Cloaking and superlensing using negative index materials

Durham Symposium on Mathematical and Computational Aspects of Maxwell's Equations

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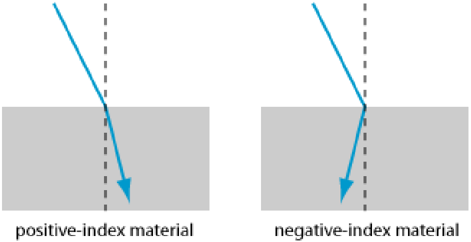
July 13, 2016

Outline

Part 1: Negative index materials

Negative index materials (NIMs)

Definition: NIMs are artificial structures where the refractive index has a negative value over some frequency range.



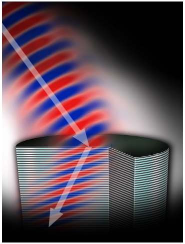


Figure: Left: RP-photonics; Right: Wikipedia.

Highlights of the development

1. Veselago (UFN 64) investigated theoretically NIMs.
2. Nicorovici, McPhedran, & Milton (PRB 94) and Pendry (PRL 00).
3. Shelby et al. (Science 01) confirmed experimentally.

Mathematical settings

Electromagnetic setting:

$$\begin{cases} \nabla \times E = ik\mu H, \\ \nabla \times H = -ik\epsilon E + j. \end{cases}$$

Negative index materials: $\epsilon < 0$ and $\mu < 0$.

Acoustic setting:

$$\mathrm{div}(A\nabla u) + k^2 \Sigma u = f.$$

Negative index materials: $A < 0$ and $\Sigma < 0$.

Remarks:

1. The ellipticity and compactness might be lost.
2. Localized resonance might appear.
3. Many surprising interesting properties.

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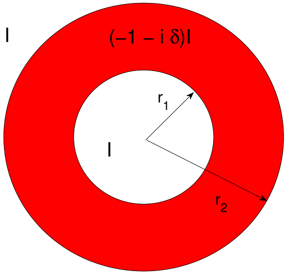
Part 2: Two examples

First example: Nicorovici, McPhedran, & Milton's result, PRB 94

Consider

$$\operatorname{div}(A_\delta \nabla u_\delta) = f \text{ in } \mathbb{R}^2 \text{ where}$$

$$A_\delta = \begin{cases} 1 & \text{in } \mathbb{R}^2 \setminus B_{r_2}, \\ -1 - i\delta & \text{in } B_{r_2} \setminus B_{r_1}, \\ 1 & \text{in } B_{r_1}. \end{cases}$$



Theorem

If $\operatorname{supp} f \cap B_{r_3} = \emptyset$ *where* $r_3 = r_2^2/r_1$, *then*

$$u_\delta \to U \text{ in } \mathbb{R}^2 \setminus B_{r_3}, \text{ where } \Delta U = f \text{ in } \mathbb{R}^2.$$

Some questions:

1. Why do the phenomena hold for $r_3 = r_2^2/r_1$?
2. Is it necessary that the geometry is radial symmetric?
3. What happens in the finite frequency case ($k \neq 0$) and in three dimensions?

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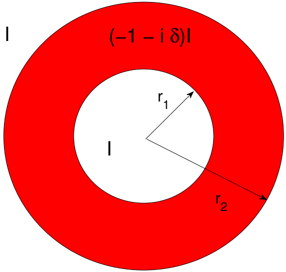

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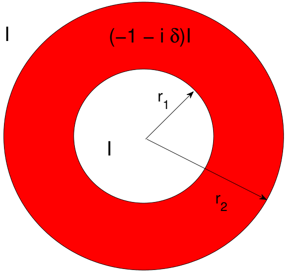

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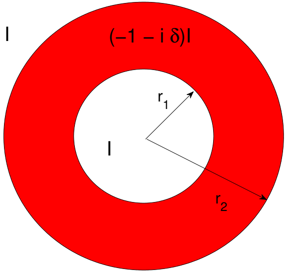

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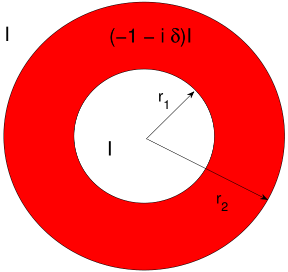

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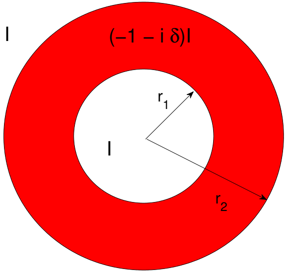

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Second example: Ng. & Loc Nguyen, M2AN 15

Set

$$\varepsilon_\delta = \begin{cases} -1 - i\delta & \text{in } B_1, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem

Let $d = 2$, $R > 1$, $g \in H^{1/2}(\partial B_R)$ *and* $u_\delta \in H^1(B_R)$ *be s.t.*

$$\operatorname{div}(\varepsilon_\delta \nabla u_\delta) = 0 \text{ in } B_R \quad \text{and} \quad u_\delta = g \text{ on } \partial B_R.$$

Case 1: g *is compatible. Then* $u_\delta \to u_0$ *weakly in* $H^1(B_R)$ *as* $\delta \to 0$.
Case 2: g *is not compatible. Then* $\lim_{\delta \to 0} \|u_\delta\|_{H^1(B_R)} = +\infty$; *however,*

$$u_\delta \to v \text{ weakly in } H^1(B_{1/R}), \text{ where } \begin{cases} \Delta v = 0 & \text{in } B_{1/R}, \\ v(x) = g(x/|x|^2) & \text{on } \partial B_{1/R}. \end{cases}$$

Moreover,

$$\limsup_{\delta \to 0} \delta \int_{B_R} |\nabla u_\delta|^2 dx < +\infty, \quad \forall g \in H^{1/2}(\partial B_R).$$

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Let $d = 2$, *and* $f \in L^2_c(\mathbb{R}^2)$ *with* $\operatorname{supp} f \cap B_1 = \emptyset$, *and* $u_\delta \in W^1(\mathbb{R}^2)$ *be the unique solution to*

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Part 3: Superlensing using complementary media

Superlensing using complementary media - State of the art

• Veselago's lens (slab lens): Veselago UFN 64 (ray theory), Pendry PRL 00 (Maxwell's equations).

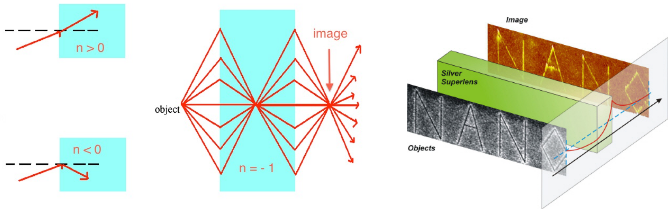
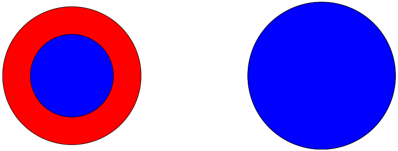


Figure: Left: Veselago's lens. Right: Yang et al.'s experiment Nature 08.

• Cylindrical lens: Nicorovici, McPhedran, Milton PRB 94 (quasistatic regime), Pendry OE 03 (finite frequency regime).
• Spherical lens: Ramakrishna & Pendry PRE 04 (finite frequency regime).

State of the art contd.

- Standard proposal:
  - Cylindrical lens: To magnify m times "an object" in B_{r_0}, one puts a plasmonic structure $-I$ in $B_{r_2} \setminus B_{r_0}$ with $r_2^2/r_0^2 = m$.
  - Spherical lens: To magnify m times "an object" in B_{r_0}, one puts a plasmonic structure $-(r_2^2/|x|^2)I$ in $B_{r_2} \setminus B_{r_0}$ with $r_2^2/r_0^2 = m$.



- Known results : "Object": a constant isotropic object, homogeneous medium via separation of variables.
- Comments: The structure in 3d is not easy to predict. This was done by searching in the set of radial isotropic structures.
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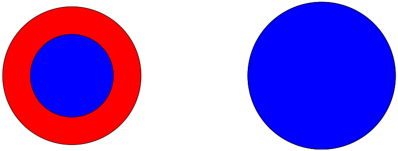

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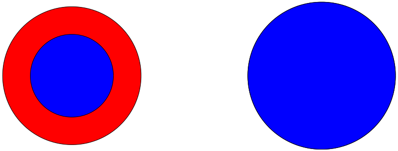

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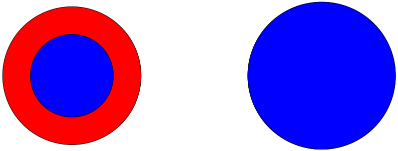

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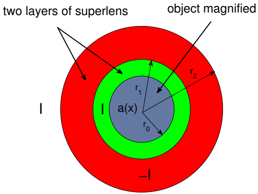
The two dimensional quasistatic regime, Ng, AIHP 15

Magnified region: B_{r_0}; Magnification: $m > 1$.
The superlensing device contains two layers:

1. The first one $-I$ in $B_{r_2} \setminus B_{r_1}$
2. The second (new) one I in $B_{r_1} \setminus B_{r_0}$.

Here $r_2 = m r_0$ and $r_1 = m^{1/2} r_0$. With loss, the medium is

$$s_\delta A = \begin{cases} 1 \cdot I & \text{in } \Omega \setminus B_{r_2}, \\ (-1 - i\delta) \cdot I & \text{in } B_{r_2} \setminus B_{r_1}, \\ 1 \cdot I & \text{in } B_{r_1} \setminus B_{r_0}, \\ 1 \cdot a & \text{in } B_{r_0}. \end{cases}$$



Theorem

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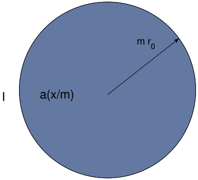


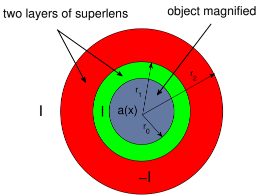
The two dimensional quasistatic regime, Ng, AIHP 15

Magnified region: B_{r_0}; Magnification: $m > 1$.
The superlensing device contains two layers:

1. The first one $-I$ in $B_{r_2} \setminus B_{r_1}$
2. The second (new) one I in $B_{r_1} \setminus B_{r_0}$.

Here $r_2 = m r_0$ and $r_1 = m^{1/2} r_0$. With loss, the medium is

$$s_\delta A = \begin{cases} 1 \cdot I & \text{in } \Omega \setminus B_{r_2}, \\ (-1 - i\delta) \cdot I & \text{in } B_{r_2} \setminus B_{r_1}, \\ 1 \cdot I & \text{in } B_{r_1} \setminus B_{r_0}, \\ 1 \cdot a & \text{in } B_{r_0}. \end{cases}$$



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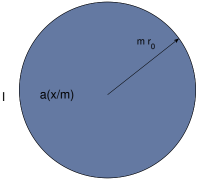

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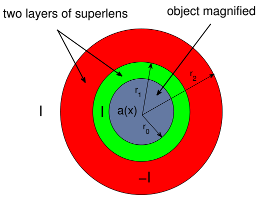

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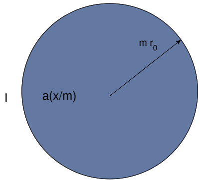

Proof.

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$$u_1(x^*) = u(x), \quad x^* = F(x) = r_2^2 x/|x|^2.$$

We have $\partial Br_3 = F(\partial B_{r_1})$ and

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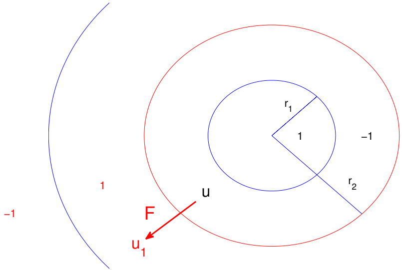

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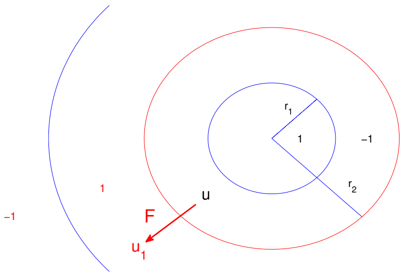

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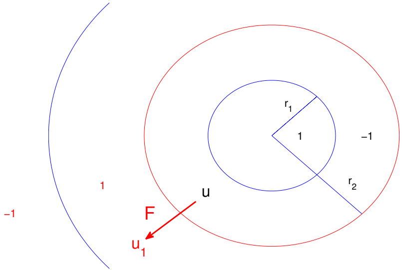

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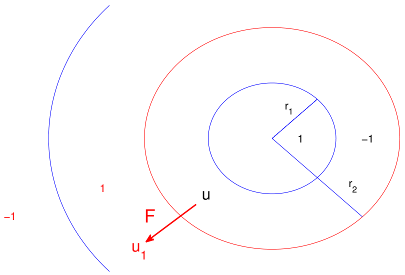

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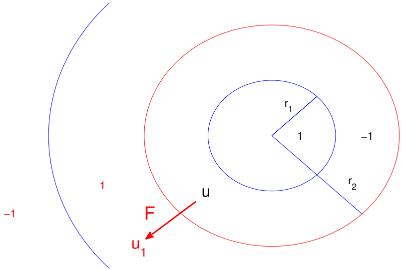

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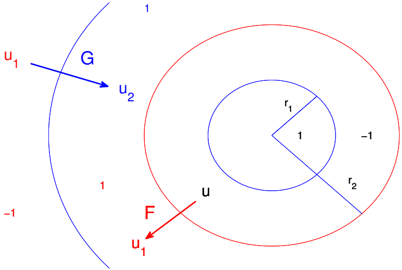

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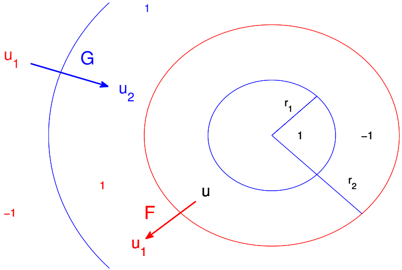

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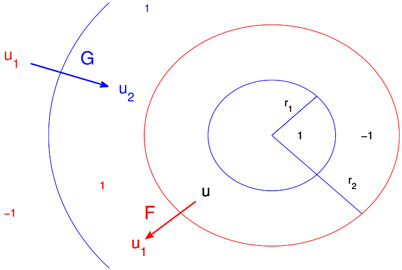

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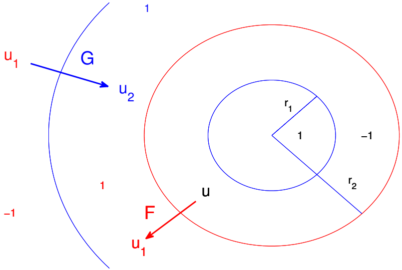

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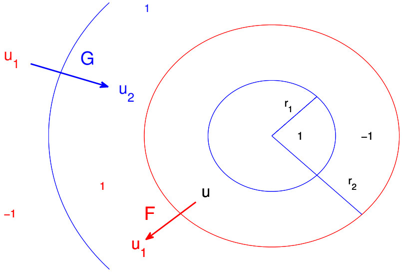

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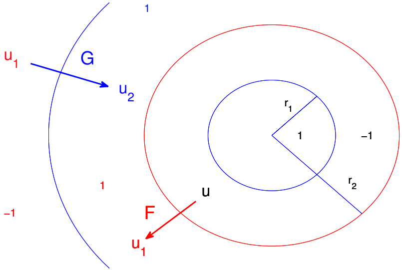

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What happens in the general case?

Transformations optics

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Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3$ *and* $T : \Omega_2 \setminus \Omega_1 \to \Omega_3 \setminus \Omega_2$. *Fix* u *defined in* $\Omega_2 \setminus \Omega_1$ *and set* $v = u \circ T^{-1}$. *We have*

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where $x = T^{-1}(y)$ and $J(x) = |\det DT(x)|$.

Reflecting complementary media : $T_* a = a$ and $T_* \sigma = \sigma$ (Ng. TRANS 15).

Some comments

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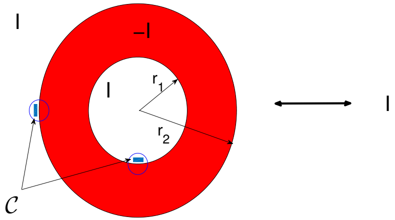
2. The green layer can be thinner (Ng. AIHP 15) but necessary (Ng. 16).



Theorem (Ng. 16)

Let $d = 2$ *and* $f \in L^2(\Omega)$ *be s.t.* $\operatorname{supp} f \cap B_{r_3} = \emptyset$ *where* $r_3 = r_2^2/r_1$. *We have*

$$u_\delta \to \hat{u} \text{ in } \Omega \setminus B_{r_3} \text{ as } \delta \to 0,$$

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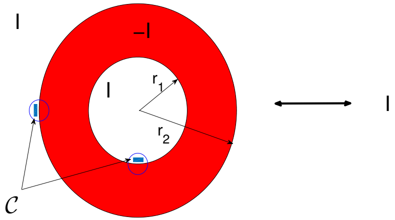

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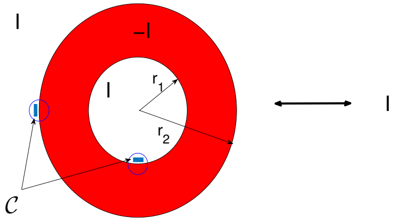

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Part 4: Cloaking using complementary media

Cloaking using complementary media

1. Suggestion: Lai et. al. PRL 09.
   Difficulty: Localized resonance + loss of ellipticity.

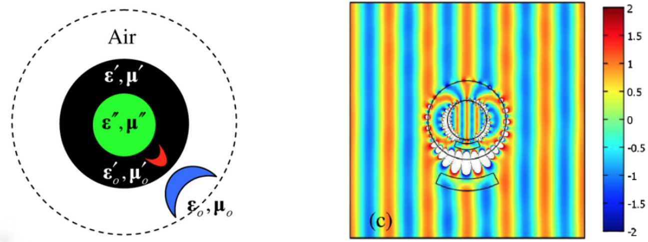

Figure: Lai et. al. PRL 09.

2. Theory confirmed: Ng. AIHP 15, Ng.-L-Nguyen TRANS B 15 (for a class of inspired schemes). Tools : removing localized singularity technique + three sphere inequality + reflecting technique.

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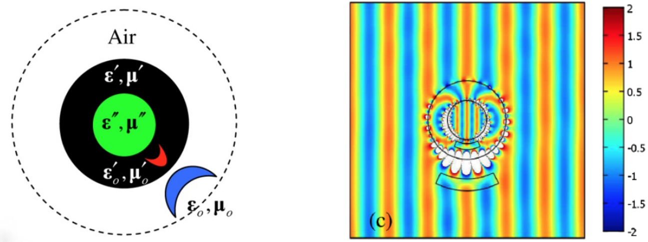

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Our proposal

Our construction: 2 parts

- The first one is to cancel the effect of the cloaked region.
- The second part is to fill the space which "disappears" from the cancellation.
- For the first part, we slightly change the strategy of Lai et. al.'s. We consider $B_{r_3} \setminus B_{r_2}$ as the cloaked region in which the medium is characterised by

$$b = \begin{cases} a & \text{in } B_{2r_2} \setminus B_{r_2}, \\ I & \text{in } B_{r_3} \setminus B_{2r_2}. \end{cases}$$

- The complementary media in $B_{r_2} \setminus B_{r_1}$ is given by

$$-\left(F^{-1}\right)_* b,$$

  Here $F : B_{r_2} \setminus \bar{B}_{r_1} \to B_{r_3} \setminus \bar{B}_{r_2}$ is the Kelvin's transform w.r.t. ∂B_{r_2}, i.e., $F(x) = r_2^2 x/|x|^2$.
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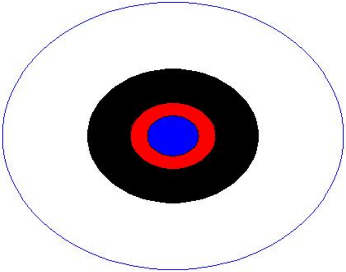
Mathematics setting

To study the problem correctly, one needs to add some loss to the medium. With the loss, the medium is characterized by $s_\delta A$, where

$$A = \begin{cases} b & \text{in } B_{r_3} \setminus B_{r_2}, \\ F_*^{-1} b & \text{in } B_{r_2} \setminus B_{r_1}, \\ \left(r_3^2/r_2^2\right)^{d-2} I & \text{in } B_{r_1}, \\ I & \text{otherwise,} \end{cases}$$

and

$$s_\delta = \begin{cases} -1 + i\delta & \text{in } B_{r_2} \setminus B_{r_1}, \\ 1 & \text{otherwise.} \end{cases}$$



Statement of the result

Let Ω be a smooth open subset of $\mathbb{R}^d$ $(d = 2, 3)$ such that $B_{r_3} \subset\subset \Omega$. Given $f \in L^2(\Omega)$, let u_δ, $u \in H^1_0(\Omega)$ be resp. the unique solution to

$$\operatorname{div}(s_\delta A \nabla u_\delta) = f \text{ in } \Omega, \tag{0.1}$$

and

$$\Delta u = f \text{ in } \Omega. \tag{0.2}$$

Theorem (Ng.)

Let $d = 2, 3$, $f \in L^2(\Omega)$ *with supp* $f \subset \Omega \setminus B_{r_3}$. *There exists* $m > 0$ *s.t. if* $r_3 > m r_2$ *then*

$$u_\delta \to u \text{ weakly in } H^1(\Omega \setminus B_{r_3}) \text{ as } \delta \to 0.$$

For an observer outside B_{r_3}, the medium in B_{r_3} looks like I: one has cloaking.

Sketch of the proof

We have

$$\|u_\delta\|_{H^1(\Omega)} \leqslant C\delta^{-1/2}\|f\|^{1/2}_{L^2(\Omega)}\|u_\delta\|^{1/2}_{L^2(\Omega\setminus B_{r_3})}(\sim \delta^{-1/2}) \qquad (0.3)$$

Let $u_{1,\delta}$ be the refl. of u_δ through ∂B_{r_2} and $u_{2,\delta}$ be the refl. of $u_{1,\delta}$ through ∂B_{r_3} (by F and G, the Kelvin's transform w.r.t. ∂B_{r_2} and ∂B_{r_3}). We have

$$\operatorname{div}(b\nabla u_{1,\delta}) = 0 \text{ in } B_{r_3} \setminus B_{r_2} \quad \text{and} \quad \Delta u_{2,\delta} = 0 \text{ in } B_{r_3}.$$

If

$$u_\delta - u_{1,\delta} \text{ would be small on } \partial B_{r_3},$$

then, $\Delta \widetilde{W}_\delta = f +$ lower order term, where $\widetilde{W}_\delta = \begin{cases} u_\delta & \text{in } \Omega \setminus B_{r_3} \\ u_{2,\delta} & \text{in } B_{r_3} \end{cases}$. Hence $\widetilde{W}_\delta \to u$. The proof would be complete. This is not true in general !!!

How to deal with this: three spheres inequality + removing localized singularity technique

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Recall

$$\operatorname{div}(b\nabla u_{1,\delta}) = 0 \text{ in } B_{r_3}\setminus B_{r_2} \quad \text{and} \quad \Delta u_{2,\delta} = 0 \text{ in } B_{r_3}.$$

Three spheres inequality, if $\operatorname{div}(A\nabla V) = 0$ in B_{r_3}, then

$$\|V\|_{L^2(\partial B_{2r_2})} \leqslant C\|V\|^{\alpha}_{L^2(\partial B_{r_2})}\|V\|^{1-\alpha}_{L^2(\partial B_{r_3})}.$$

Since $u_\delta = u_{1,\delta}$ and $\partial_r u_\delta = (1 - i\delta)\partial_r u_{1,\delta}$ on ∂B_{r_2}, it follows that if $r_3 >> r_2$,

$$u_\delta - u_{1,\delta} \text{ is small on } \partial B_{2r_2}.$$

Define

$$W_\delta = \begin{cases} u_\delta & \text{in } \Omega\setminus B_{r_3}, \\ u_{2,\delta} - (u_{1,\delta} - u_\delta) & \text{in } B_{r_3}\setminus B_{2r_2}, \\ u_{2,\delta} & \text{in } B_{2r_2}. \end{cases}$$

Then $\Delta W_\delta = f$ in $\Omega \setminus (\partial B_{r_3} \cup \partial B_{2r_2})$, $[W_\delta]$ and $[A\nabla W_\delta \cdot \nu]$ are small on $\partial B_{r_3} \cup \partial B_{2r_2}$ and $W_\delta = u_\delta$ in $\Omega \setminus B_{r_3}$. The conclusion follows. □

Summary

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- Two interesting examples.
- Superlensing using complementary media
- Cloaking using complementary media.

Thank you for your attention!

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