Virtual Hybrid Edge Detection: Propagation and recovery of singularities in Calderón's inverse conductivity problem

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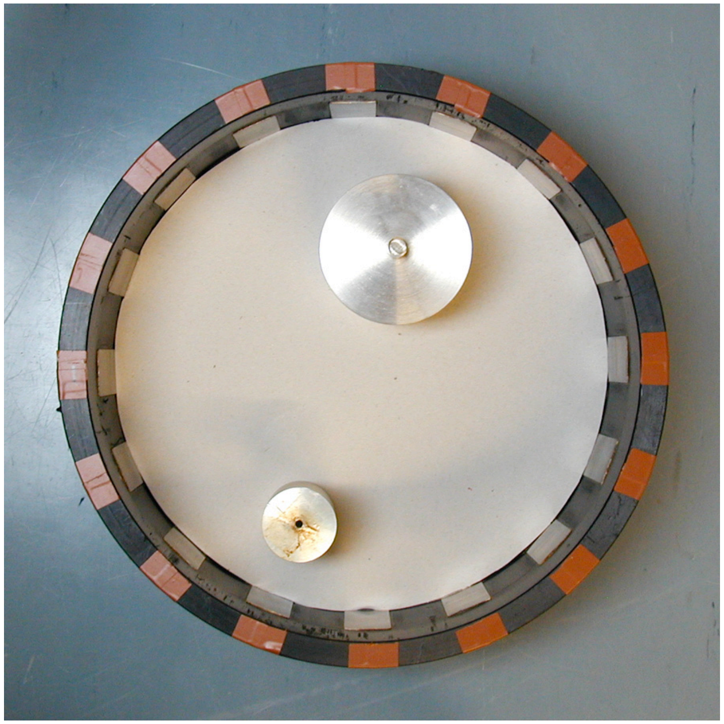
Electrical impedance tomography (EIT)

Problem: EIT is high contrast, but low resolution:



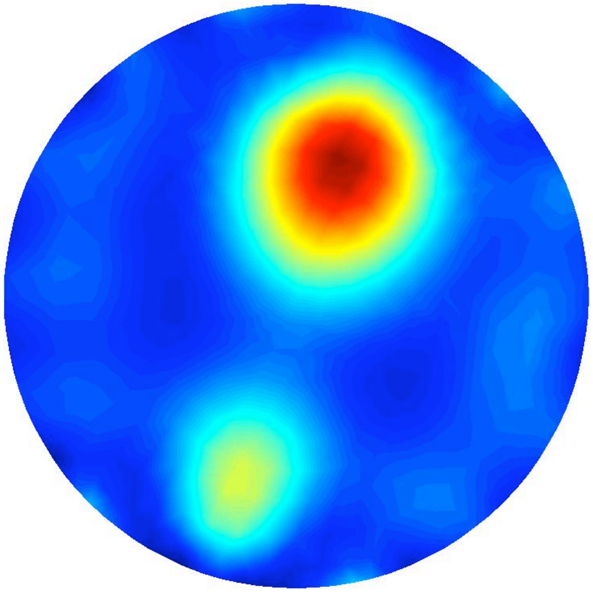


Figure 1: EIT tank and measurements. Source: Kaipio lab, Univ. of Kuopio, Finland

Hybrid imaging

"Multi-wave" methods often combine illumination and measurement modalities, one having

- **high contrast sensitivity/low resolution (EIT,.)**

and the other exhibiting

- **low contrast/high resolution (ultrasound. . .)**

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Mathematically: couple an elliptic PDE with a hyperbolic / real principal type PDE

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Exploit complex principal type operator geometry underlying CGO solutions $\Longrightarrow$ Singularities propagate efficiently along 2D characteristics to all of $\partial\Omega$ (in $\mathbb{R}^2$).

Formally: if σ is pws with jumps (edges), can stably reconstruct leading singularities. Can image inclusions within inclusions.

Astala-Päivärinta CGO solutions in 2D

$\Omega \subset \mathbb{R}^2 = \mathbb{C}$, $(x, y) = x + iy = z$, $(\xi, \eta) = \xi + i\eta = \zeta$

$\sigma \in L^\infty(\Omega)$, $0 < c_1 \le \sigma(z) \le c_2 < \infty$**,**

$\sigma \equiv 1$ **near** $\partial\Omega$**, extended to** $\mathbb{R}^2$**.**

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Exponentially growing/decaying solutions:

For $k \in \mathbb{C}$ **a complex frequency,** $\exists\ u_1$, u_2 **s.t.**

$$\nabla \cdot \sigma \nabla u_1 = 0, \quad \nabla \cdot \sigma^{-1} \nabla u_2 = 0, \ \textbf{on } \mathbb{R}^2,$$

$$u_1, u_2 \sim e^{ikz}\Big(1 + O\big(1/|z|\big)\Big), \ |z| \to \infty.$$

Beltrami equations

Let $\mu = \mu_\sigma = \frac{1-\sigma}{1+\sigma}$**, so** $|\mu| \leq 1 - \epsilon$**,** $\mu_{\sigma^{-1}} = -\mu_\sigma$**.**

Look for CGO solns $f_\mu(z)$ **of** $\overline{\partial}_z f_\mu = \mu \overline{\partial_z f_\mu}$**, similarly** $f_{-\mu}$ **for** $-\mu$ **:**

$$f_{\pm\mu}(z,k) = e^{ikz}\big(1 + \omega^{\pm}(z,k)\big), \quad \omega^{\pm} = O\big(1/|z|\big).$$

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$(u_1,\, u_2) \leftrightarrow (f_\mu,\, f_{-\mu})$

$\omega^{\pm}$ **can be computed from D2N data for** σ**.**

Focus on $\omega^{+} =: \omega$**.**

Huhtanen and Perämäki solutions (2012)

Let

$$e_k(z) = e^{i(kz+\overline{kz})} = e^{i\,2\,Re\,(kz)},$$

so that $|e_k(z)| = 1$**,** $\quad \overline{e_k} = e_{-k}$**.**

Define

$$\alpha(z,k) = -i\overline{k}e_{-k}(z)\mu(z), \qquad \beta(z,k) = e_{-k}(z)\mu(z)$$

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Then $\omega(z,k)$ **satisfies a** $\mathbb{R}$**-Beltrami equation:**

$$(\mathbf{1}) \qquad \overline{\partial}\omega - \beta\overline{\partial\omega} - \alpha\overline{\omega} = \alpha.$$

H.-P. show that $\exists!\ \omega \in W^{1,p}(\mathbb{C})$, $2 < p < p_\epsilon$**.**

Solid Cauchy and Beurling transforms

$$Pf(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(z')}{z-z'}\, d^2z', \qquad Sf(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(z')}{(z'-z)^2}\, d^2z'$$

so that $\overline{\partial}P = I$**,** $S = \partial P$ **and** $S\overline{\partial} = \partial$ **on** $C_0^\infty(\mathbb{C})$**.**

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so that $\overline{\partial}P = I$, $S = \partial P$ and $S\overline{\partial} = \partial$ on $C_0^\infty(\mathbb{C})$.

Define $u = -\partial\overline{\omega} = -\overline{(\overline{\partial}\omega)} \in L^p$. Then

$\omega = -P\overline{u}$ and $\partial\omega = -S\overline{u}$ and (1) becomes

$$(\mathbf{1'}) \qquad \big(I + A\rho\big)u = -\overline{\alpha},$$

where $\rho =$ complex conjugation and

$$A = -(\overline{\alpha}P + \overline{\beta}S)$$

Neumann series

Expand $u \sim \sum_{n=0}^{\infty} u_n$, $u_0 = -\overline{\alpha}$, $u_{n+1} = -A\overline{u_n}$

$\implies \; \omega = -P\overline{u} \sim \sum_{n=0}^{\infty} \omega_n$, $\omega_n = -P\overline{u_n}$ **.**

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Focus on

$$
\begin{aligned}
u_0 &= -\overline{\alpha}, & \omega_0 &= P\alpha, \\
u_1 &= A\alpha = -(\overline{\alpha}P + \overline{\beta}S)(\alpha), & \omega_1 &= P(\alpha\overline{P\alpha} + \beta\overline{S\alpha}).
\end{aligned}
$$

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Focus on

$$u_0 = -\overline{\alpha}, \qquad \omega_0 = P\alpha,$$

$$u_1 = A\alpha = -(\overline{\alpha}P + \overline{\beta}S)(\alpha), \qquad \omega_1 = P(\alpha\overline{P\alpha} + \beta\overline{S\alpha}).$$

$\omega_0|_{z\in\partial\Omega}$**: Stably determines singularities of** μ**.**

$\omega_n|_{z\in\partial\Omega,}\, n \geq 1$**: Contribute scattering, which explains artifacts in numerics.**

Note: ω_n **is an** $(n+1)$**-linear operator of** μ**.**

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- **WF set analysis: all $\widetilde{\omega}_n$ for general $\sigma \in L^\infty$.**

- **Operator theory: σ pws with jumps across curved interfaces $\implies$ $\widetilde{\omega}_1$, $\widetilde{\omega}_2$ are in $I^{p,l}$ spaces.**

- **Higher order terms in Neumann series create a strong artifact at $t = 0$ and weaker ones via multiple scattering of points in $WF(\mu)$.**

$$\omega_0(z,k) = \frac{ik}{\pi}\int_{\mathbb{C}} \frac{e^{i\,2\,Re(kz')}\mu(z')}{z-z'}\,d^2z'$$

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1. Polar coordinates in k: write $k = \tau e^{i\varphi}$

2. Partial Fourier transform $\tau \to t$:

$$\begin{aligned}
\widetilde{\omega}_0(z,t,e^{i\varphi}) &:= \int_{\mathbb{R}} e^{-it\tau} \omega_0(z, \tau e^{i\varphi})\, d\tau \\
&= \frac{e^{i\varphi}}{\pi} \int_{\mathbb{R}} \int_{\mathbb{C}} (i\tau)\, \frac{e^{-i\tau(t - 2\,Re(e^{i\varphi} z'))}}{z - z'}\, \mu(z')\, d^2 z'\, d\tau \\
&= -2e^{i\varphi} \int_{\mathbb{C}} \frac{\delta'(t - 2\,Re(e^{i\varphi} z'))}{z - z'}\, \mu(z')\, d^2 z'
\end{aligned}$$

Recall: $\sigma \in L^{\infty}$, $\sigma \equiv 1$ **near** $\partial\Omega$**,** $\mu \equiv 0$ **near** $\partial\Omega$**.**
Assume $supp(\mu) \subset \Omega_0 \subset\subset \Omega$**.**

Define $T_0 : \mathcal{E}'(\Omega_0) \to \mathcal{D}'(\mathbb{C} \times \mathbb{R} \times \mathbb{S}^1)$**,**

$$\mu(z') \longrightarrow (T_0\mu)(z, t, e^{i\varphi}) := \widetilde{\omega}_0(z, t, e^{i\varphi}).$$

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Schwartz kernel of T_0**:**

$$K_0(z,t,e^{i\varphi},z') = \Big(\frac{-2e^{i\varphi}}{z-z'}\Big)\delta'(t - 2\,Re(e^{i\varphi}z')).$$

First factor is smooth for $z \notin \Omega_0$, $z' \in \Omega_0 \implies$

T_0 **is a generalized Radon transform and thus a Fourier integral operator (FIO).**

Define $T_0^{z_0} : \mathcal{E}'(\Omega_0) \to \mathcal{D}'(\mathbb{R} \times \mathbb{S}^1)$ **by**

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• $T_0^{z_0}$ **is a weighted and differentiated version of the Radon transform on** $\mathbb{C} \simeq \mathbb{R}^2$**.**

• $T_0^{z_0}$ **is an FIO of order** $\frac{1}{2}$**,** $\quad T_0^{z_0} \in I^{\frac{1}{2}}(C)$**, with same canonical relation as std. Radon transf.**

$$C = N^*\big\{t = 2\, Re(e^{i\varphi} z')\big\}' \subset T^*(\mathbb{R} \times \mathbb{S}^1) \times T^*\Omega_0.$$

• C **is a canonical graph.**

• C **is independent of** z_0**, but symbol**

$$\sigma_{prin}(T_0^{z_0}) = \frac{(-ie^{i\varphi})\, sgn(\tau)|\tau|^{\frac{1}{2}}}{z_0 - z'},$$

is not.

• **The factor** $(z_0 - z')^{-1}$ **is smooth and** $\neq 0$**, but causes**

(i) A fall-off in detectability of jumps, at rate $\sim d(z', z_0)^{-1}$**.**

(ii) Artifacts, esp. when μ **has singularities at** z' **close to** z_0**, due to the large magnitude and phase gradient of** $(z_0 - z')^{-1}$**.**

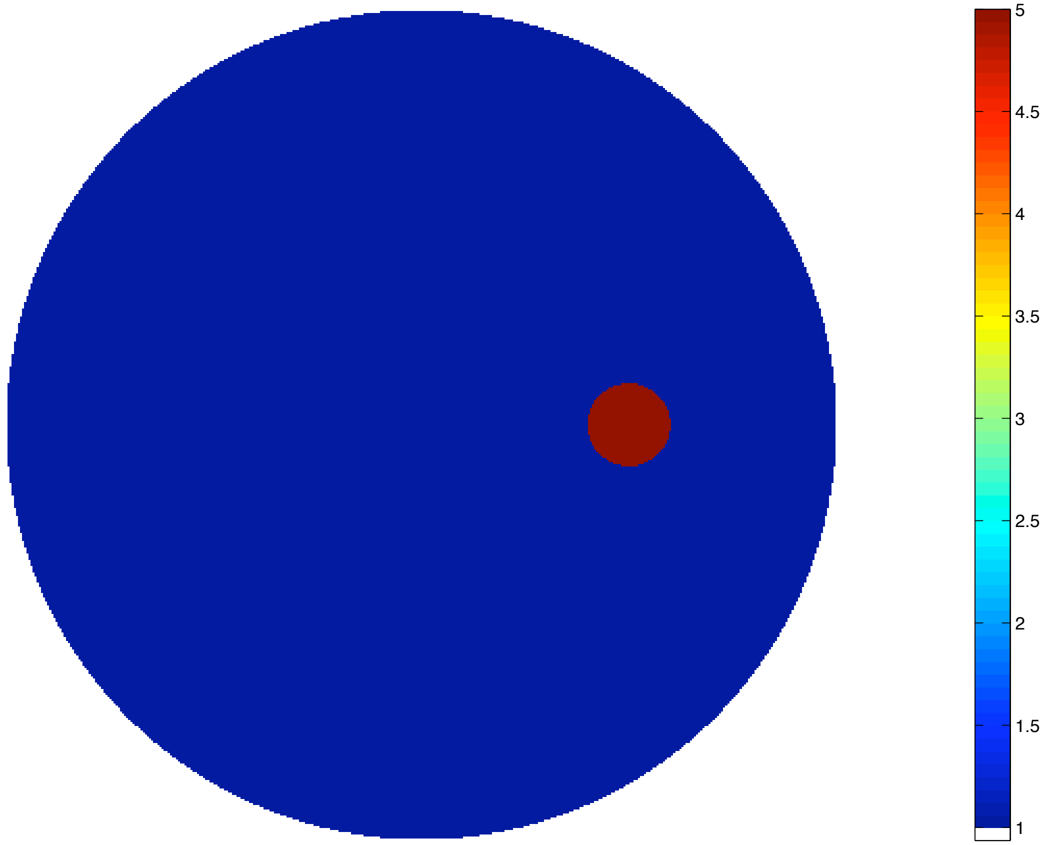

Figure 2: Conductivity phantom: a small circular inclusion.

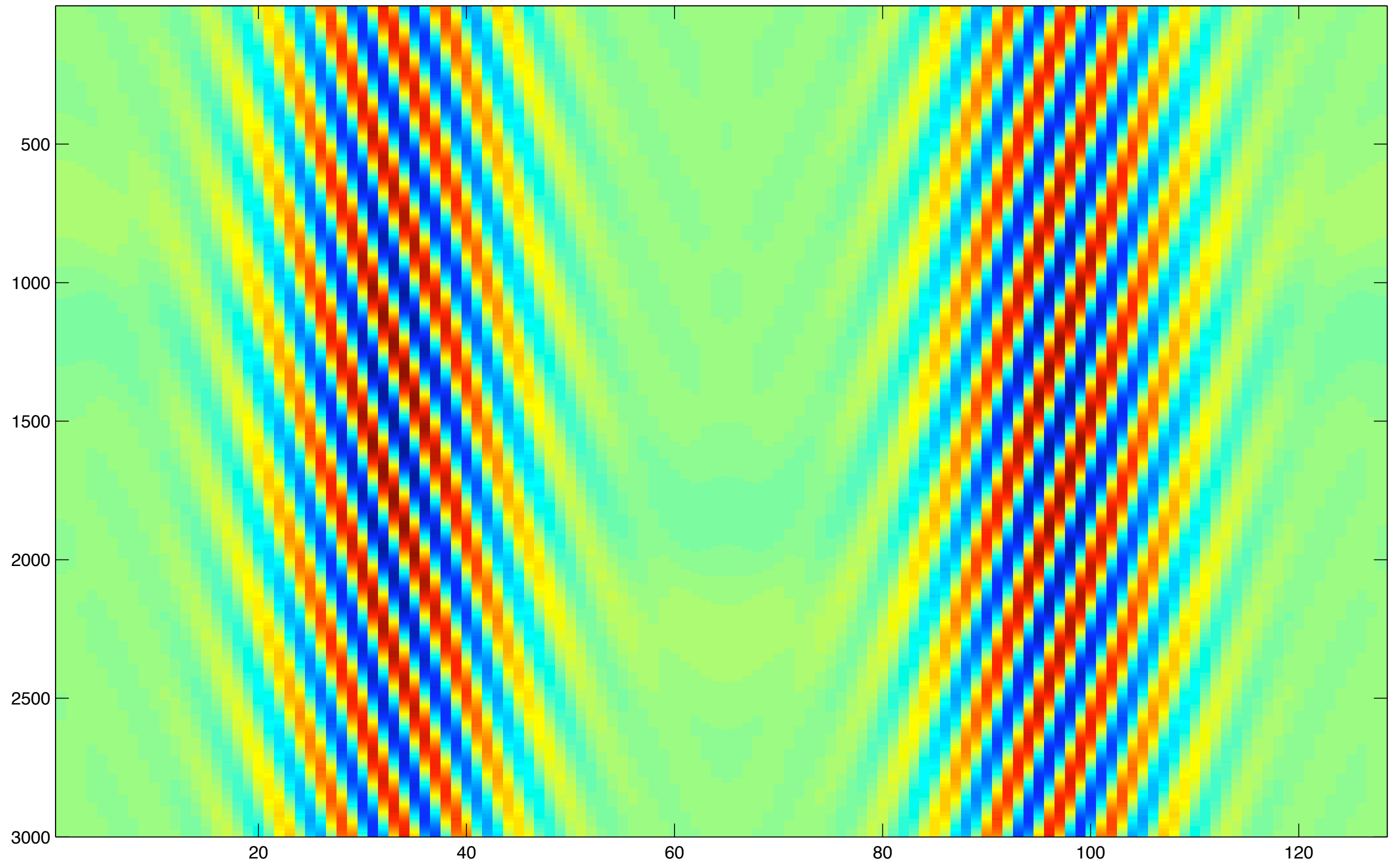

Figure 3: $Re\,\widetilde{\omega}(z_0, t, e^{i\varphi})$ (axes: $\varphi =$ horiz., $t =$ vert.)

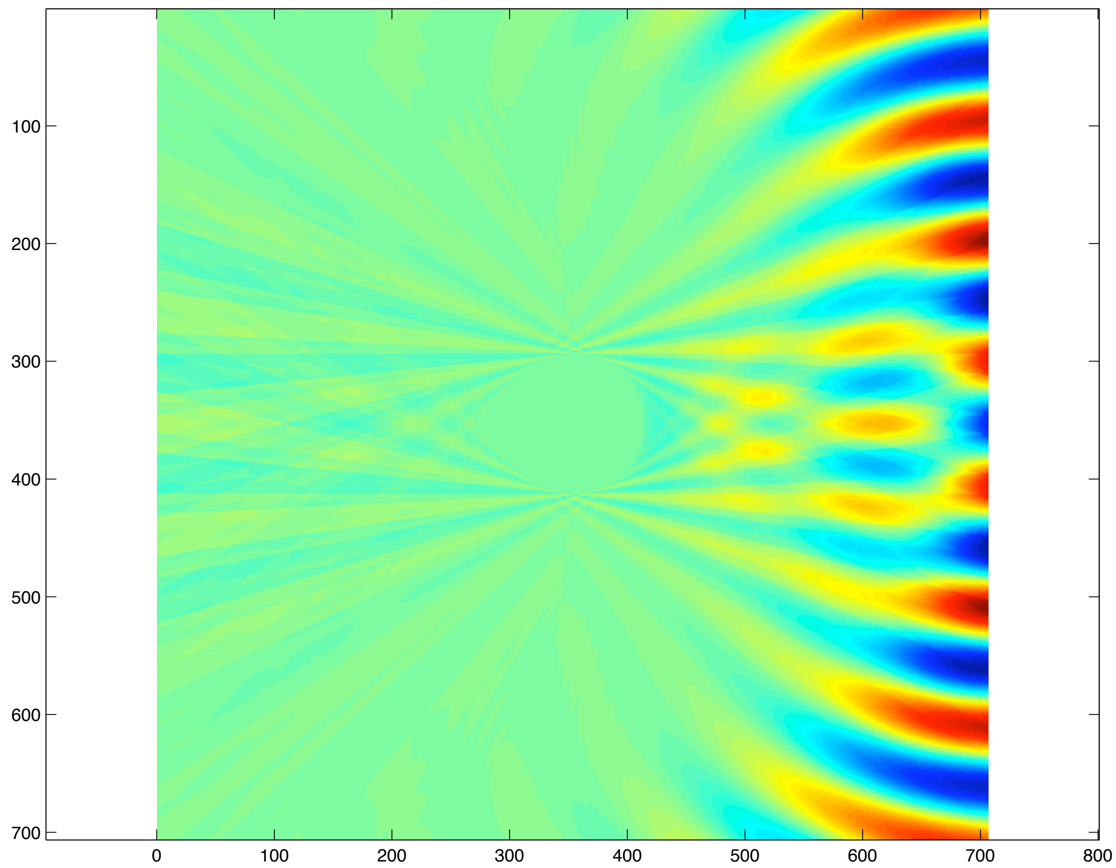

Figure 4: Backprojected reconstruction from $\omega(z_0, \cdot, \cdot)$.

Weighted 'averages' in z_0

Let $a(z_0)$ be a $\mathbb{C}$-valued weight on $\partial\Omega$. Form

$$(3) \qquad \widetilde{\omega}_0^a(t,\varphi) := \frac{1}{2\pi i}\int_{\partial\Omega} \widetilde{\omega}_0(z_0,t,\varphi)\, a(z_0)\, dz_0,$$

Let T_0^a be the operator $\mu \to \widetilde{\omega}_0^a$.

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Let T_0^a be the operator $\mu \to \widetilde{\omega}_0^a$.

Then $T_0^a \in I^{\frac{1}{2}}(C)$ and $(T_0^a)^* T_0^a \in \Psi^1(\Omega_0)$, with

$$\sigma_{prin}\left((T_0^a)^* T_0^a\right)(z', \zeta') = 2\pi^2 |\alpha(z')|^2 |\zeta'|,\ z' \in \Omega_0,$$

where

$$\alpha(z') = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{a(z_0)\, dz_0}{z_0 - z'}, \qquad z' \in \Omega$$

is the Cauchy (line) integral of $a(\cdot)$

Pick $a \equiv 1/\sqrt{2}$ **on** $\partial\Omega$ **in (2).** **(**$\int_{\partial\Omega} a\, dz_0 = 0$ **!)**

Then $\alpha(z') \equiv 1/\sqrt{2}$ **on** Ω_0 **and**

$$(T_0^a)^* T_0^a = (-\Delta)^{\frac{1}{2}}\ mod\ \Psi^0(\Omega_0),$$

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$$(T_0^a)^* T_0^a = (-\Delta)^{\frac{1}{2}} \; mod \; \Psi^0(\Omega_0),$$

Gives local-tomography type imaging of μ**, good for detection of singularities of** σ **from the singularities of** $\widetilde{\omega}$ **(which correspond to high frequency behavior of** ω**).**

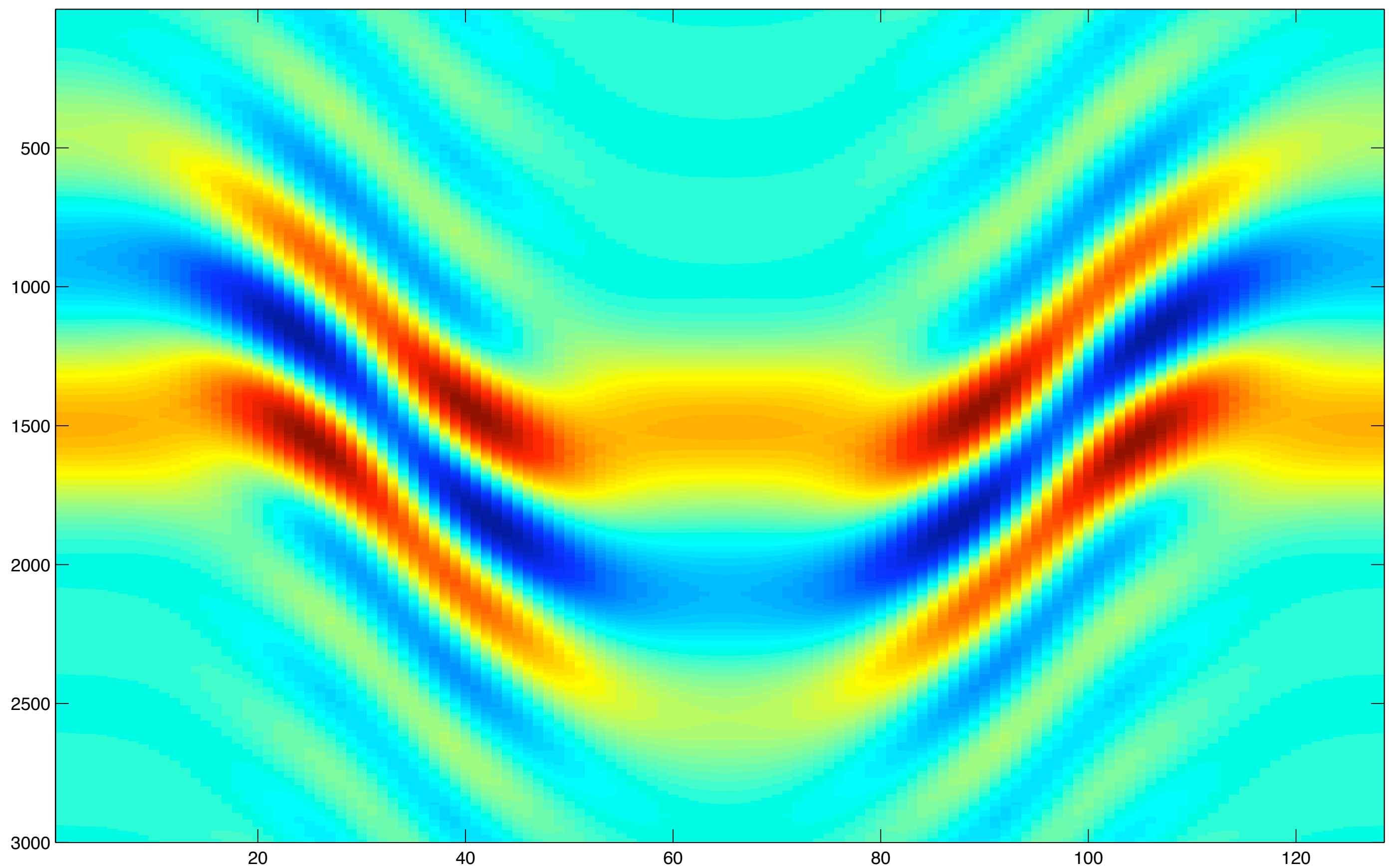

Figure 5: $Re\,\widetilde{\omega}^a(z_0, t, e^{i\varphi})$ for $a \equiv 1/(2\sqrt{2})$

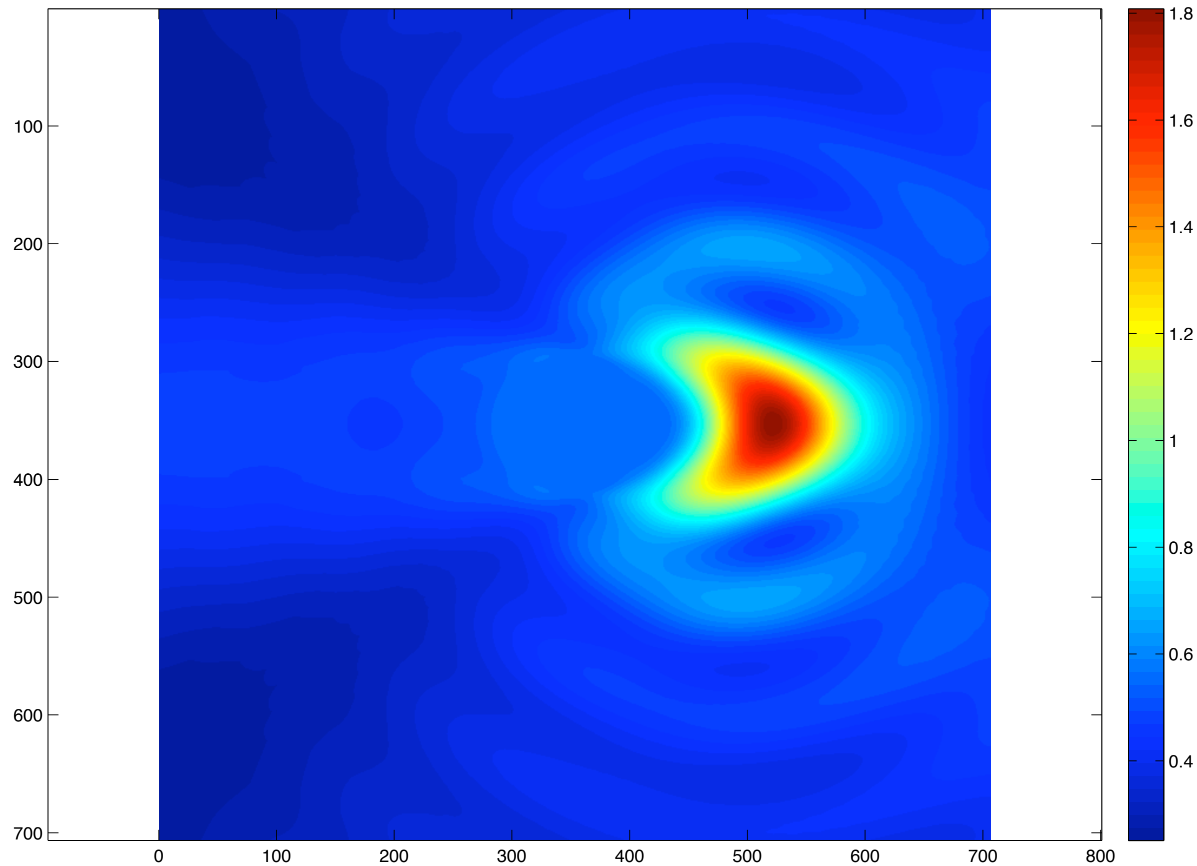

Figure 6: Reconstruction from $\widetilde{\omega}^a(\cdot,\cdot)$ for $a \equiv 1/\sqrt{2}$.

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However: the figures above were created after filtering out certain artifacts.

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Singularities of $\widetilde{\omega}^{z_0}$, $\widetilde{\omega}^a$ occur at

(i) $t = 0$ for any μ with singularities, and

(ii) at other values of t, φ, depending on μ.

Explained by wave-front set analysis of the higher order terms in the Neumann series, which are multilinear FIOs.

$$\widetilde{\omega}_1^{z_0}(t,\varphi) = \int e^{-it\tau}\omega_1(z_0,\tau e^{i\varphi})\,d\tau$$
$$= \int_\Omega\int_\Omega K_1^{z_0}(t,e^{i\varphi};z',z'')\cdot\mu(z')\cdot\mu(z'')\,d^2z'd^2z''$$

Bilinear operator acting on $\mu \otimes \mu$, w/ kernel

$$K_1^{z_0}(t,e^{i\varphi};z',z'') = \frac{1}{\pi^2}\Big(\frac{e^{2i\varphi}\delta''(t+2\,Re(e^{i\varphi}(z'-z'')))}{(z'-z_0)(\overline{z}''-\overline{z}')}$$
$$+\frac{e^{i\varphi}\delta'(t+2\,Re(e^{i\varphi}(z'-z'')))}{(z'-z_0)(\overline{z}''-\overline{z}')^2}\Big)$$

- $WF(\widetilde{\omega}_n)$ **can be described in terms of** $(n+1)$**-fold scatterings of** $WF(\mu)$**.**

- **Still to do: estimates to control smoothness of** $\widetilde{\omega}_n$ **for** $n \geq 3$**.**

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- **Still to do: estimates to control smoothness of** $\widetilde{\omega}_n$ **for** $n \geq 3$**.**

- **Can currently do this for** $n = 1, 2$ **under prior on** σ **which includes pws with jumps across curved interfaces.**

- **Rigorous justification of the Neumann series will require a mixture of multilinear microlocal and harmonic analysis.**

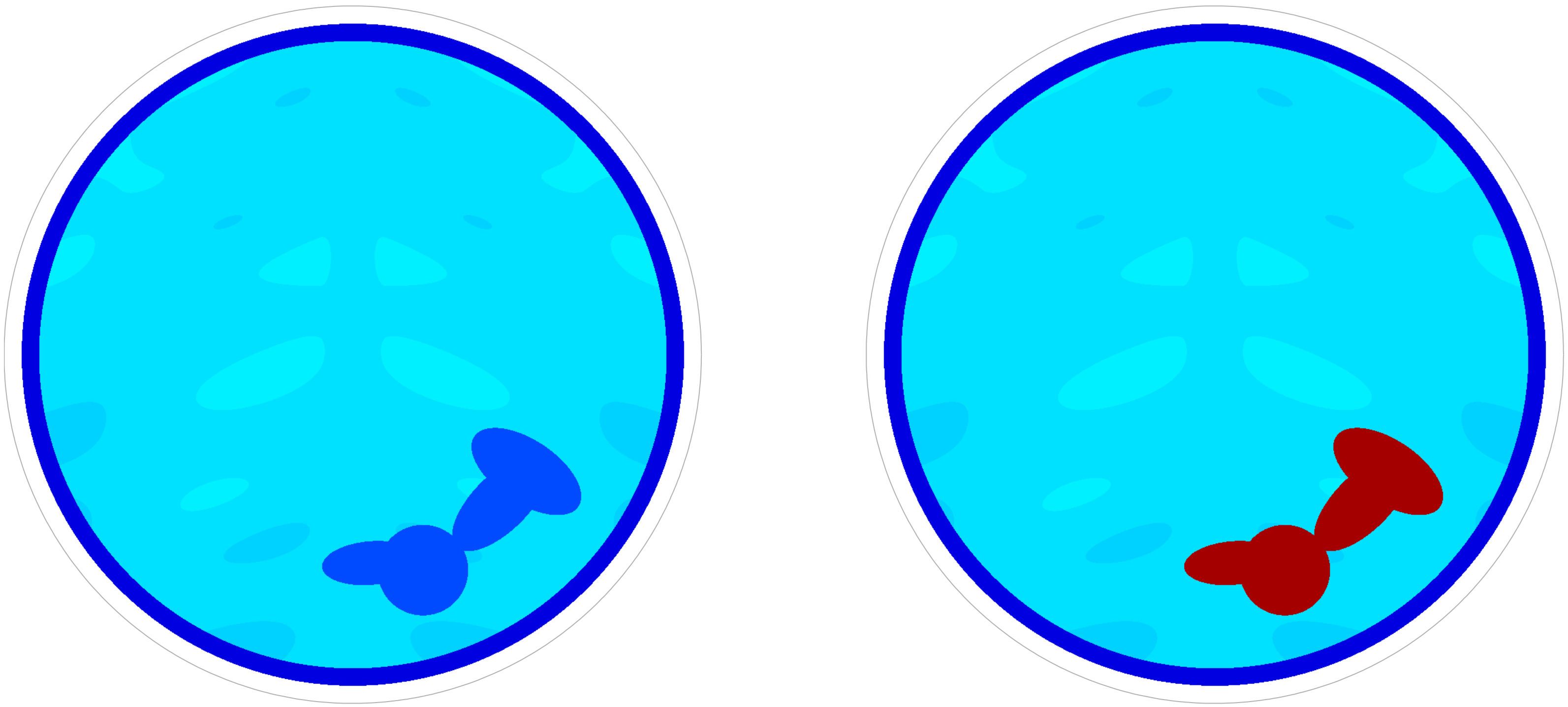

Figure 7: Stroke phantoms within low conductivity skull: Clot **(left)**, haemorrhage **(right)**.

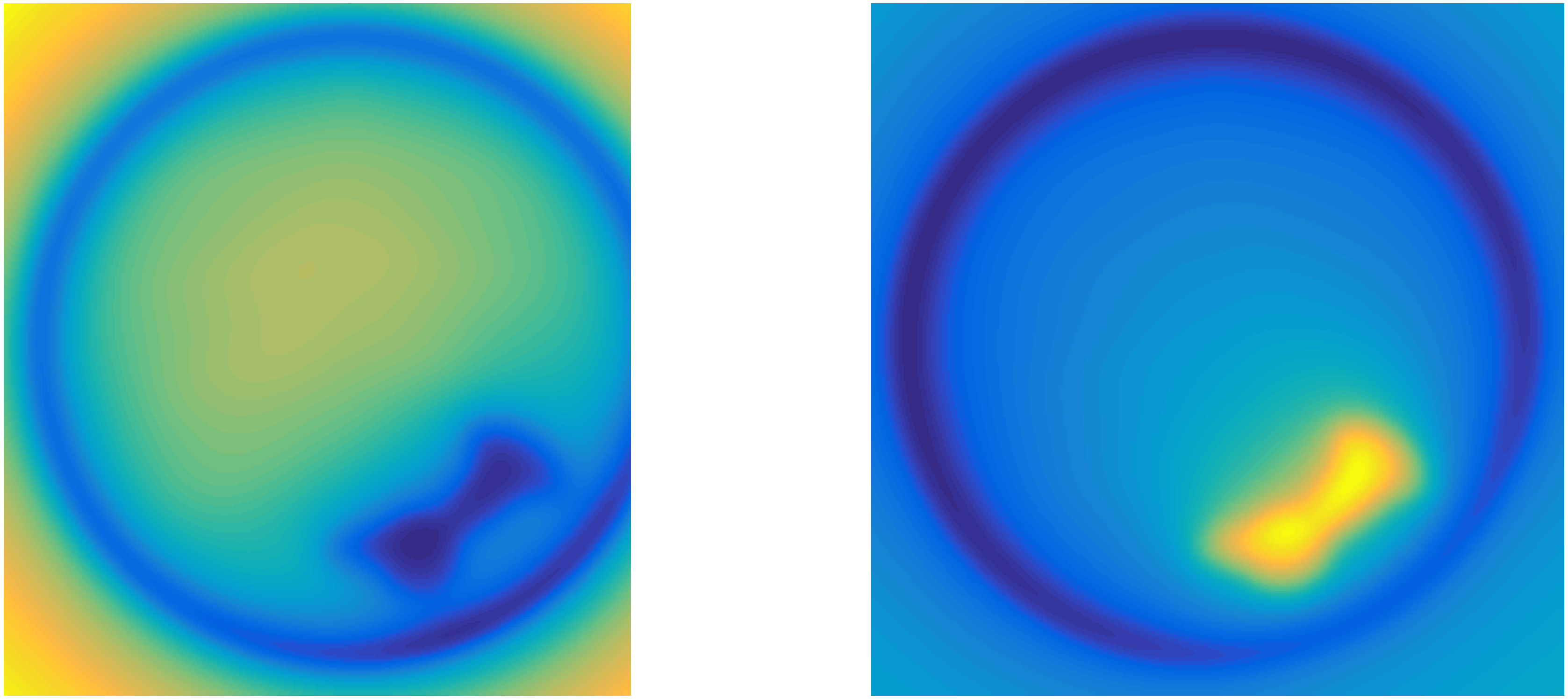

Figure 8: Stroke phantoms reconstructions: Clot **(left)**, haemorrhage **(right)**.