Discontinuous Galerkin method for hyperbolic equations with δ-singularities

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Outline

- Introduction to discontinuous Galerkin (DG) method for hyperbolic equations
- DG method for hyperbolic equations with δ-singularities
- High order accuracy bound-preserving limiters
- Applications

DG method for hyperbolic equations

We are interested in solving a hyperbolic conservation law

$$u_t + f(u)_x = 0$$

In 2D it is

$$u_t + f(u)_x + g(u)_y = 0$$

and in system cases u is a vector, and the Jacobian $f'(u)$ is diagonalizable with real eigenvalues.

Several properties of the solutions to hyperbolic conservation laws.

- The solution u may become discontinuous regardless of the smoothness of the initial conditions.
- Weak solutions are not unique. The unique, physically relevant entropy solution satisfies additional entropy inequalities

$$U(u)_t + F(u)_x \leq 0$$

in the distribution sense, where $U(u)$ is a convex scalar function of u and the entropy flux $F(u)$ satisfies $F'(u) = U'(u)f'(u)$.

To solve the hyperbolic conservation law:

$$u_t + f(u)_x = 0, \tag{1}$$

we multiply the equation with a test function v, integrate over a cell $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, and integrate by parts:

$$\int_{I_j} u_t v dx - \int_{I_j} f(u) v_x dx + f(u_{j+\frac{1}{2}}) v_{j+\frac{1}{2}} - f(u_{j-\frac{1}{2}}) v_{j-\frac{1}{2}} = 0$$

Now assume both the solution u and the test function v come from a finite dimensional approximation space V_h, which is usually taken as the space of piecewise polynomials of degree up to k:

$$V_h = \left\{ v : v|_{I_j} \in P^k(I_j),\ j = 1, \cdots, N \right\}$$

However, the boundary terms $f\left(u_{j+\frac{1}{2}}\right)$, $v_{j+\frac{1}{2}}$ etc. are not well defined when u and v are in this space, as they are discontinuous at the cell interfaces.

From the conservation and stability (upwinding) considerations, we take

- A single valued monotone numerical flux to replace $f\left(u_{j+\frac{1}{2}}\right)$:

$$\hat{f}_{j+\frac{1}{2}} = \hat{f}(u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}})$$

  where $\hat{f}(u,u) = f(u)$ (consistency); $\hat{f}(\uparrow,\downarrow)$ (monotonicity) and $\hat{f}$ is Lipschitz continuous with respect to both arguments.

- Values from inside I_j for the test function v

$$v^-_{j+\frac{1}{2}}, \qquad v^+_{j-\frac{1}{2}}$$

Hence the DG scheme is: find $u \in V_h$ such that

$$\int_{I_j} u_t v dx - \int_{I_j} f(u) v_x dx + \hat{f}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0 \qquad (2)$$

for all $v \in V_h$.

Notice that, for the piecewise constant $k = 0$ case, we recover the well known first order monotone finite volume scheme:

$$(u_j)_t + \frac{1}{h}\left(\hat{f}(u_j, u_{j+1}) - \hat{f}(u_{j-1}, u_j)\right) = 0.$$

Time discretization could be by the TVD Runge-Kutta method (Shu and Osher, JCP 88). For the semi-discrete scheme:

$$\frac{du}{dt} = L(u)$$

where $L(u)$ is a discretization of the spatial operator, the third order TVD Runge-Kutta is simply:

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n) \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}) \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}) \end{aligned}$$

DG method for hyperbolic equations with δ-singularities

We develop and analyze DG methods for solving hyperbolic conservation laws

$$\begin{aligned} u_t + f(u)_x &= g(x,t), && (x,t) \in R \times (0,T], \\ u(x,0) &= u_0(x), && x \in R, \end{aligned} \tag{3}$$

where the initial condition u_0, or the source term $g(x,t)$, or the solution $u(x,t)$ contains δ-singularities.

- Such problems appear often in applications and are difficult to approximate numerically, especially for finite difference schemes.
- Many numerical techniques rely on modifications with smooth kernels (mollification) and hence may severely smear such singularities, leading to large errors in the approximation.

In Yang and Shu, Num Math 2013 and Yang, Wei and Shu, JCP 2013, we develop, analyze and apply DG methods for solve hyperbolic equations with δ-singularities. The DG methods are based on weak formulations and can be designed directly to solve such problems without modifications, leading to very accurate results.

Linear equations with singular initial condition

We consider the linear model equation

$$\begin{aligned} u_t + \beta u_x &= 0 \\ u(x,0) &= u^0(x) \end{aligned}$$

where β is a constant, $u_0(x)$ has compact support, has a sole δ-singularity at $x = 0$ and is sufficiently smooth everywhere else.

Even though the initial condition $u^0(x)$ is no longer in L^2, it does have an L^2-projection to the DG space V_h, which we use as the initial condition for the DG scheme. For problems involving δ-singularities, negative-order norm estimates are more natural. We have the following theorem in Yang and Shu, Num Math 2013:

Theorem: By taking $\Omega_0 + 2supp(K_h^{2k+2,k+1}) \subset\subset \Omega_1 \subset\subset \Omega \backslash \mathcal{R}_T$, we have

$$\|u(T) - u_h(T)\|_{-(k+1)} \leq Ch^k, \tag{4}$$

$$\|u(T) - u_h(T)\|_{-(k+2)} \leq Ch^{k+1/2}, \tag{5}$$

$$\|u(T) - u_h(T)\|_{-(k+1),\Omega_1} \leq Ch^{2k+1}, \tag{6}$$

$$\|u(T) - K_h^{2k+2,k+1} * u_h(T)\|_{\Omega_0} \leq Ch^{2k+1}, \tag{7}$$

where the positive constant C does not depend on h. Here the mesh is assumed to be uniform for (7) but can be regular and non-uniform for the other three inequalities.

Several comments:

- We use the results about the pollution region in Zhang and Shu, Num Math 2014, which is also valid in the current case with δ-singularities.
- We follow the proof of negative-order error estimates and post-processing for DG methods solving linear hyperbolic equations with smooth solutions in Cockburn, Luskin, Shu and Süli Math Comp 2003 with suitable adjustments.

Numerical example: We solve the following problem

$$\begin{aligned} &u_t + u_x = 0, && (x,t) \in [0,\pi] \times (0,1], \\ &u(x,0) = \sin(2x) + \delta(x - 0.5), && x \in [0,\pi], \end{aligned} \tag{8}$$

with periodic boundary condition $u(0,t) = u(\pi,t)$. Clearly, the exact solution is

$$u(x,t) = \sin(2x - 2t) + \delta(x - t - 0.5).$$

Table 1: L^2-norm of the error between the numerical solution and the exact solution for equation (8) after post-processing in the region away from the singularity.

		$\mathcal{P}^1$ polynomial		$\mathcal{P}^2$ polynomial		$\mathcal{P}^3$ polynomial	
N	d	error	order	error	order	error	order
200	0.2	6.88E-05	-	8.40e-07	-	1.48E-09	-
300	0.2	1.41E-05	3.92	3.56e-10	19.2	3.98E-13	20.3
400	0.2	5.89E-06	3.02	1.98e-11	10.1	4.42E-16	23.7
500	0.2	3.01E-06	3.01	6.13e-12	5.25	7.49E-17	7.95
600	0.2	1.74E-06	3.00	2.37e-12	5.21	1.76E-17	7.94

We consider the following two dimensional problem

$$\begin{aligned} &u_t + u_x + u_y = 0, && (x, y, t) \in [0, 2\pi] \times [0, 2\pi] \\ &u(x, 0) = \sin(x + y) + \delta(x + y - 2\pi), && (x, y) \in [0, 2\pi] \times [0, 2\pi], \end{aligned} \tag{9}$$

with periodic boundary condition. Clearly, the exact solution is

$$u(x, t) = \sin(x + y - 2t) + \delta(x + y - 2t) + \delta(x + y - 2t - 2\pi).$$

We use $\mathcal{Q}^k$ polynomial approximation spaces with $k = 1$ and 2.

Table 2: L^2-norm of the error between the numerical solution and the exact solution for equation (9) after post-processing in the region away from the singularity.

		$\mathcal{Q}^1$ polynomial		$\mathcal{Q}^2$ polynomial	
N	d	error	order	error	order
400	0.4	2.60E-05	-	3.23e-08	-
500	0.4	1.24E-05	3.32	2.47e-10	20.0
600	0.4	7.16E-06	3.01	1.19e-11	16.6
700	0.4	4.50E-06	3.01	5.11e-12	5.47
800	0.4	3.01E-06	3.02	2.53e-12	5.29

The theory generalizes to linear systems in a straightforward way. We solve the following linear system

$$\begin{aligned} &u_t - v_x = 0, && (x,t) \in [0,2] \times (0,0.4], \\ &v_t - u_x = 0, && (x,t) \in [0,2] \times (0,0.4], \\ &u(x,0) = \delta(x-1), v(x,0) = 0, && x \in [0,2]. \end{aligned} \tag{10}$$

Clearly, the exact solution (the Green's function) is

$$u(x,t) = \frac{1}{2}\delta(x-1-t) + \frac{1}{2}\delta(x-1+t),$$

$$v(x,t) = \frac{1}{2}\delta(x-1+t) - \frac{1}{2}\delta(x-1-t).$$

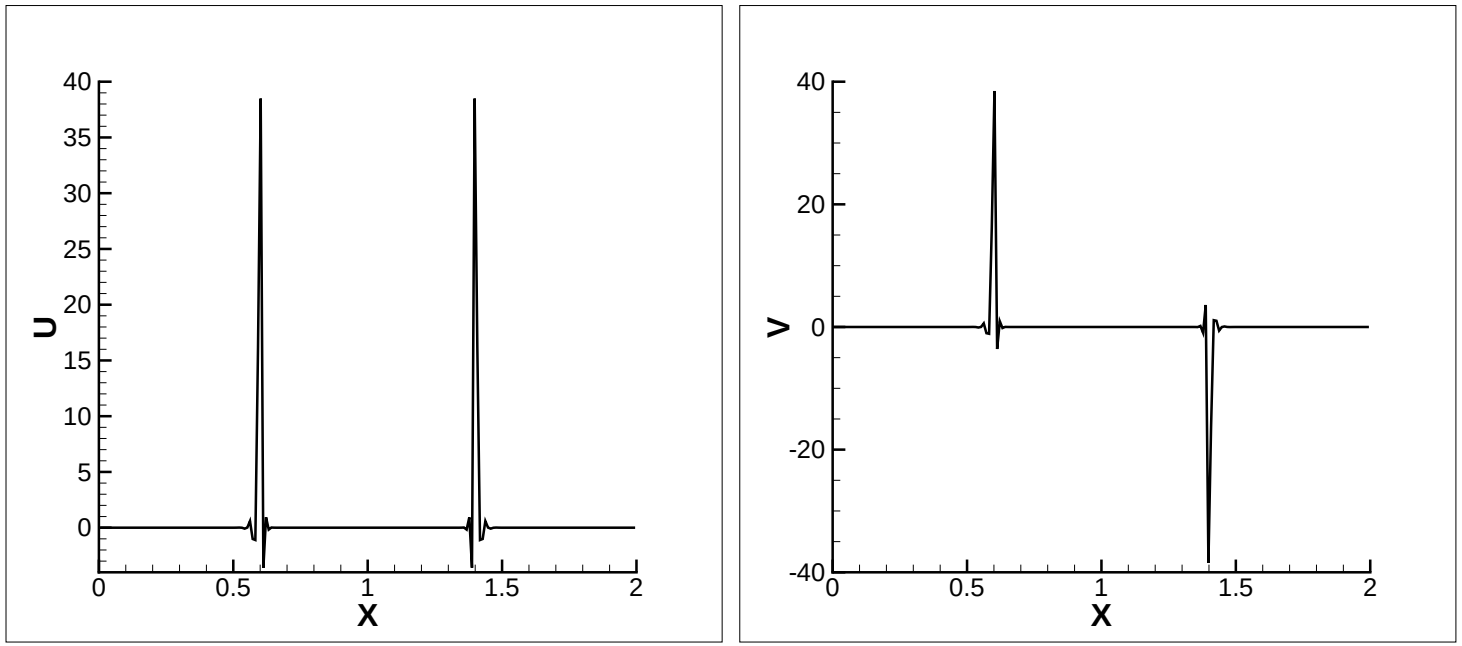

Figure 1: Solutions of u (left) and v (right) for (10) at $t = 0.4$.

Linear equations with singular source terms

We consider the linear model equation

$$\begin{aligned} u_t(x,t) + Lu(x,t) &= g(x,t), \qquad (x,t) \in \Omega \times (0,\infty), \\ u(x,0) &= 0, \qquad x \in \Omega, \end{aligned}$$

with L being a linear differential operator that does not involve time derivatives and $g(x,t)$ is a singular source term, for example $g(x,t) = \delta(x)$. The singular source term can be implemented in the DG scheme in a straightforward way, since it involves only the integrals of the singular source term with test functions in V_h.

By using Duhamel's principle, we can prove the following theorem (Yang and Shu, Num Math 2013):

Theorem: Denote $\mathcal{R}_T = I_i \cup (T - C\log(1/h)h^{1/2}, T + C\log(1/h)h^{1/2})$, where I_i is the cell which contains the concentration of the δ-singularity on the source term. Then we have the following estimates

$$\|u(T) - u_h(T)\|_{-(k+1)} \leq Ch^k, \tag{11}$$

$$\|u(T) - u_h(T)\|_{-(k+2)} \leq Ch^{k+1/2}, \tag{12}$$

$$\|u - u_h\|_{-(k+1),\Omega_1} \leq Ch^{2k+1}, \tag{13}$$

$$\|u(T) - K_h^{2k+2,k+1} * u_h(T)\|_{\Omega_0} \leq Ch^{2k+1}, \tag{14}$$

where $\Omega_0 + 2supp(K_h^{2k+2,k+1}) \subset\subset \Omega_1 \subset\subset \mathbb{R}\backslash\mathcal{R}_T$. Here the mesh is assumed to be uniform for (14) but can be regular and non-uniform for the other three inequalities.

Numerical example: We solve the following problem

$$\begin{aligned} &u_t + u_x = \delta(x-\pi), && (x,t) \in [0,2\pi] \times (0,1], \\ &u(x,0) = \sin(x), && x \in [0,2\pi], \\ &u(0,t) = 0, && t \in (0,1]. \end{aligned} \tag{15}$$

Clearly, the exact solution is

$$u(x,t) = \sin(x-t) + \chi_{[\pi,\pi+t]},$$

where $\chi_{[a,b]}$ denotes the indicator function of the interval $[a,b]$.

Table 3: L^2-norm of the error between the numerical solution and the exact solution for equation (15) after post-processing in the region away from the singularity.

		$\mathcal{P}^1$ polynomial		$\mathcal{P}^2$ polynomial	
N	d	error	order	error	order
401	0.2	1.74E-06	-	4.29E-08	-
801	0.2	5.92E-09	8.22	6.80E-13	15.9
1601	0.2	7.36E-10	3.03	1.34E-17	12.3
3201	0.2	9.19E-11	3.01	3.86E-18	5.13
6401	0.2	1.15E-11	3.01	1.16E-19	5.07

High order accuracy bound-preserving limiters

We notice that the unmodulated DG scheme for solving problems with δ-function singularities generates spurious oscillations (Gibbs phenomenon). While such oscillations also happen when the DG scheme is used to solve discontinuous solutions, they are more severe for δ-function singularities.

While in some of the applications, e.g. the linear problems mentioned above, such oscillations are just unpleasant visually but do not lead to instabilities of the scheme, for other applications, especially nonlinear systems to be studied below, these oscillations may lead to unphysical regimes in which the PDE is ill-posed, hence causing immediate blowups of the scheme.

Many slope limiters such as minmod or MUSCL limiters, which are designed for discontinuous solution, are too severe for δ-function singularities (which would prefer slope to be very large), leading to serious deterioration of resolution when used.

Therefore, a milder limiter which can enforce the physical bounds yet can maintain the originally designed high order accuracy of the DG scheme is desired.

For the scalar conservation laws

$$u_t + \nabla \cdot \mathbf{F}(u) = 0, \qquad u(\mathbf{x}, 0) = u_0(\mathbf{x}). \tag{16}$$

An important property of the entropy solution (which may be discontinuous) is that it satisfies a strict maximum principle: If

$$M = \max_{\mathbf{x}} u_0(\mathbf{x}), \quad m = \min_{\mathbf{x}} u_0(\mathbf{x}), \tag{17}$$

then $u(\mathbf{x}, t) \in [m, M]$ for any $\mathbf{x}$ and t.

First order monotone schemes can maintain the maximum principle. For the one-dimensional conservation law

$$u_t + f(u)_x = 0,$$

the first order monotone scheme

$$\begin{aligned} u_j^{n+1} &= H_\lambda(u_{j-1}^n, u_j^n, u_{j+1}^n) \\ &= u_j^n - \lambda[h(u_j^n, u_{j+1}^n) - h(u_{j-1}^n, u_j^n)] \end{aligned}$$

where $\lambda = \frac{\Delta t}{\Delta x}$ and $h(u^-, u^+)$ is a monotone flux ($h(\uparrow, \downarrow)$), satisfies

$$H_\lambda(\uparrow, \uparrow, \uparrow)$$

under a suitable CFL condition

$$\lambda \leq \lambda_0.$$

Also, for any constant c,

$$H_\lambda(c,c,c) = c - \lambda[h(c,c) - h(c,c)] = c.$$

Therefore, if

$$m \le u_{j-1}^n, u_j^n, u_{j+1}^n \le M$$

then

$$u_j^{n+1} = H_\lambda(u_{j-1}^n, u_j^n, u_{j+1}^n) \ge H_\lambda(m,m,m) = m,$$

and

$$u_j^{n+1} = H_\lambda(u_{j-1}^n, u_j^n, u_{j+1}^n) \le H_\lambda(M,M,M) = M.$$

However, for higher order linear schemes, i.e. schemes which are linear for a linear PDE

$$u_t + a u_x = 0 \tag{18}$$

for example the second order accurate Lax-Wendroff scheme

$$u_j^{n+1} = \frac{a\lambda}{2}(1 + a\lambda)u_{j-1}^n + (1 - a^2\lambda^2)u_j^n - \frac{a\lambda}{2}(1 - a\lambda)u_{j+1}^n$$

where $\lambda = \frac{\Delta t}{\Delta x}$ and $|a|\lambda \leq 1$, the maximum principle is not satisfied. In fact, no linear schemes with order of accuracy higher than one can satisfy the maximum principle (Godunov Theorem).

Therefore, nonlinear schemes, namely schemes which are nonlinear even for linear PDEs, have been designed to overcome this difficulty. These include roughly two classes of schemes:

- TVD schemes. Most TVD (total variation diminishing) schemes also satisfy strict maximum principle, even in multi-dimensions. TVD schemes can be designed for any formal order of accuracy for solutions in smooth, monotone regions. However, all TVD schemes will degenerate to first order accuracy at smooth extrema.
- TVB schemes, ENO schemes, WENO schemes. These schemes do not insist on strict TVD properties, therefore they do not satisfy strict maximum principles, although they can be designed to be arbitrarily high order accurate for smooth solutions.

Remark: If we insist on the maximum principle interpreted as

$$m \leq u_j^{n+1} \leq M, \qquad \forall j$$

if

$$m \leq u_j^{n} \leq M, \qquad \forall j,$$

where u_j^n is either the approximation to the point value $u(x_j, t^n)$ for a finite difference scheme, or to the cell average $\frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t^n) dx$ for a finite volume or DG scheme, then the scheme can be at most second order accurate (proof due to Harten, see Zhang and Shu, Proceedings of the Royal Society A, 2011).

"Proof": Consider finite difference, solving

$$u_t + u_x = 0$$

with the initial condition

$$u(x, 0) = \sin(x).$$

Put the mesh so that

$$x_{j_0} = \frac{\pi}{2} - \frac{\Delta x}{2}, \qquad x_{j_0+1} = \frac{\pi}{2} + \frac{\Delta x}{2},$$

then we have

$$M = \max_j u_j^0 = u_{j_0}^0 = \sin\left(\frac{\pi}{2} - \frac{\Delta x}{2}\right) \leq 1 - C\Delta x^2$$

for a constant C, when $\Delta x \ll 1$.

By the definition of the the maximum principle given above, we have

$$\max_j u_j^1 \leq \max_j u_j^0 \leq 1 - C\Delta x^2.$$

Therefore, if we take $\Delta t = \frac{\Delta x}{2}$, the error at x_{j_0} and the first time step satisfies

$$|u^{exact}(x_{j_0}, \Delta t) - u_{j_0}^1| = |\sin\left(\frac{\pi}{2}\right) - u_{j_0}^1| \geq 1 - (1 - C\Delta x^2) = C\Delta x^2.$$

That is, even after one time step, the accuracy is at most second order.

The same proof works for finite volume schemes.

A high order discontinuous Galerkin scheme has the following algorithm flowchart:

(1) Given $u^n(x)$ (piecewise polynomial with the cell average $\bar{u}_j^n$)

(2) evolve by, e.g. Runge-Kutta time discretization to get $u^{n+1}(x)$ (with the cell average $\{\bar{u}_j^{n+1}\}$

(3) return to (1)

Therefore, instead of requiring

$$m \leq \bar{u}_j^{n+1} \leq M, \qquad \forall j$$

if

$$m \leq \bar{u}_j^{n} \leq M, \qquad \forall j,$$

we will require

$$m \leq u^{n+1}(x) \leq M, \qquad \forall x$$

if

$$m \leq u^{n}(x) \leq M, \qquad \forall x.$$

Maximum-principle-preserving for scalar conservation laws

The flowchart for designing a high order finite volume or DG scheme which obeys a strict maximum principle is as follows:

1. Start with $u^n(x)$ which is high order accurate

$$|u(x,t^n) - u^n(x)| \leq C\Delta x^p$$

and satisfies

$$m \leq u^n(x) \leq M, \qquad \forall x$$

therefore of course we also have

$$m \leq \bar{u}_j^n \leq M, \qquad \forall j.$$

2. Evolve for one time step to get

$$m \leq \bar{u}_j^{n+1} \leq M, \qquad \forall j. \tag{19}$$

3. Given (19) above, obtain $u^{n+1}(x)$ (reconstruction or evolution) which

   - satisfies the maximum principle

$$m \leq u^{n+1}(x) \leq M, \qquad \forall x;$$

   - is high order accurate

$$|u(x, t^{n+1}) - u^{n+1}(x)| \leq C\Delta x^p.$$

Three major difficulties

1. The first difficulty is how to evolve in time for one time step to guarantee

$$m \leq \bar{u}_j^{n+1} \leq M, \qquad \forall j. \tag{20}$$

This is very difficult to achieve. Previous works use one of the following two approaches:

- Use exact time evolution. This can guarantee

$$m \leq \bar{u}_j^{n+1} \leq M, \qquad \forall j.$$

However, it can only be implemented with reasonable cost for linear PDEs, or for nonlinear PDEs in one dimension. This approach was used in, e.g., Jiang and Tadmor, SISC 1998; Liu and Osher, SINUM 1996; Sanders, Math Comp 1988; Qiu and Shu, SINUM 2008; Zhang and Shu, SINUM 2010; to obtain TVD schemes or maximum-principle-preserving schemes for linear and nonlinear PDEs in one dimension or for linear PDEs in multi-dimensions, for second or third order accurate schemes.

- Use simple time evolution such as SSP Runge-Kutta or multi-step methods. However, additional limiting will be needed on $u^n(x)$ which will destroy accuracy near smooth extrema.

We have figured out a way to obtain

$$m \leq \bar{u}_j^{n+1} \leq M, \qquad \forall j$$

with simple Euler forward or SSP Runge-Kutta or multi-step methods without losing accuracy on the limited $u^n(x)$:

The evolution of the cell average for a higher order finite volume or DG scheme satisfies

$$\begin{aligned}\bar{u}_j^{n+1} &= G(\bar{u}_j^n, u^-_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}}, u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}) \\ &= \bar{u}_j^n - \lambda[h(u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}) - h(u^-_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}})],\end{aligned}$$

where

$$G(\uparrow, \uparrow, \downarrow, \downarrow, \uparrow)$$

therefore there is no maximum principle. The problem is with the two arguments $u^+_{j-\frac{1}{2}}$ and $u^-_{j+\frac{1}{2}}$ which are values at points inside the cell I_j.

The polynomial $p_j(x)$ (either reconstructed in a finite volume method or evolved in a DG method) is of degree k, defined on I_j such that $\bar{u}_j^n$ is its cell average on I_j, $u^+_{j-\frac{1}{2}} = p_j(x_{j-\frac{1}{2}})$ and $u^-_{j+\frac{1}{2}} = p_j(x_{j+\frac{1}{2}})$.

We take a Legendre Gauss-Lobatto quadrature rule which is exact for polynomials of degree k, then

$$\bar{u}_j^n = \sum_{\ell=0}^{m} \omega_\ell p_j(y_\ell)$$

with $y_0 = x_{j-\frac{1}{2}}$, $y_m = x_{j+\frac{1}{2}}$. The scheme for the cell average is then rewritten as

$$
\begin{aligned}
\bar{u}_j^{n+1} &= \omega_m \left[u^-_{j+\frac{1}{2}} - \frac{\lambda}{\omega_m} \left(h(u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}) - h(u^+_{j-\frac{1}{2}}, u^-_{j+\frac{1}{2}}) \right) \right] \\
&\quad + \omega_0 \left[u^+_{j-\frac{1}{2}} - \frac{\lambda}{\omega_0} \left(h(u^+_{j-\frac{1}{2}}, u^-_{j+\frac{1}{2}}) - h(u^-_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}}) \right) \right] \\
&\quad + \sum_{\ell=1}^{m-1} \omega_\ell p_j(y_\ell) \\
&= \omega_m H_{\lambda/\omega_m}(u^+_{j-\frac{1}{2}}, u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}) + \omega_0 H_{\lambda/\omega_0}(u^-_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}}, u^-_{j+\frac{1}{2}}) \\
&\quad + \sum_{\ell=1}^{m-1} \omega_\ell p_j(y_\ell).
\end{aligned}
$$

Therefore, if

$$m \leq p_j(y_\ell) \leq M$$

at all Legendre Gauss-Lobatto quadrature points and a reduced CFL condition

$$\lambda/\omega_m = \lambda/\omega_0 \leq \lambda_0$$

is satisfied, then

$$m \leq \bar{u}_j^{n+1} \leq M.$$

2. The second difficulty is: given

$$m \leq \bar{u}_j^{n+1} \leq M, \qquad \forall j$$

how to obtain an accurate $u^{n+1}(x)$ (reconstruction or limited DG evolution) which satisfies

$$m \leq u^{n+1}(x) \leq M, \qquad \forall x.$$

Previous work was mainly for relatively lower order schemes (second or third order accurate), and would typically require an evaluation of the extrema of $u^{n+1}(x)$, which, for a piecewise polynomial of higher degree, is quite costly.

We have figured out a way to obtain such $u^{n+1}(x)$ with a very simple scaling limiter, which only requires the evaluation of $u^{n+1}(x)$ at certain pre-determined quadrature points and does not destroy accuracy:

We replace $p_j(x)$ by the limited polynomial $\tilde{p}_j(x)$ defined by

$$\tilde{p}_j(x) = \theta_j(p_j(x) - \bar{u}_j^n) + \bar{u}_j^n$$

where

$$\theta_j = \min\left\{\left|\frac{M - \bar{u}_j^n}{M_j - \bar{u}_j^n}\right|, \left|\frac{m - \bar{u}_j^n}{m_j - \bar{u}_j^n}\right|, 1\right\},$$

with

$$M_j = \max_{x \in S_j} p_j(x), \qquad m_j = \min_{x \in S_j} p_j(x)$$

where S_j is the set of Legendre Gauss-Lobatto quadrature points of cell I_j.

Clearly, this limiter is just a simple scaling of the original polynomial around its average.

The following lemma, guaranteeing the maintenance of accuracy of this simple limiter, is proved in Zhang and Shu, JCP 2010a:

Lemma: Assume $\bar{u}_j^n \in [m, M]$ and $p_j(x)$ is an $O(\Delta x^p)$ approximation, then $\tilde{p}_j(x)$ is also an $O(\Delta x^p)$ approximation.

We have thus obtained a high order accurate scheme satisfying the following maximum principle: If

$$m \leq u^n(x) \leq M, \qquad \forall x \in S_j,$$

then

$$m \leq u^{n+1}(x) \leq M, \qquad \forall x \in S_j.$$

Recall that S_j is the set of Legendre Gauss-Lobatto quadrature points of cell I_j.

3. The third difficulty is how to generalize the algorithm and result to 2D (or higher dimensions). Algorithms which would require an evaluation of the extrema of the reconstructed polynomials $u^{n+1}(x, y)$ would not be easy to generalize at all.

   Our algorithm uses only explicit Euler forward or SSP (also called TVD) Runge-Kutta or multi-step time discretizations, and a simple scaling limiter involving just evaluation of the polynomial at certain quadrature points, hence easily generalizes to 2D or higher dimensions on structured or unstructured meshes, with strict maximum-principle-satisfying property and provable high order accuracy.

The technique has been generalized to the following situations maintaining uniformly high order accuracy:

- 2D scalar conservation laws on rectangular or triangular meshes with strict maximum principle (Zhang and Shu, JCP 2010a; Zhang, Xia and Shu, JSC 2012).
- 2D incompressible equations in the vorticity-streamfunction formulation (with strict maximum principle for the vorticity), and 2D passive convections in a divergence-free velocity field, i.e.

$$\omega_t + (u\omega)_x + (v\omega)_x = 0,$$

with a given divergence-free velocity field (u, v), again with strict maximum principle (Zhang and Shu, JCP 2010a; Zhang, Xia and Shu, JSC 2012).

Positivity-preserving for systems

The framework of establishing maximum-principle-satisfying schemes for scalar equations can be generalized to hyperbolic systems to preserve the positivity of certain physical quantities, such as density and pressure of compressible gas dynamics.

Compressible Euler equations:

$$u_t + f(u)_x = 0$$

with

$$u = \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}, \qquad f(u) = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix},$$

where $E = e + \frac{1}{2}\rho v^2$. The internal energy e is related to density and pressure through the equation of states (EOS). For the ideal gas, we have $e = \frac{p}{\gamma - 1}$ with $\gamma = 1.4$ for air.

The main ingredients for designing positivity-preserving schemes for systems are:

- A first order explicit scheme which can keep the positivity of the desired quantities (e.g. density and pressure) under a suitable CFL condition.

  Examples include the Godunov scheme, Lax-Friedrichs scheme, kinetic scheme, HLLC scheme, etc.

- The quantity for which positivity is desired is one of the components of the conserved variable u (for example the density ρ), or is a concave function of the conserved variable u (for example the pressure p or the internal energy e). Under this assumption, the region of positivity of the desired quantities is a convex region in the u space.

With these ingredients, the technique to enforce maximum-principle for scalar equations can be directly generalized to enforce positivity of the desired quantities without affecting the high order accuracy of the finite volume or DG schemes. That is, we have a high order scheme satisfying positivity-preserving in the following sense: If $u^n(x)$ has positive density and pressure for all $x \in S_j$, then $u^{n+1}(x)$ also has positive density and pressure for all $x \in S_j$. Recall that S_j is the set of Legendre Gauss-Lobatto quadrature points of cell I_j.

Positivity-preserving finite volume or DG schemes have been designed for:

- One and multi-dimensional compressible Euler equations maintaining positivity of density and pressure (Zhang and Shu, JCP 2010b; Zhang, Xia and Shu, JSC 2012).
- One and two-dimensional shallow water equations maintaining non-negativity of water height and well-balancedness for problems with dry areas (Xing, Zhang and Shu, Advances in Water Resources 2010; Xing and Shu, Advances in Water Resources 2011).

- One and multi-dimensional compressible Euler equations with source terms (geometric, gravity, chemical reaction, radiative cooling) maintaining positivity of density and pressure (Zhang and Shu, JCP 2011).
- One and multi-dimensional compressible Euler equations with gaseous detonations maintaining positivity of density, pressure and reactant mass fraction, with a new and simplified implementation of the pressure limiter. DG computations are stable without using the TVB limiter (Wang, Zhang, Shu and Ning, JCP 2012).
- A minimum entropy principle satisfying high order scheme for gas dynamics equations (Zhang and Shu, Num Math 2012).

Applications

Rendez-vous algorithm

Even though our theory is established only for linear equations, the DG algorithm can be easily implemented for nonlinear hyperbolic equations involving δ-singularities.

In Canuto, Fagnani and Tilli, SIAM J Control and Optimization 2012, the following problem

$$\begin{aligned} &\rho_t + F_x = 0, \qquad x \in [0,1], t > 0, \\ &\rho(0,t) = u_0(x), \quad t > 0, \end{aligned} \tag{21}$$

is studied. Here ρ is the density function, which is always positive.

The flux F is given by

$$F(t,x) = v(t,x)\rho(t,x),$$

and the velocity v is defined by

$$v(t,x) = \int_{\mathbb{R}^n} (y-x)\xi(y-x)\rho(t,y)dy,$$

where $\xi(x)$ is a positive function and supported on a ball centered at zero with radius R. Canuto et al. proved that when t tends to infinity, the density function ρ will converge to some δ-singularities, and the distances between any of them cannot be less than R. Some computational results are shown in Canuto et al. based on a first order finite volume method.

We use the DG algorithm with the positivity-preserving limiter in Zhang and Shu JCP 2010, which can maintain positivity without affecting the high order accuracy, to both the one and two dimensional Rendez-vous algorithms, in Yang and Shu, Num Math 2013 and Yang, Wei and Shu, JCP 2013.

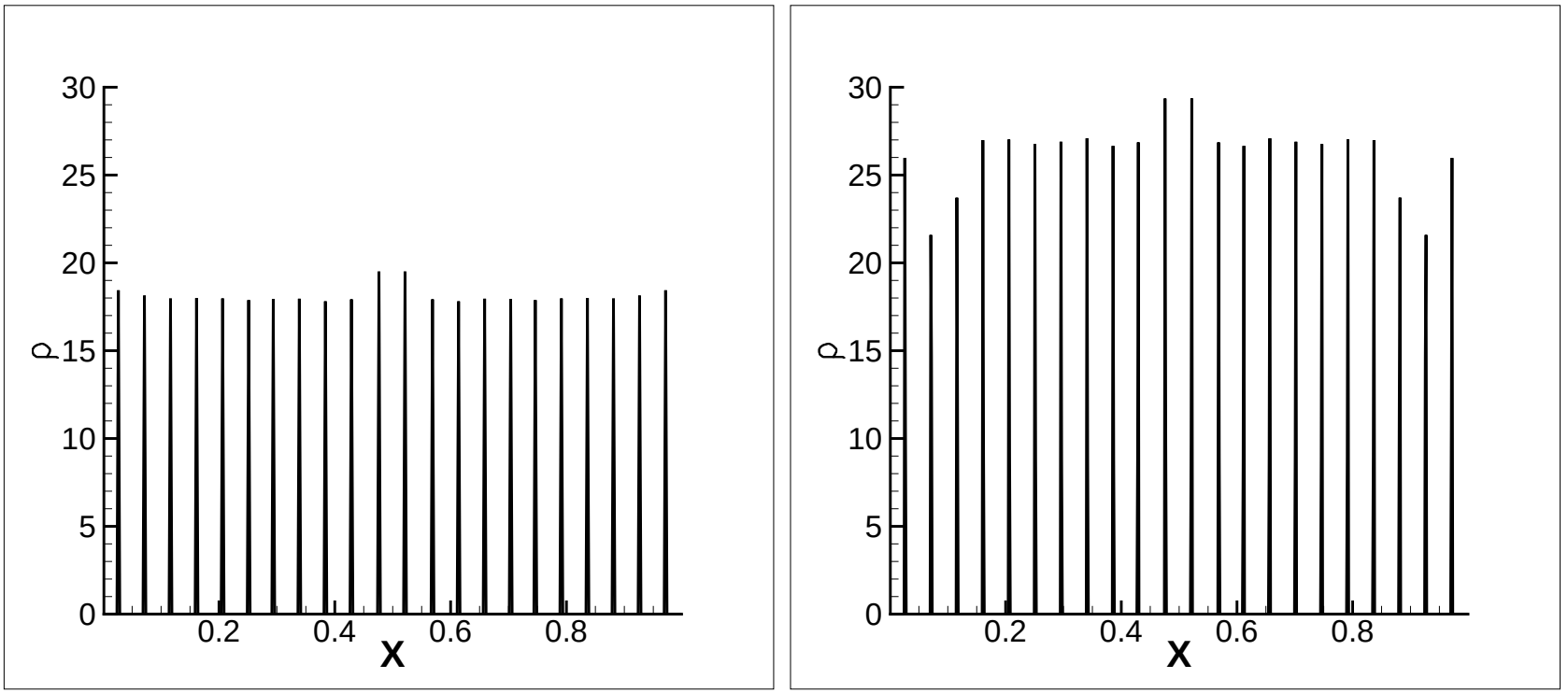

Figure 2: Numerical density at $t = 1000$ with $N = 400$ when using $\mathcal{P}^0$ (left) and $\mathcal{P}^1$ (right) polynomials.

In 2D, the model is

$$\begin{aligned} &\rho_t + \text{div}(\mathbf{v}\rho) = 0, \qquad && \mathbf{x} \in [-1,1]^2, t > 0, \\ &\rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}), \qquad && t > 0, \end{aligned} \tag{22}$$

where the velocity $\mathbf{v}$ is defined by

$$\mathbf{v}(\mathbf{x}, t) = \int_{B_R(\mathbf{x})} (\mathbf{y} - \mathbf{x}) \rho(\mathbf{y}, t) d\mathbf{y}.$$

In this example, we take $R = 0.1$ and

$$\rho_0(\mathbf{x}) = \begin{cases} 1 & r < 0.5, \\ 0 & r > 0.5, \end{cases}$$

where $r = \|\mathbf{x}\|$ is the Euclidean norm of $\mathbf{x}$.

In Canuto et al., the authors demonstrated that the exact solution should be a single delta placed at the origin.

However, when we use rectangle meshes, we observe more than one delta singularity for R sufficiently small. This is because the meshes are not invariant under rotation.

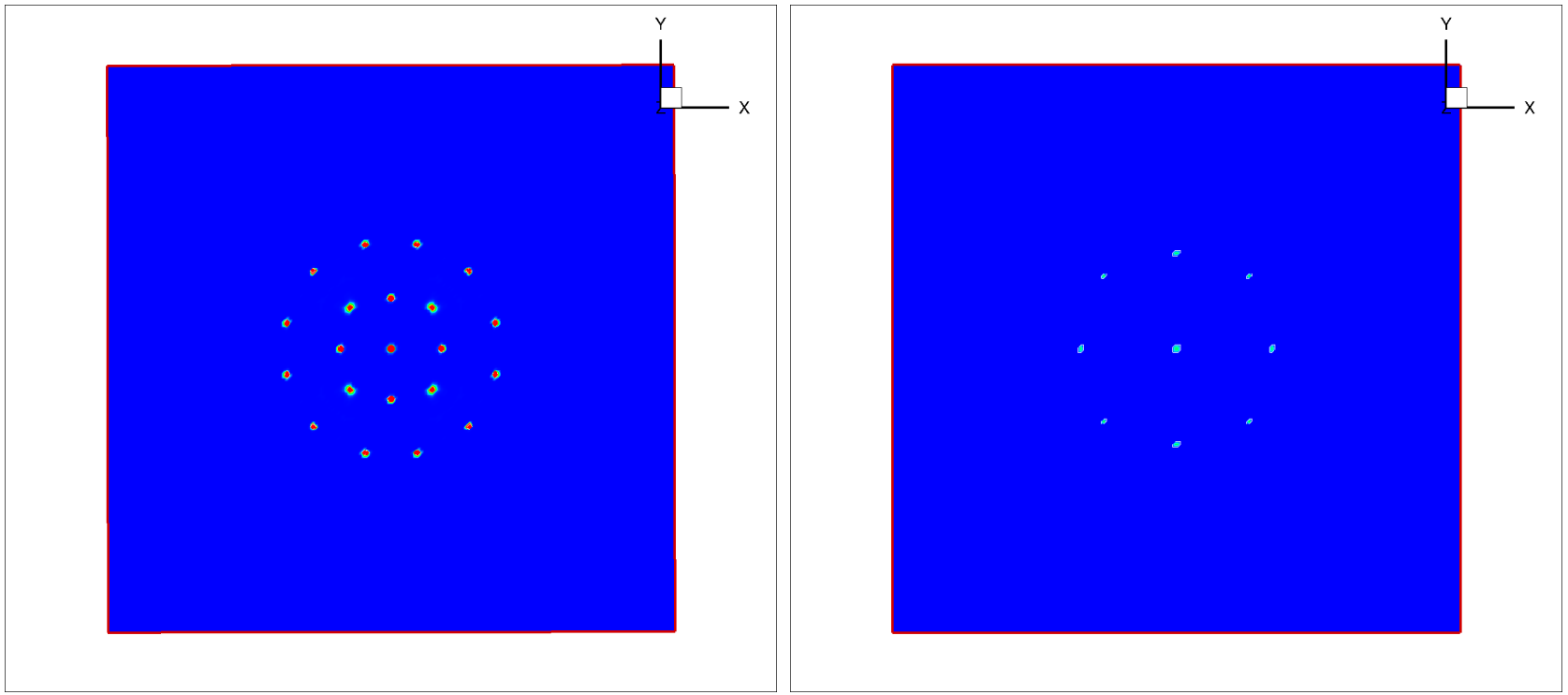

Figure 3: Numerical density ρ with a rectangular 100×100 mesh using $\mathcal{P}^0$ elements. $R = 0.08$ (left) and $R = 0.1$ (right).

To tackle this problem, we follow the same ideas in Cheng and Shu, JCP 2010; CiCP 2012, and construct a special equal-angle-zoned mesh. The structure of the mesh is given in figure 4. By using such a special mesh, the limit density is a single delta placed at the origin.

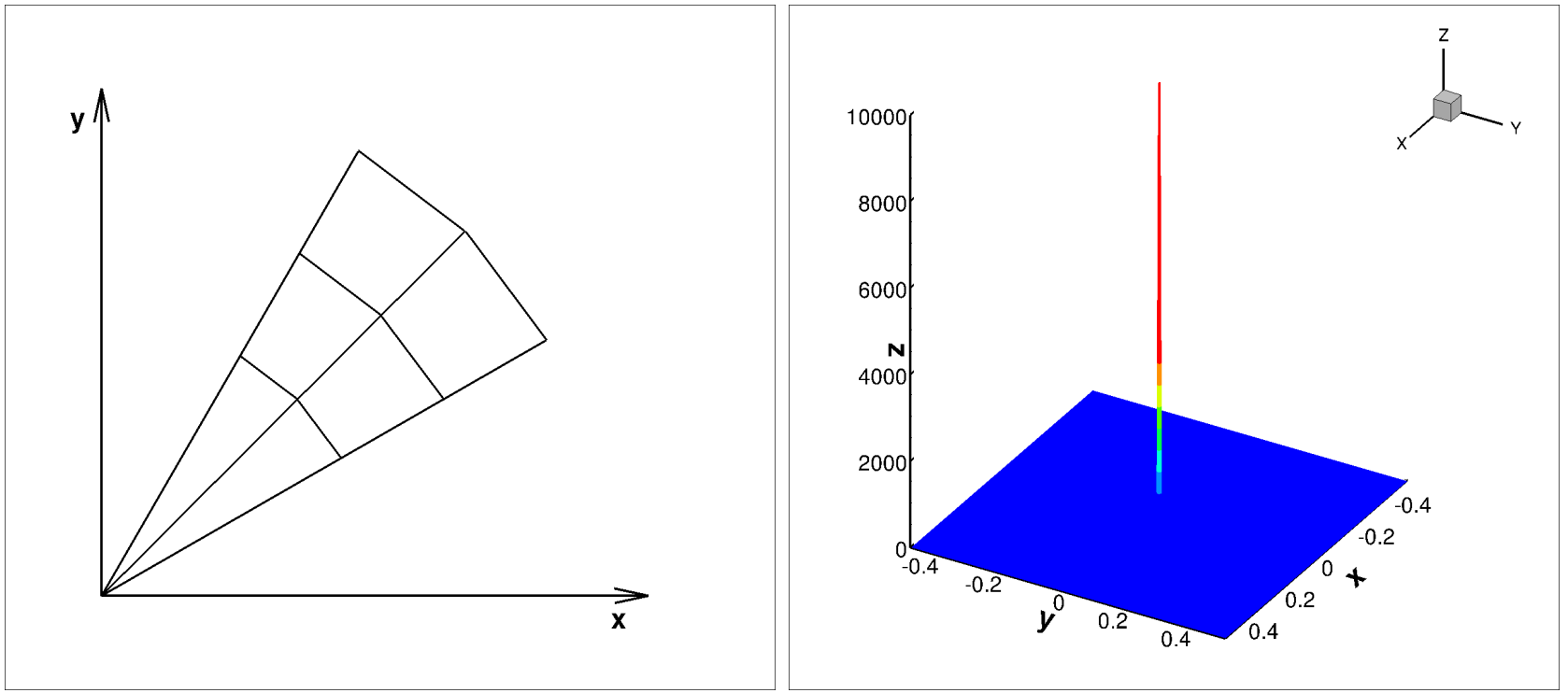

Figure 4: Left: Equal-angle-zoned mesh. Right: Numerical density ρ for (22) at $t = 2000$ with $N = 200$ using $\mathcal{P}^0$ elements.

Pressureless Euler equations

Another important system admitting δ-singularities in its solutions is the pressureless Euler equation

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_x = 0, \quad t > 0,\ x \in \mathbb{R}, \tag{23}$$

$$\mathbf{w} = \begin{pmatrix} \rho \\ m \end{pmatrix}, \quad \mathbf{f}(\mathbf{w}) = \begin{pmatrix} m \\ \rho u^2 \end{pmatrix},$$

with $m = \rho u$, where ρ is the density function and u is the velocity.

It is quite difficult to obtain stable schemes for solve this system, especially for high order schemes.

A good property of this system is that the density is always positive and the velocity satisfies a maximum-principle. Thus, in 1D, the convex set

$$G = \left\{ \mathbf{w} = \begin{pmatrix} \rho \\ m \end{pmatrix} : \rho > 0, a\rho \leq m \leq b\rho \right\},$$

where

$$a = \min u_0(x), \qquad b = \max u_0(x), \tag{24}$$

with u_0 being the initial velocity, is invariant. In Yang, Wei and Shu, JCP 2013, we adapt the techniques in Zhang and Shu, JCP 2010 to design a limiter to guarantee that our DG solution stays in set G without affecting high order accuracy. This is also generalized to 2D. Our scheme is thus very robust, stable and high order accurate for this pressureless Euler system.

We consider the following initial data

$$\rho_0(x) = \sin(x) + 2,\ u_0(x) = \sin(x) + 2, \tag{25}$$

with periodic boundary condition. Clearly, the exact solution is

$$u(x,t) = u_0(x_0), \quad \rho(x,t) = \frac{\rho_0(x_0)}{1 + u_0'(x_0)},$$

where x_0 is given implicitly by

$$x_0 + t u_0(x_0) = x.$$

Table 4: L^2-norm of the error between the numerical density and the exact density for initial condition (25).

	k=1		k=2		k=3	
N	error	order	error	order	error	order
20	1.41E-02	-	6.84E-04	-	3.40e-5	-
40	4.18E-03	1.76	1.04E-04	2.72	2.82e-6	3.59
80	1.30E-03	1.68	1.55E-05	2.74	2.26e-7	3.64
160	4.24E-04	1.62	2.41E-06	2.69	1.83e-8	3.62
320	1.51E-04	1.49	3.80E-07	2.67	1.49e-9	3.63

We consider the following initial condition

$$\rho_0(x) = \begin{cases} 1 & x < 0, \\ 0.25 & x > 0, \end{cases} \qquad u_0(x) = \begin{cases} 1 & x < 0, \\ 0 & x > 0. \end{cases} \tag{26}$$

Clearly, the exact solution is

$$(\rho(x,t), u(x,t)) = \begin{cases} (1,1) & x < 2t/3, \\ (0.25, 0) & x > 2t/3, \end{cases}$$

and at $x = \frac{2t}{3}$, the density should be a δ-function.

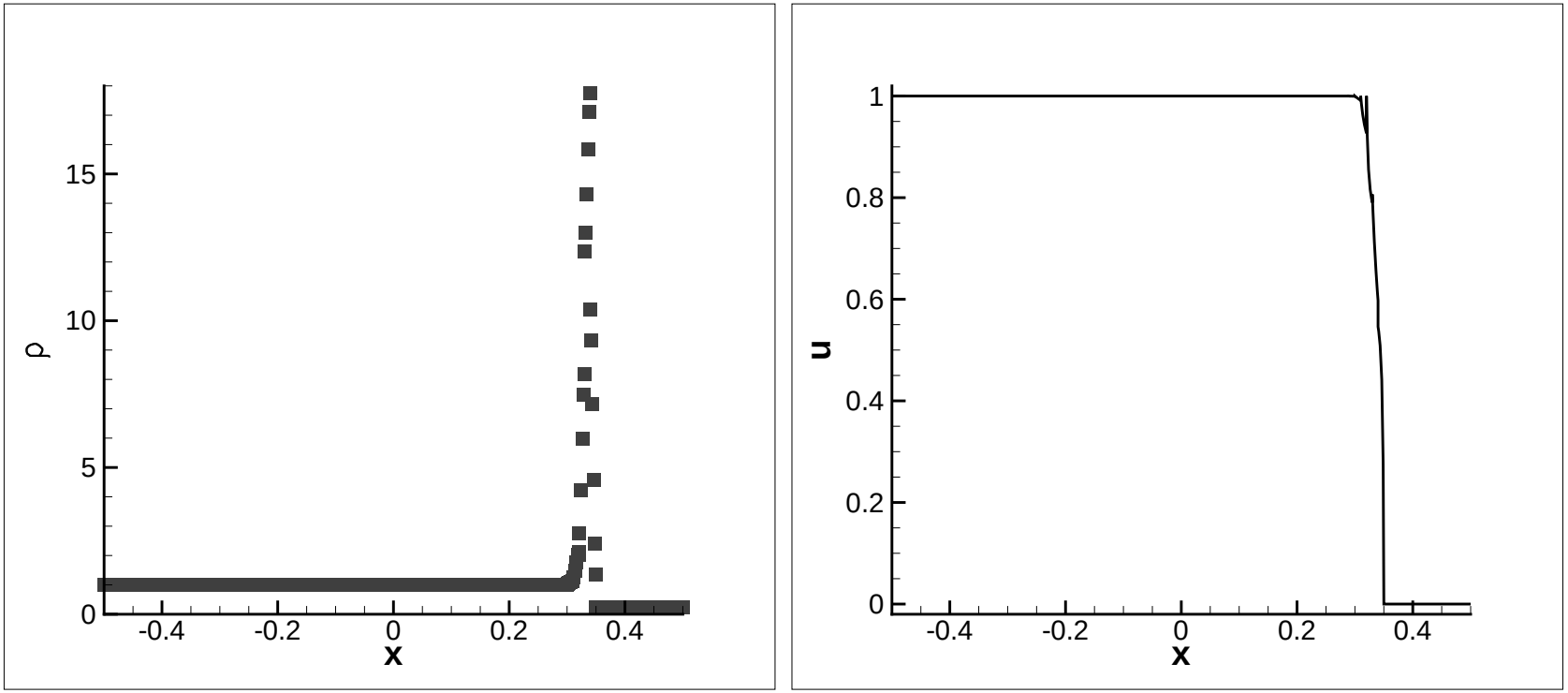

Figure 5: Numerical density (left) and velocity (right) at $t = 0.5$ with $\mathcal{P}^1$ polynomials for initial condition (26).

We consider the following initial condition

$$\begin{aligned} \rho(x,y,0) &= \rho_0(x+y) = \exp(\sin(x+y)), \\ u(x,y,0) &= u_0(x+y) = \tfrac{1}{3}(\cos(x+y)+2), \\ v(x,y,0) &= v_0(x+y) = \tfrac{1}{3}(\sin(x+y)+2). \end{aligned} \tag{27}$$

The exact solution is

$$u(x,y,t) = u_0(z_0), \quad v(x,y,t) = v_0(z_0), \quad \rho(x,y,t) = \frac{\rho_0(z_0)}{1+u_0'(z_0)+v_0'(z$$

where z_0 is given implicitly by

$$z_0 + t(u_0(z_0) + v_0(z_0)) = x + y.$$

Table 5: L^2-norm of the error between the numerical density and the exact density for initial condition (27).

	k=1		k=2		k=3	
N	error	order	error	order	error	order
10	0.512	-	0.107	-	3.42E-02	-
20	0.176	1.54	3.12E-02	1.78	3.57E-03	3.26
40	6.48E-02	1.44	8.52E-03	1.87	4.86E-04	2.88
80	2.32E-02	1.48	1.39E-03	2.62	3.97E-05	3.61
160	9.08E-03	1.35	1.92E-04	2.86	3.65E-06	3.45

We consider the following initial condition

$$\rho(x, y, 0) = \frac{1}{100}, \quad (u, v)(x, y, 0) = (-\frac{1}{10}\cos\theta, -\frac{1}{10}\sin\theta), \quad (28)$$

where θ is the polar angle.

Since all the particles are moving towards the origin, the density function at $t > 0$ should be a single delta at the origin.

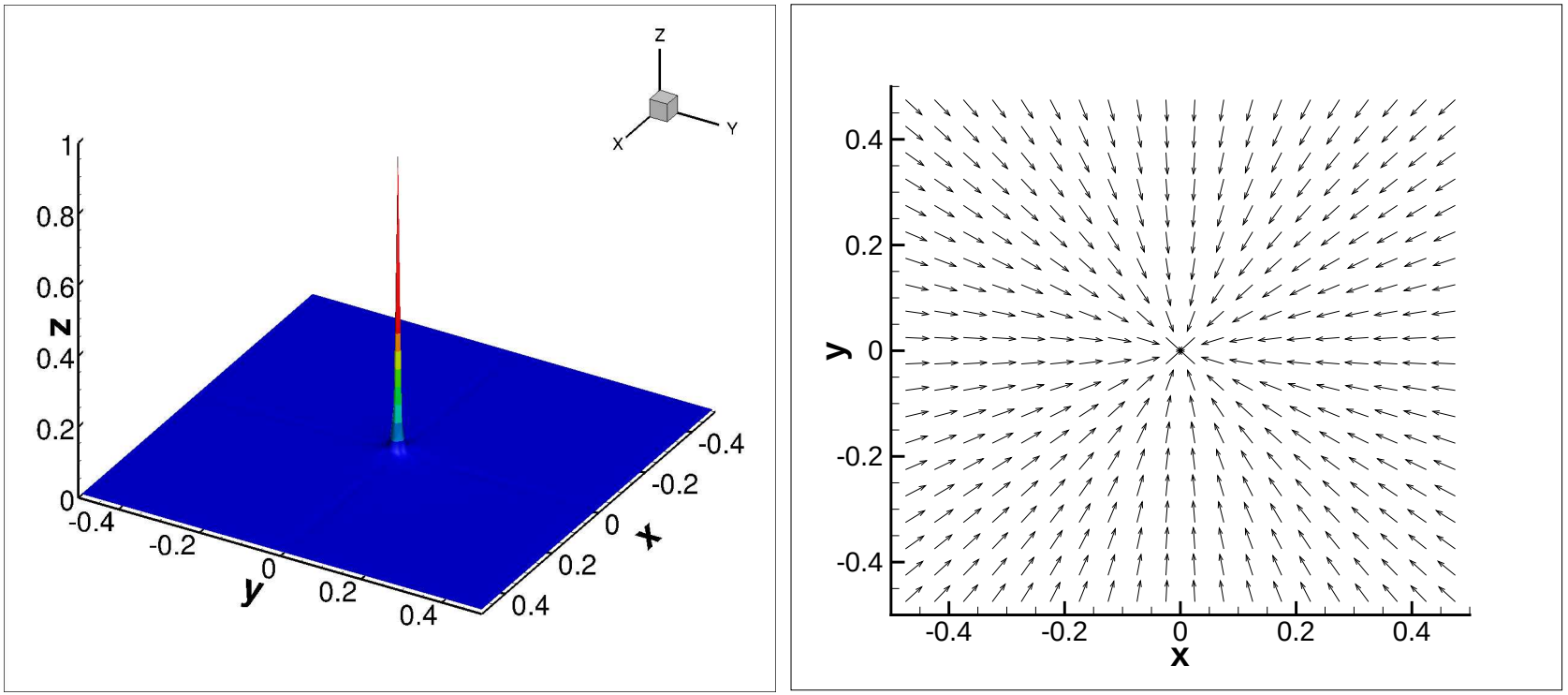

Figure 6: Numerical density (left) and velocity field (right) at $t = 0.5$ for the initial condition (28).

We consider the following initial condition

$$\rho(x,y,0)=\frac{1}{10},\quad (u,v)(x,y,0)=\begin{cases}(-0.25,-0.25) & x>0,y>0,\\ (0.25,-0.25) & x<0,y>0,\\ (0.25,0.25) & x<0,y<0,\\ (-0.25,0.25) & x>0,y<0.\end{cases} \tag{29}$$

Figure 7 shows the numerical density and velocity field at $t=0.5$. From the figure, we can observe δ-singularities located at the origin and the two axes.

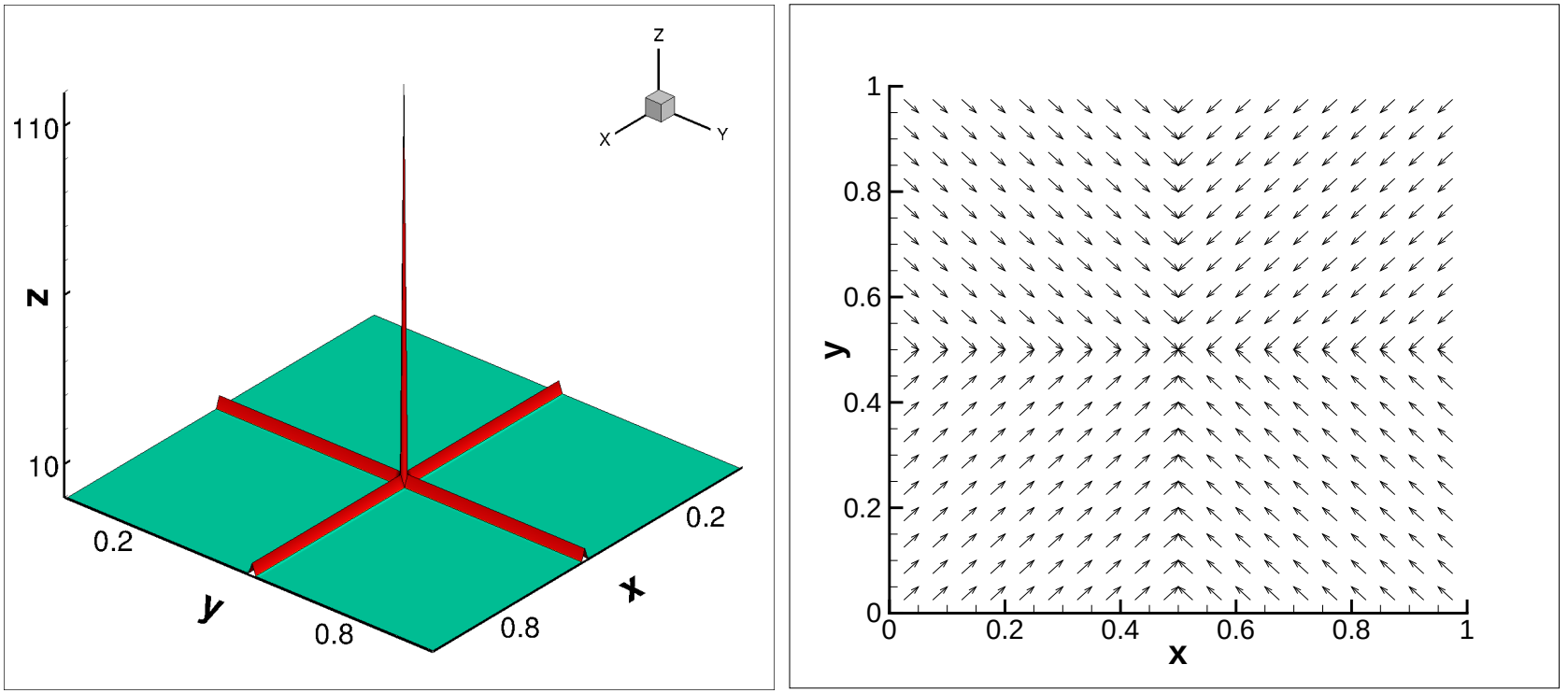

Figure 7: Numerical density (left) and velocity field (right) at $t = 0.5$ for initial condition (29).

We consider the following initial condition

$$\rho(x,y,0)=\frac{1}{100},\quad (u,v)(x,y,0)=\begin{cases}(\cos\theta,\sin\theta) & r<0.3,\\ (-\frac{1}{2}\cos\theta,-\frac{1}{2}\sin\theta) & r>0.3,\end{cases} \tag{30}$$

where $r=\sqrt{x^2+y^2}$ and θ is the polar angle.

Figure 8 shows the numerical density (contour plot) and velocity field at $t=0.5$. From the figure, we can observe δ-shocks located on a circle and vacuum inside.

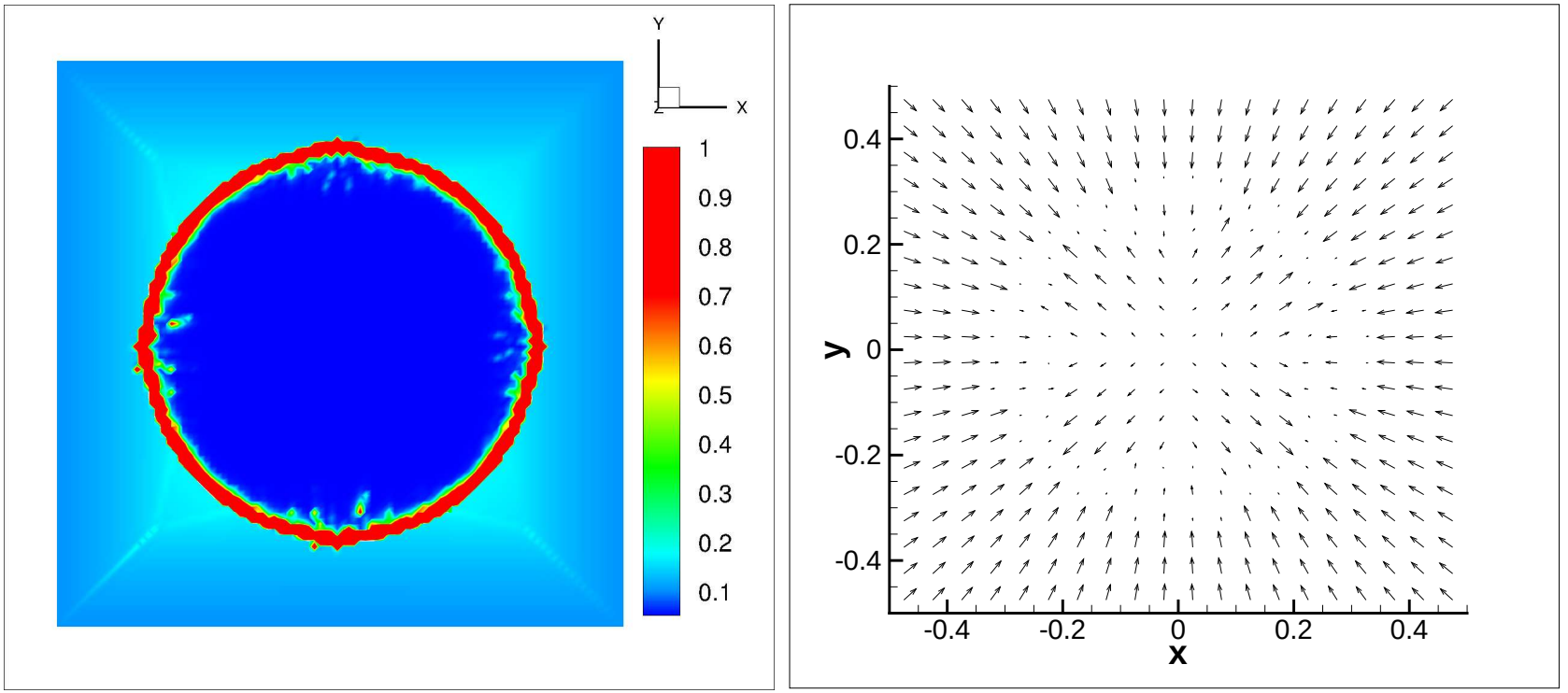

Figure 8: Numerical density (left) and velocity field (right) at $t = 0.5$ for initial condition (30).

We consider the following initial condition

$$\rho(x,y,0)=0.5, \quad (u,v)(x,y,0)=\begin{cases}(0.3,0.4) & x>0, y>0,\\ (-0.4,0.3) & x<0, y>0,\\ (-0.3,-0.4) & x<0, y<0,\\ (0.4,-0.3) & x>0, y<0.\end{cases} \tag{31}$$

Figure 9 shows the numerical density (contour plot) and velocity field with $N=50$ at $t=0.4$. From the figure, we can observe that the numerical solution approximates the vacuum quite well.

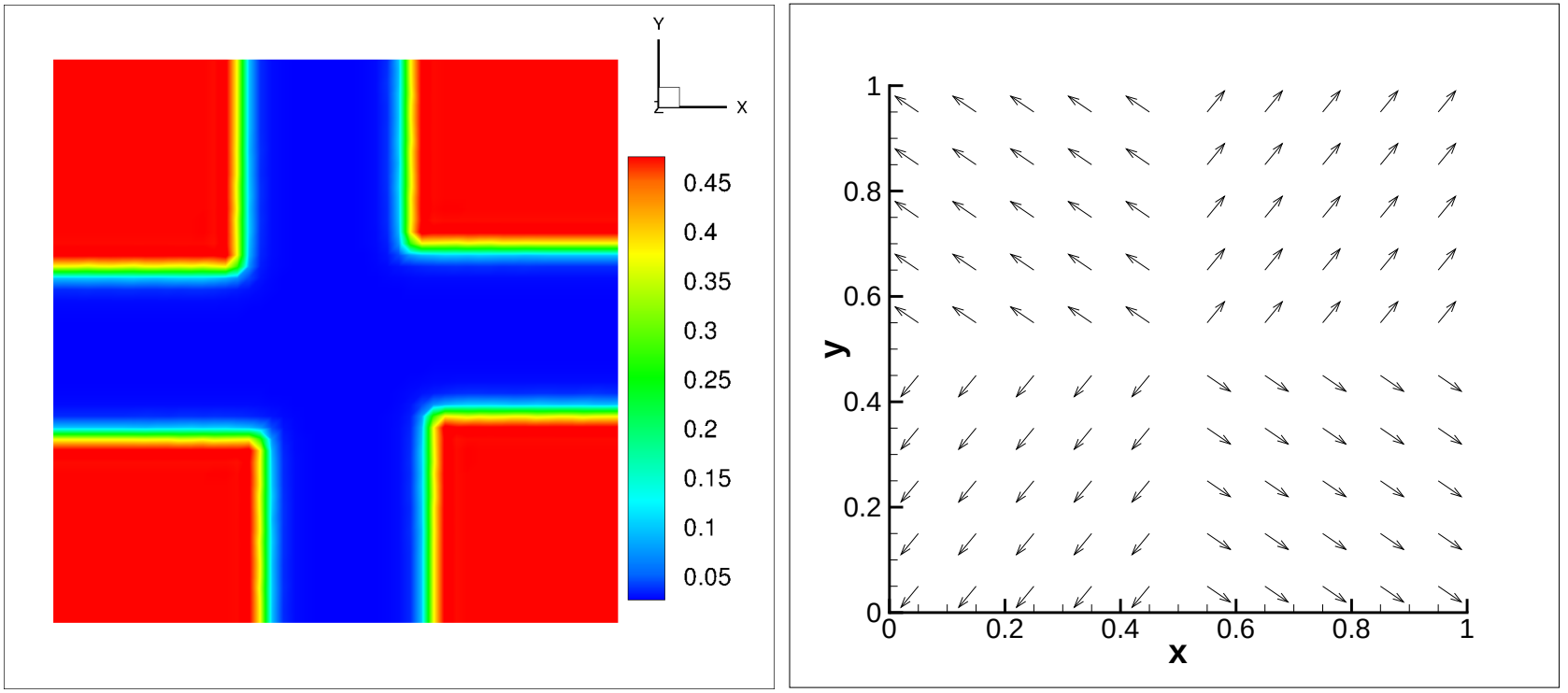

Figure 9: Numerical density (left) and velocity field (right) at $t = 0.4$ with $N = 50$ for initial condition (31).

Concluding remarks

- DG methods are suitable for computing solutions with δ-singularities, because it satisfies a cell entropy inequality and is based on a weak formulation.
- For linear model equations, the DG methods can be shown to converge in negative norms when either the initial condition or the source term contains δ-singularities. This convergence is of $\mathcal{O}(h^{k+1/2})$ order in the whole domain and of $\mathcal{O}(h^{2k+1})$ order $\mathcal{O}(h^{1/2}\log(1/h))$ away from the singularities. Post-processing then produces $\mathcal{O}(h^{2k+1})$ order superconvergence in the strong L^2 norm $\mathcal{O}(h^{1/2}\log(1/h))$ away from the singularities.

- DG methods work well for nonlinear problems containing δ-singularities. It is however important to design and apply a bound-preserving limiter which keeps high order accuracy and can effectively prevent nonlinear instability caused by overshoots of the numerical solution into the ill-posed regime of the nonlinear PDEs.

The End

THANK YOU!