

Asymptotic behaviour of the spectra of systems of Maxwell equations in periodic composite media with high contrast

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Joint work with Shane Cooper

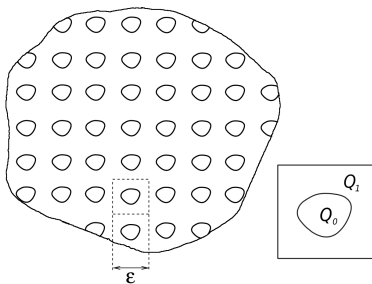
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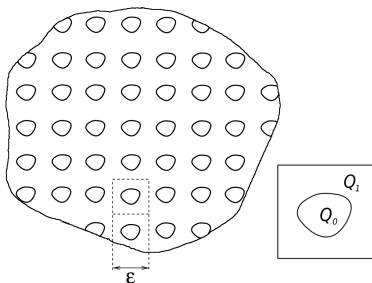
Mathematical and Computational Aspects of Maxwell's Equations

LMS Durham Symposium



Homogenisation setting

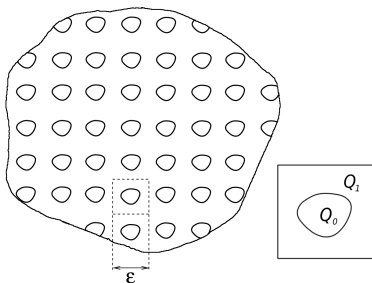




“Classical” homogenisation

$$-\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u\right) + u = f, \quad f \in L^2(\mathbb{R}^d),$$

$$A \geq \nu I > 0$$



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Convergence (two-scale expansions, compensated compactness, two-scale convergence, periodic unfolding, Bloch decomposition):

$$u = u_\varepsilon \rightharpoonup u_0 \text{ in } H^1(\mathbb{R}^d), \quad -\operatorname{div}A^{\operatorname{hom}}\nabla u_0 + u_0 = f.$$

Problem under study: high contrast

We study the problem

$$\operatorname{curl} \left(\epsilon_\eta^{-1} \left(\frac{x}{\eta} \right) \operatorname{curl} H^\eta \right) = \omega_\eta^2 H^\eta, \quad \epsilon_\eta(y) = \begin{cases} 1, & y \in Q_1, \\ \eta^{-2}, & y \in Q_0, \end{cases}$$

$\eta \in (0, 1)$ period, $Q_1 := Q \setminus \overline{Q_0}$ simply connected Lipschitz set

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Define: "Homogenised matrix"

$$A^{\text{hom}} := \int_{Q_1} (\operatorname{curl} N(y) + I) \, dy,$$

$\operatorname{curl}(\operatorname{curl} N(y) + I) = 0$ in Q_1 , $(\operatorname{curl} N(y) + I) \times n = 0$ on ∂Q_0 , N is Q -periodic.

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Consider: $\omega \in \mathbb{R}_+$, $H^0(x, y) := u(x) + \nabla_y v(x, y) + z(x, y)$,
 $(u, v, z) \in [H_{\# \operatorname{curl}}^1(\mathbb{T})]^3 \times L^2(\mathbb{R}^3; H_{\#}^1(Q)) \times [L^2(\mathbb{T}; H_0^1(Q_0))]^3$, solution to

$$\operatorname{curl}_x (A^{\text{hom}} \operatorname{curl}_x u(x)) = \omega^2 \left(u(x) + \int_{Q_0} z(x, y) dy \right), \quad x \in \mathbb{T},$$

$$\operatorname{div}_y (\nabla_y v(x, y) + z(x, y)) = 0, \quad (x, y) \in \mathbb{T} \times Q,$$

$$\operatorname{curl}_y (\operatorname{curl}_y z(x, y)) = \omega^2 (u(x) + \nabla_y v(x, y) + z(x, y)), \quad (x, y) \in \mathbb{T} \times Q_0.$$

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Our result: A) \exists at least one eigenfrequency ω_η such that $|\omega_\eta - \omega| < C\eta$,
B) $\operatorname{dist}(H^0, X_\eta) < \widehat{C}\eta$, where $\widehat{C} > 0$, $X_\eta := \operatorname{span}\{H^\eta : \omega_\eta \text{ as above}\}$.

Spectral decomposition of the limit problem

Write $v(x, \cdot) = G * (\operatorname{div}_y z)(x, \cdot)$, then

$$\operatorname{curl}_y \operatorname{curl}_y z(x, y) = \omega^2 \left(u(x) + \nabla_y \int_{Q_0} G(y - y') \operatorname{div}_{y'} z(x, y') dy' + z(x, y) \right),$$
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Note that

$$\nabla_y \int_{Q_0} G(y - y') \operatorname{div}_{y'} z(x, y') dy' + z(x, y) = \omega^2 B(y) u(x),$$

B is 3×3 matrix function with columns $B^j \in [H^1_{\#}(Q)]^3$, $j = 1, 2, 3$:

$$\operatorname{curl} \operatorname{curl} B^j = e_j + \omega^2 B^j \quad \text{in } Q_0,$$

$$\operatorname{curl} B^j(y) = 0, \quad y \in Q_1, \tag{1}$$

$$\operatorname{div} B^j(y) = 0, \quad y \in Q, \tag{2}$$

$$a(B^j) = 0, \tag{3}$$

$a(B^j)$ "circulation" of B^j :

$$H^1\text{-continuous extension of } a(\phi)_i = \int_0^1 \phi_i(te_i) dt, \quad \phi \in [C^\infty(Q)]^3.$$

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Equivalent **variational formulation**

$$\int_{Q_0} \operatorname{curl} B^j \cdot \operatorname{curl} \varphi = \int_Q e_j \cdot \varphi + \omega^2 \int_Q B^j \cdot \varphi, \quad \forall \varphi \in [H_{\#}^1(Q)]^3 \text{ subject to (1)-(3)}.$$

“Macroscopic” equation

Operator-pencil spectral problem

$$\operatorname{curl} (A^{\text{hom}} \operatorname{curl} u(x)) = \Gamma(\omega)u(x), \quad x \in \mathbb{T}, \quad (4)$$

where Γ is a matrix-valued function that vanishes at $\omega = 0$, and for $\omega \neq 0$ has elements

$$\Gamma_{ij}(\omega) = \omega^2 \left(\delta_{ij} + \omega^2 \int_Q B_i^j \right), \quad i, j = 1, 2, 3. \quad (5)$$

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Alternative representation for $\Gamma(\omega)$:

Consider $\phi^k \in [H_0^1(Q_0)]^3$, $k \in \mathbb{N}$, solutions to non-local problems

$$\operatorname{curl} \operatorname{curl} \phi^k(y) = \alpha_k \left(\nabla \int_{Q_0} G(y - y') \operatorname{div} \phi^k(y') dy' + \phi^k(y) \right), \quad y \in Q_0, \quad (6)$$

subject to orthonormality conditions

$$\int_{Q_0} \int_{Q_0} (\nabla^2 G(y - y') + I) \phi_j(y) \cdot \overline{\phi^k(y')} dy dy' = \delta_{jk}, \quad j, k = 1, 2, \dots,$$

where $\nabla^2 G$ is the Hessian matrix of G . Then

$$\Gamma_{ij}(\omega) = \omega^2 \delta_{ij} + \omega^4 \sum_{k=1}^{\infty} \frac{\left(\int_{Q_0} \phi_i^k \right) \left(\int_{Q_0} \phi_j^k \right)}{\alpha_k - \omega^2}, \quad i, j = 1, 2, 3, \quad \omega^2 \notin \{0\} \cup \{\alpha_k\}_{k=1}^{\infty}. \quad (7)$$

Structure of the limit spectrum

$$u(x) = \sum_{m \in \mathbb{Z}^3} \exp(2\pi i m \cdot x) \hat{u}(m), \quad \hat{u}(m) := \int_{\mathbb{T}} \exp(-2\pi i m \cdot x) u(x) dx,$$

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Then

$$\begin{aligned} \mathcal{M}(m) \hat{u}(m) &= \Gamma(\omega) \hat{u}(m), \quad m \in \mathbb{Z}^3, \\ \mathcal{M}_{lp}(m) &= 4\pi^2 \varepsilon_{ils} m_s A_{ij}^{\text{hom}} \varepsilon_{jpt} m_t = 4\pi^2 (e_l \times m) \cdot A^{\text{hom}}(e_p \times m). \end{aligned}$$

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$$\begin{aligned} \hat{u}(m) &= C(\tilde{m})^\top \tilde{u}(\tilde{m}) + \alpha(\tilde{m}) \tilde{m}, \\ \tilde{u}(\tilde{m}) \in \mathbb{R}^2, \quad \alpha(\tilde{m}) \in \mathbb{R}, \quad C(\tilde{m}) &= \begin{pmatrix} \tilde{e}_{11}(\tilde{m}) & \tilde{e}_{12}(\tilde{m}) & \tilde{e}_{13}(\tilde{m}) \\ \tilde{e}_{21}(\tilde{m}) & \tilde{e}_{22}(\tilde{m}) & \tilde{e}_{23}(\tilde{m}) \end{pmatrix}. \end{aligned}$$

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Determine $(\tilde{u}(\tilde{m}), \alpha(\tilde{m})) \in \mathbb{R}^3 \setminus \{0\}$ such that

$$\begin{aligned} |m|^2 \Lambda(\tilde{m}) \tilde{u}(\tilde{m}) &= C(\tilde{m}) \Gamma(\omega) C(\tilde{m})^\top \tilde{u}(\tilde{m}) + \alpha(\tilde{m}) C(\tilde{m}) \Gamma(\omega) \tilde{m}, \\ \Gamma(\omega) C(\tilde{m})^\top \tilde{u}(\tilde{m}) \cdot \tilde{m} &= -\alpha(\tilde{m}) \Gamma(\omega) \tilde{m} \cdot \tilde{m}, \end{aligned} \tag{8}$$

where

$$\Lambda(\tilde{m}) := \begin{pmatrix} \lambda_1(\tilde{m}) & 0 \\ 0 & \lambda_2(\tilde{m}) \end{pmatrix}.$$

Different admissible propagation regimes

If the inclusion Q_0 is symmetric under a rotation by π around at least two of the three coordinate axes, then matrices A^{hom} and $\Gamma(\omega)$ are diagonal:

$$A^{\text{hom}} = \text{diag}(a_1, a_2, a_3), \quad \Gamma(\omega) = \text{diag}(\beta_1(\omega), \beta_2(\omega), \beta_3(\omega)).$$

The eigenvalues $\lambda_{1,2}(\tilde{m})$ of $\mathcal{M}(\tilde{m})$ are the solutions to the quadratic equation

$$\lambda^2 - \lambda \{ (a_2 + a_3) \tilde{m}_1^2 + (a_1 + a_3) \tilde{m}_2^2 + (a_1 + a_2) \tilde{m}_3^2 \} + (a_1 a_2 \tilde{m}_3^2 + a_2 a_3 \tilde{m}_1^2 + a_1 a_3 \tilde{m}_2^2) = 0. \quad (9)$$

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Suppose Q_0 is symmetric by a $\pi/2$ rotation around at least two of the three axes, say x_1 and x_2 , then $a = a_1 = a_2 = a_3$ and $\beta(\omega) = \beta_1(\omega) = \beta_2(\omega) = \beta_3(\omega)$.

If $\beta(\omega) \neq 0$, then $\tilde{u}(\tilde{m})$ is an arbitrary element of \mathbb{R}^2 and $\hat{u}(m) = C(\tilde{m})^\top \tilde{u}(\tilde{m})$ is an arbitrary vector of the (2-dimensional) eigenspace spanned by the vectors $\tilde{e}_1(\tilde{m})$ and $\tilde{e}_2(\tilde{m})$.

Isotropic propagation (no “weak” band gaps)

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Isotropic propagation (no “weak” band gaps)

Suppose Q_0 is symmetric by a $\pi/2$ rotation around one of the three coordinate axis, say x_1 , and by a π rotation around another axis, say x_2 , one has $a = a_1$, $b = a_2 = a_3$ and $\beta_2(\omega) = \beta_3(\omega)$.

Generically, propagation is restricted to the direction of $\tilde{e}_1(\tilde{m})$ (*resp.* $\tilde{e}_2(\tilde{m})$) which is orthogonal to the eigenvector(s) corresponding to the negative eigenvalue of $\Gamma(\omega)$.

Directional propagation (existence of “weak” band gaps)

Proposition

The spectrum of the limit problem is the union of the following sets.

- 1 The elements of $\{\alpha_k : k \in \mathbb{Z}\}$ such that at least one of the corresponding ϕ^k has zero mean over Q . These are eigenvalues of infinite multiplicity and the corresponding eigenfunctions $H^0(x, y)$ are of the form

$$w(x) \left(\nabla \int_{Q_0} G(y - y') \operatorname{div} \phi^k(y') dy' + \phi^k(y) \right), \quad w \in L^2(\mathbb{T}).$$

- 2 The set $\{\omega^2 : \exists m \in \mathbb{Z}^3 \text{ such that (8) holds}\}$, with eigenfunctions $H^0(x, y)$ of the limit problem having the form $u(x) + \nabla_y v(x, y) + z(x, y)$, where $u(x) = \exp(2\pi i m \cdot x) \hat{u}(m)$ is an eigenfunction of macroscopic problem and

$$\nabla_y v(x, y) + z(x, y) = \omega^2 B(y) u(x, y) \quad \text{a.e. } (x, y) \in \mathbb{T} \times Q,$$

that is $H^0(x, y) = (I + \omega^2 B(y)) \exp(2\pi i m \cdot x) \hat{u}(m)$.

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Corollary

If the matrix $\Gamma(\omega)$ is negative-definite, the value $\lambda = \omega^2$ does not belong to the spectrum of the limit problem.

Proof of main result

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A) We seek solutions in the form of an asymptotic expansion

$$H^\eta(x) = H^0\left(x, \frac{x}{\eta}\right) + \eta H^1\left(x, \frac{x}{\eta}\right) + \eta^2 H^2\left(x, \frac{x}{\eta}\right) + \dots, \quad (10)$$

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B) \mathcal{A}_η operator in $L^2_{\#\text{sol}}(\mathbb{T})$ defined by the form

$$\int_{\mathbb{T}} \varepsilon_\eta^{-1}(\frac{\cdot}{\eta}) \operatorname{curl} u \cdot \operatorname{curl} v, \quad u, v \in [H^1_{\#}(\mathbb{T})]^3 \cap L^2_{\#\text{sol}}(\mathbb{T}) =: \mathcal{H}.$$

For fixed ω in the spectrum of limit problem let H^0 be a corresponding eigenfunction. Consider solution $\tilde{H}^\eta \in \mathcal{H}$ to

$$(\mathcal{A}_\eta + I)\tilde{H}^\eta = (\omega^2 + 1)H^0(\cdot, \frac{\cdot}{\eta}). \quad (11)$$

and

$$H^{(2)}(\cdot, \eta) := H^0(\cdot, \frac{\cdot}{\eta}) + \eta H^1(\cdot, \frac{\cdot}{\eta}) + \eta^2 H^2(\cdot, \frac{\cdot}{\eta}), \quad (12)$$

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where $H^j(x, y)$, $j = 0, 1, 2, \dots$, are Q -periodic in the variable y .

B) \mathcal{A}_η operator in $L^2_{\#\text{sol}}(\mathbb{T})$ defined by the form

$$\int_{\mathbb{T}} \varepsilon_\eta^{-1}(\frac{\cdot}{\eta}) \operatorname{curl} u \cdot \operatorname{curl} v, \quad u, v \in [H^1_{\#}(\mathbb{T})]^3 \cap L^2_{\#\text{sol}}(\mathbb{T}) =: \mathcal{H}.$$

For fixed ω in the spectrum of limit problem let H^0 be a corresponding eigenfunction. Consider solution $\tilde{H}^\eta \in \mathcal{H}$ to

$$(\mathcal{A}_\eta + I)\tilde{H}^\eta = (\omega^2 + 1)H^0(\cdot, \frac{\cdot}{\eta}). \quad (11)$$

and

$$H^{(2)}(\cdot, \eta) := H^0(\cdot, \frac{\cdot}{\eta}) + \eta H^1(\cdot, \frac{\cdot}{\eta}) + \eta^2 H^2(\cdot, \frac{\cdot}{\eta}), \quad (12)$$

C) There exists a constant $\hat{C} > 0$ such that the estimate

$$\mathfrak{b}_\eta(\tilde{H}^\eta - H^{(2)}(\cdot, \eta), \varphi) \leq \hat{C}\eta\sqrt{\mathfrak{b}_\eta(\varphi, \varphi)} \quad \forall \varphi \in [H^1_{\#}(\mathbb{T})]^3,$$

where

$$\mathfrak{b}_\eta(u, v) := \int_{\mathbb{T}} \varepsilon_\eta^{-1}(\frac{\cdot}{\eta}) \operatorname{curl} u \cdot \operatorname{curl} v + \int_{\mathbb{T}} u \cdot v$$

Theorem

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Proof

Setting $\varphi = \tilde{H}^\eta - H^{(2)}(\cdot, \eta)$:

$$\hat{C}^2 \eta^2 \geq \mathfrak{b}_\eta(\tilde{H}^\eta - H^{(2)}(\cdot, \eta), \tilde{H}^\eta - H^{(2)}(\cdot, \eta)) \geq \|\tilde{H}^\eta - H^{(2)}(\cdot, \eta)\|_{L^2(\mathbb{T})}^2.$$

Note that

$$\|H^{(2)}(\cdot, \eta) - H^0(\cdot, \cdot/\eta)\|_{L^2(\mathbb{T})} \leq \tilde{C}\eta, \quad \tilde{C} > 0,$$

and hence

$$\begin{aligned} \|\tilde{H}^\eta - H^0(\cdot, \cdot/\eta)\|_{L^2(\mathbb{T})} &\leq \|\tilde{H}^\eta - H^{(2)}(\cdot, \eta)\|_{L^2(\mathbb{T})} \\ &\quad + \|H^{(2)}(\cdot, \eta) - H^0(\cdot, \cdot/\eta)\|_{L^2(\mathbb{T})} \leq (\hat{C} + \tilde{C})\eta. \end{aligned}$$

Proposition

Suppose that σ is a rotation such that $\sigma Q = Q$ and assume that

$$a(y) = \sigma^{-1} a(\sigma y) \sigma, \quad y \in Q. \quad (13)$$

Then A^{hom} inherits the same symmetry, i.e. one has

$$A^{\text{hom}} = \sigma^{-1} A^{\text{hom}} \sigma.$$

In particular, $A_{kl}^{\text{hom}} = A_{lk}^{\text{hom}} = 0$, for all $l \neq k$.

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Corollary

1. If (13) holds for $\sigma = \sigma_k$, where σ_k is the rotation by π around the x_k -axis, then $A_{kl}^{\text{hom}} = 0$, for all $l \neq k$.
2. If (13) holds for $\sigma = \sigma_k$, where σ_k is the rotation by $\pi/2$ around the x_k -axis, then $A_{kl}^{\text{hom}} = 0$, for all $l \neq k$ and $A_{ii}^{\text{hom}} = A_{jj}^{\text{hom}}$, $i, j \neq k$.

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Similar results can be demonstrated for $\Gamma(\omega)$.