theorem. *The discrete bicycle monodromy is also a Möbius transformation.*



The next two results hold in the continuous and discrete settings.

Theorem. *If closed curves* Γ_1 *and* Γ_2 *are in the bicycle correspondence, then* $M_{\Gamma_1,\lambda}$ *and* $M_{\Gamma_2,\lambda}$ *are conjugated for every value of* λ.

Thus the conjugacy invariants of $M_{\Gamma,\lambda}$, as functions of λ (the spectral parameter), are integrals of the bicycle correspondence.

Theorem. (Bianchi permutability). *Let* Γ_1, Γ_2 *and* Γ_3 *be three closed curves such that* $\mathcal{B}_\ell(\Gamma_1, \Gamma_2)$ *and* $\mathcal{B}_\lambda(\Gamma_1, \Gamma_3)$ *hold. Then there exists a closed curve* Γ_4 *such that* $\mathcal{B}_\lambda(\Gamma_2, \Gamma_4)$ *and* $\mathcal{B}_\ell(\Gamma_3, \Gamma_4)$ *hold.*

Conjecture. *The bicycle correspondence in* $\mathbf{R}^n$, *continuous and discrete, is Liouville integrable; likewise, in the elliptic and hyperbolic geometries.*

(Pre)symplectic geometry (S.T., to appear in JGP)

Two differential 2-forms on the space of smooth curves in $\mathbf{R}^3$:

$$\omega(u,v) = \int u'(x) \cdot v(x)\ dx, \quad \Omega(u,v) = \int \mathrm{det}(\Gamma'(x), u(x), v(x))\ dx,$$

where $u(x), v(x)$ are vector fields along a curve $\Gamma(x)$. Both forms are closed (in fact, exact).

The form ω depends on the metric, but exists in all dimensions; Ω depends on the volume form, but is 3-dimensional only. Kernels: the constant vector fields, for ω, and the tangent vector fields (reparameterizations), for Ω.

Theorem. *The bicycle transformation preserves the forms ω and Ω.*

Conjecture. *The monodromy integrals commute in the corresponding quotient spaces with respect to the Poisson bracket induced by ω (all dimensions) and Ω (dimension three).*

Furthermore, the forms ω and Ω are *compatible* (as if they defined Poisson structures that form a pencil)...

Conjecture. *The forms ω and Ω are compatible if and only if M^3 has constant curvature.*

Filament equation $\dot{\Gamma} = \Gamma' \times \Gamma''$

It is Hamiltonian flow with respect to Ω, with Poisson commuting integrals $F_1, F_2, \ldots$

$$\int 1\ dx,\ \int \tau\ dx,\ \int \kappa^2\ dx,\ \int \kappa^2 \tau\ dx, \int \left((\kappa')^2 + \kappa^2 \tau^2 - \frac{1}{4}\kappa^4 \right)\ dx, \ldots$$

and their commuting Hamiltonian vector fields $X_0, X_1, X_2, \ldots$

$$-T,\ \kappa B, \frac{\kappa^2}{2} T + \kappa' N + \kappa\tau B, \kappa^2 \tau T + (2\kappa'\tau + \kappa\tau')N + \left(\kappa\tau^2 - \kappa'' - \frac{\kappa^3}{2} \right) B, \ldots$$

where (T, N, B) is the Frenet frame along the curve Γ.

The vector fields X_i satisfy the recurrence relation

$$T \times X_n = X'_{n-1}.$$

Theorem. *The functions F_i are integrals of the bicycle transformation (for all length parameters ℓ). The bicycle transformation commutes with the flows of the vector fields X_i.*

Outline of proof: one has

$$\omega(X_{n-1}, \cdot) = \Omega(X_n, \cdot) = dF_n.$$

Induction: if X_{n-1} is preserved, then so is dF_n, and then X_n is preserved as well.

Two Riccati equations and two sets of integrals

Let $\Gamma(x)$ be a space curve, the front track. The motion of the bicycle is described by a differential equation. In the moving Frenet frame, via the stereographic projection from T, this is the Riccati equation

$$z' = \frac{1}{2}k(1+z^2) + \left(\frac{1}{\ell} - i\tau\right)z,$$

where k and τ are the curvature and torsion of Γ.

Let $u(x)$ be the linearization at a periodic solution $z(x)$. Then

$$\frac{u'}{u} = \left(\frac{1}{\ell} - i\tau\right) + kz.$$

The monodromy integrals are the integrals of the RHS, expanded in the powers of ℓ.

The first two integrals are $\int 1dx$ and $\int \tau dx$, and the next ones are $\int kz_j dx,\ j \geq 1$. They are real for odd j and imaginary for even j.

Conjecture. *These integrals coincide, up to constants, with the filament integrals* $F_1, F_2, \ldots$

Recall the filament hierarchy of the commuting Hamiltonian vector fields X_n. Consider the generating function

$$X = \sum_{n \geq 0} \ell^n X_n$$

where ℓ is a variable. Then the recurrence implies:

$$T \times X = \ell X', \quad X \cdot X = 1.$$

Rewrite this as a Riccati equation via the same stereographic projection:

$$w' = \frac{1}{2}k(1 + w^2) + i\left(\frac{1}{\ell} - \tau\right)w.$$

This is the same equation, with ℓ replaced by $i\ell$. In particular, the monodromy integrals are (almost) the same.

Derivative of the bicycle monodromy at a fixed point in terms of the rear track.

Let γ be a closed rear track, L its signed length (negative, when the bicycle moves backwards), and τ its torsion.

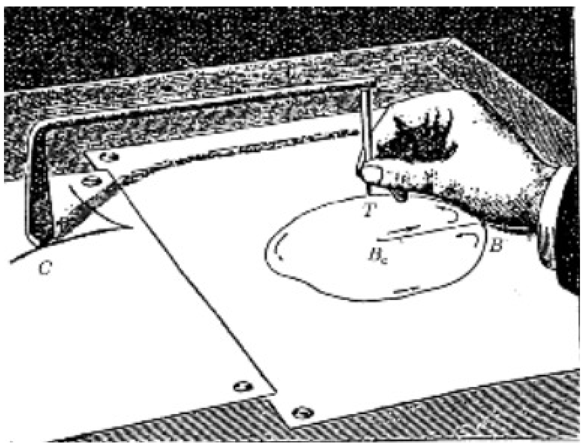
Theorem. *The derivative of the bicycle monodromy at the fixed point corresponding to γ equals*

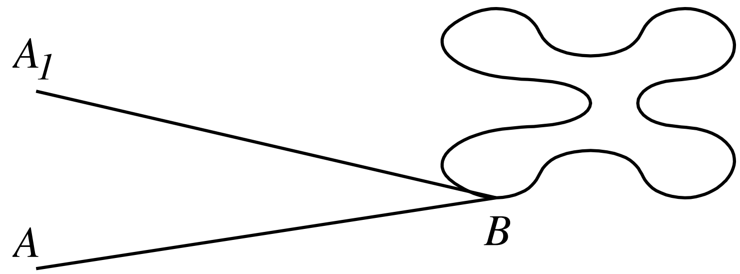
$$e^{-(L+i\int \tau dx)}.$$

In particular, in the plane, the monodromy is parabolic if and only if the signed length of the rear track is zero.

Bicycles and planimeters

In dimension 2, *hatchet, or Prytz, planimeter*:





How it works: Area $\approx |AB||AA_1|$, the more accurate, the larger $|AB|$ is (a power series expansion in $1/\ell$).

A consequence for *parallel parking*: maximize the area bounded by the front wheel trajectory.

In dimension 3, let $\ell = 1/\varepsilon$, let $\Gamma(x)$ be the front track, and $v(x)$ the unit vector along the bicycle frame. Then

$$\varepsilon \Gamma' \times v = v' \times v.$$

Write $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots$ Then v_0 is a constant vector.

Proposition. *The lowest in ε non-trivial component of the monodromy $v(L) - v(0)$ is*

$$\frac{1}{2}\, v_0 \times \left(\int_0^L (\Gamma' \times \Gamma)\ dx \right)$$

(space planimeter).

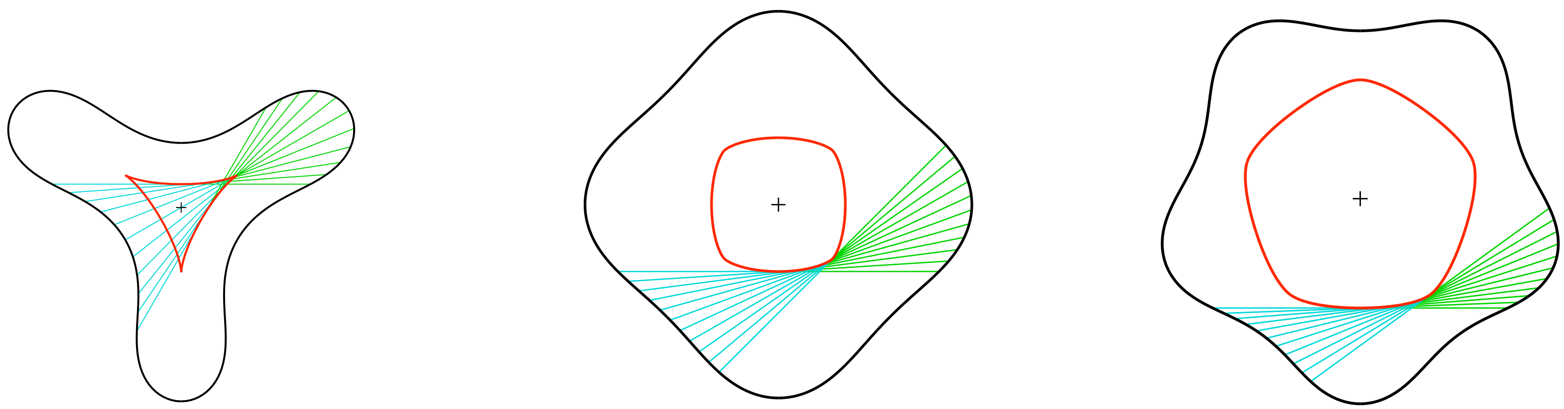
The area (bi)vector is an integral of the bicycle transformation (and that of the filament equation too).

Which way did the bicycle go?

In "The Adventure of the Priory School" by A. Conan Doyle, Sherlock Holmes did not do very well:

> No, no, my dear Watson. The more deeply sunk impression is, of course, hind wheel, upon which the weight rests. You perceive several places where it has passed across and obliterated the more shallow mark of the front one. It was undoubtedly heading away from the school.

Usually, you can tell which way the bicycle went, but sometimes you cannot. Trivial example: concentric circles. But also:



F. Wegner, *Three Problems – One Solution*
`http://www.tphys.uni-heidelberg.de/~wegner/Fl2mvs/Movies.html`.

Ulam's problem: which bodies float in equilibrium in all positions?

In dimension two (floating log), it's the same problem!

(The role of relative density is played by the relative length of the arc subtended by the moving segment.)

Wegner's curves: in polar coordinates $r = r(\psi)$,

$$\frac{1}{\sqrt{r^2 + r_\psi^2}} = ar^2 + b + \frac{c}{r^2}$$

with parameters a, b, c. One has, for the curvature,

$$k = 4ar^2 + 2b.$$

Let $X_2 = \frac{k^2}{2}T + k'N$ be the planar filament vector field.

Theorem. *The Wegner curves are solitons: under X_2, they evolve by rigid rotation and parameter shift. Their curvature satisfies*

$$k'' + \frac{1}{2}k^3 + \lambda k = \mu$$

with $\lambda = 8ac - 2b^2, \mu = 8a$ (the Euler-Lagrange equation for pressurized elastica).

A special class of solutions. Consider the n-fold unit circle. Its bicycle monodromy is trivial for

$$\ell_{k,n} = \frac{1}{\sqrt{1-\frac{k^2}{n^2}}}, \quad k = 1, 2, \ldots, n-1.$$

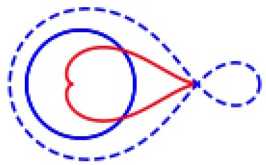
This gives a family of curves $\Gamma_{k,n}$ that admit a trigonometric parameterization.

The bicycle correspondence rotates theses curves.

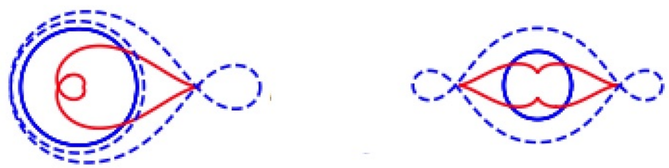
Theorem. *The curve $\Gamma_{k,n}$ with $1 \le k \le n-2$ is in the bicycle correspondence with itself for $n-k-1$ values of the "density" ρ satisfying*

$$n \tan(k\pi\rho) = k \tan(n\pi\rho).$$

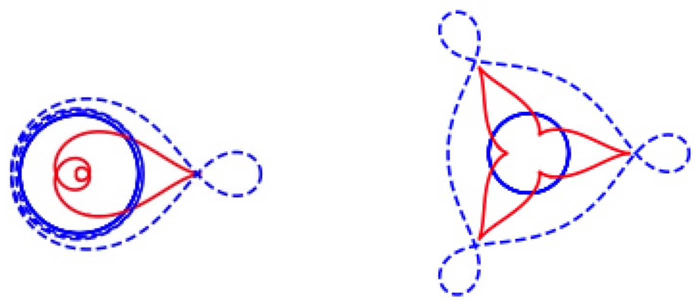
n=2,
k=1



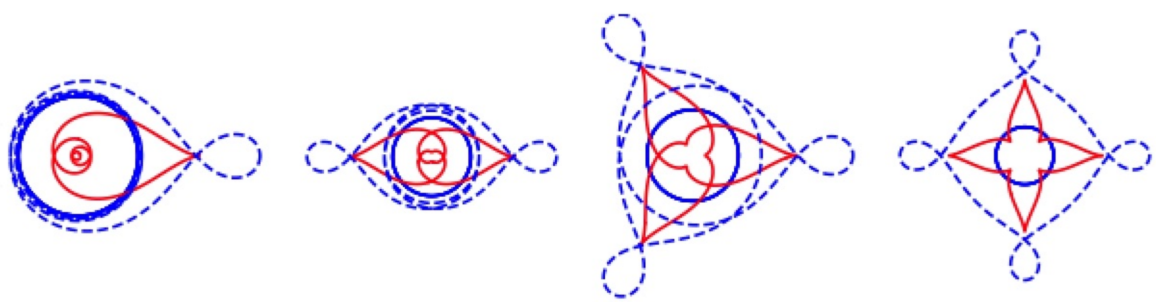
n=3
k=1,2



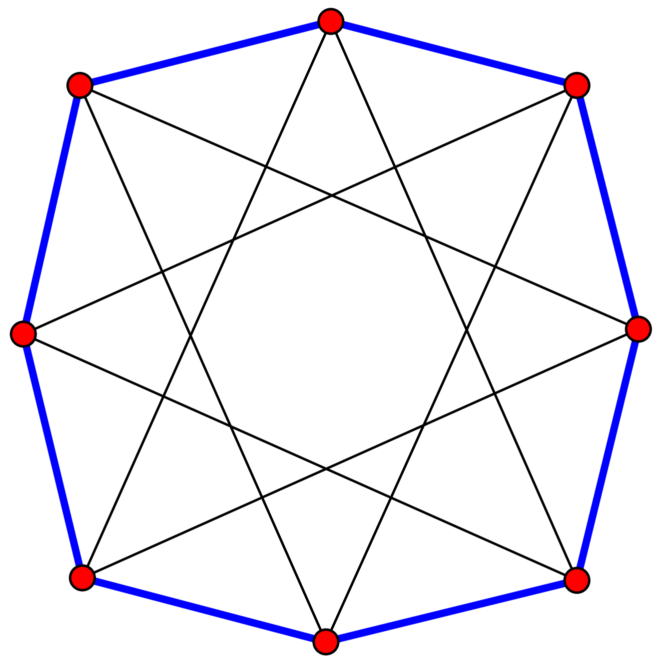
n=4
k=1,3



n=5,
k=1,2,3,4



What about polygons? For even n and odd k, there exist non-regular self-bicycle (n, k)-gons:



Theorem. *An infinitesimal deformation of a regular polygon as a self-bicycle (n, k)-gon exists if and only if*

$$\tan\left(kr\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right) = \tan\left(k\frac{\pi}{n}\right)\tan\left(r\frac{\pi}{n}\right)$$

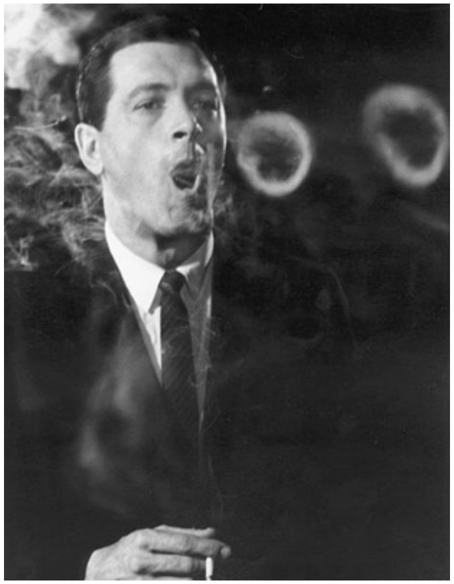
for some $2 \le r \le n-2$.

In addition to the described polygons ($n = 2r$, and k odd), there may be others.

Theorem [R. Connelly and B. Csikos, 2009]. *For $2 \le r \le n/2$, all other solutions of the above equation are*

$$k + r = \frac{n}{2} \quad \text{and} \quad n|(k-1)(r-1)$$

Problem. *Do such polygons exist?*



Thank you!