The Hybridizable discontinuous Galerkin methods

Bernardo Cockburn

School of Mathematics
University of Minnesota

London Mathematical Society
EPSRC Durham Symposium
July 07-17, 2014

Outline I

Joint work with:

O.Dubois, B.Dong, B.Chabaud, F.Celiker, A. Cesmelioglu, Y.Chen, J. Cui, G.Fu, J.Gopalakrishnan, J.Guzmán, M.Kirby, L. Ji, R.Lazarov, F.Li, N.C. Nguyen, R.Nochetto, J. Peraire, V.Queneville-Bélair, W.Qiu, S.Rheberghen, F.Reitich, F.-J.Sayas, J.Shen, S.Sherwin, K.Shi, M.Solano, S.-C.Soon, H.Stolarski, S.Tan, H.Wang, W.Zhang.

The DG methods are attracting the interest of many scientists because:

- They enforce the equations in an element-by-element fashion through a Galerkin formulation which can give rise to locally conservative methods.
- They can handle any type of mesh, element shape and basis functions: They are ideally suited for *hp*-adaptivity.
- They have a built-in stabilization mechanism which does not degrade their (high-order) accuracy.
- They can be applied to a wide variety of partial differential equations.

The HDG methods.

Motivation.

However, the DG methods (for second-order elliptic equations) have been criticized because:

- For the same mesh and the same polynomial degree, the number of globally coupled degrees of freedom of the DG methods is much bigger than those of the CG method. Moreover, the orders of convergence of both the vector and scalar variables are also the same.
- For the same mesh and the same index, the number of globally coupled degrees if freedom of the DG methods are much bigger than those of the hybridized version of the RT and BDM methods. Moreover, the orders of convergence of both the vector and the local average of the scalar variables are smaller by one.

The HDG methods.

The main features of the HDG methods.

- The HDG methods are obtained by discretizing characterizations of the exact solution written in terms of many local problems, one for each element of the mesh Ω_h, with suitably chosen data, and in terms of a single global problem that actually determines them.
- This permits an efficiently implementation since they inherit the above-mentioned structure of the exact solution. This is what renders them efficiently implementable, especially within the framework of $hp-$adaptive methods, as is typical of DG methods.

- The way in which they are defined allows them to be, in some instances, more accurate than already existing DG methods. In fact, in some cases when standard DG methods do not converge, HDG methods do.
- The HDG methods can be used for steady-state problems and for time-dependent problems when implicit time-marching methods are used. However, they might also be defined for explicit time-marching schemes.

The HDG methods.

Guidelines for devising the methods.

- Use a characterization of the exact solution in terms of solutions of local problems and transmission conditions.
- Use discontinuous approximations for both the solution inside each element and its trace on the element boundary.
- Define the local solvers by using a Galerkin method to weakly enforce the equations on each element.
- Define a global problem by weakly imposing the transmission conditions.

We provide two different characterizations of the solution of the following second-order elliptic model problem:

$$\begin{aligned} \mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } \Omega, \\ \nabla \cdot \mathbf{q} &= f && \text{in } \Omega, \\ \widehat{u} &= u_D && \text{on } \partial\Omega. \end{aligned}$$

Here c is a matrix-valued function which is symmetric and uniformly positive definite on Ω.

The main idea.

The general approach: Local problems and transmission conditions.

We have that the exact solution satisfies the local problems

$$\begin{aligned} \mathrm{c}\,\mathbf{q} + \nabla u &= 0 \quad \text{in } K, \\ \nabla \cdot \mathbf{q} &= f \quad \text{in } K, \end{aligned}$$

the transmission conditions

$$\begin{aligned} [\![\widehat{u}]\!] &= 0 \quad \text{if } F \in \mathcal{E}_h^o, \\ [\![\widehat{\mathbf{q}}]\!] &= 0 \quad \text{if } F \in \mathcal{E}_h^o, \end{aligned}$$

and the Dirichlet boundary condition

$$\widehat{u} = u_D \quad \text{if } F \in \mathcal{E}_h^\partial.$$

We can obtain $(\mathbf{q}, u)$ in K in terms of $\widehat{u}$ on ∂K and f by solving

$$\begin{aligned} \mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } K,\\ \nabla \cdot \mathbf{q} &= f && \text{in } K,\\ u &= \widehat{u} && \text{on } \partial K. \end{aligned}$$

The function $\widehat{u}$ can now be determined as the solution, on each $F \in \mathcal{E}_h$, of the equations

$$\begin{aligned} [\![\widehat{\mathbf{q}}]\!] &= 0 && \text{if } F \in \mathcal{E}_h^o,\\ \widehat{u} &= u_D && \text{if } F \in \mathcal{E}_h^\partial, \end{aligned}$$

where $\widehat{\mathbf{q}}$ is the trace of $\mathbf{q} = \mathbf{q}(\widehat{u}, f)$ on ∂K.

The main idea.

A first approach: Characterization of the solution.

We have that $(\mathbf{q}, u) = (\mathbf{Q}_{\widehat{u}}, \mathsf{U}_{\widehat{u}}) + (\mathbf{Q}_f, \mathsf{U}_f)$, where

$$\begin{aligned} \mathrm{c}\,\mathbf{Q}_{\widehat{u}} + \nabla \mathsf{U}_{\widehat{u}} &= 0 \quad \text{in } K, & \mathrm{c}\,\mathbf{Q}_f + \nabla \mathsf{U}_f &= 0 \quad \text{in } K,\\ \nabla \cdot \mathbf{Q}_{\widehat{u}} &= 0 \quad \text{in } K, & \nabla \cdot \mathbf{Q}_f &= f \quad \text{in } K,\\ \mathsf{U}_{\widehat{u}} &= \widehat{u} \quad \text{on } \partial K, & \mathsf{U}_f &= 0 \quad \text{on } \partial K. \end{aligned}$$

The function $\widehat{u}$ can now be determined as the solution, on each $F \in \mathcal{E}_h$, of the equations

$$\begin{aligned} -[\![\widehat{\mathbf{Q}}_{\widehat{u}}]\!] &= [\![\widehat{\mathbf{Q}}_f]\!] \quad &\text{if } F \in \mathcal{E}_h^o,\\ \widehat{u} &= u_D \quad &\text{if } F \in \mathcal{E}_h^\partial. \end{aligned}$$

The main idea.

A first approach: The one-dimensional case $K = (x_{i-1}, x_i)$ for $i = 1, \ldots, I$, with $\mathrm{c} = 1$.

We have that $(\mathbf{q}, u) = (\mathbf{Q}_{\widehat{u}}, \mathrm{U}_{\widehat{u}}) + (\mathbf{Q}_f, \mathrm{U}_f)$, where

$$\begin{aligned} \mathbf{Q}_{\widehat{u}} + \frac{d}{dx}\mathrm{U}_{\widehat{u}} &= 0 \quad \text{in } (x_{i-1}, x_i), \\ \frac{d}{dx}\mathbf{Q}_{\widehat{u}} &= 0 \quad \text{in } (x_{i-1}, x_i), \\ \mathrm{U}_{\widehat{u}} &= \widehat{u} \quad \text{on } \{x_{i-1}, x_i\}, \end{aligned} \qquad \begin{aligned} \mathbf{Q}_f + \frac{d}{dx}\mathrm{U}_f &= 0 \quad \text{in } (x_{i-1}, x_i), \\ \frac{d}{dx}\mathbf{Q}_f &= f \quad \text{in } (x_{i-1}, x_i), \\ \mathrm{U}_f &= 0 \quad \text{on } \{x_{i-1}, x_i\}. \end{aligned}$$

The function $\widehat{u}$ is the solution of

$$\begin{aligned} \widehat{\mathbf{Q}}_{\widehat{u}}(x_i^+) - \widehat{\mathbf{Q}}_{\widehat{u}}(x_i^-) &= -\widehat{\mathbf{Q}}_f(x_i^+) + \widehat{\mathbf{Q}}_f(x_i^-) \qquad \text{for } i = 1, \ldots, I-1, \\ \widehat{u}(x_i) &= u_D(x_i) \qquad \text{for } i = 0, I. \end{aligned}$$

The main idea.

A first approach: The one-dimensional case $K = (x_{i-1}, x_i)$ for $i = 1, \ldots, I$, with $\mathrm{c} = 1$.

We have that $(\mathbf{q}, u) = (\mathbf{Q}_{\widehat{u}}, \mathrm{U}_{\widehat{u}}) + (\mathbf{Q}_f, \mathrm{U}_f)$, where, for $x \in (x_{i-1}, x_i)$,

$$\mathbf{Q}_{\widehat{u}}(x) = -\frac{1}{h}(\widehat{u}_i - \widehat{u}_{i-1}), \qquad \mathbf{Q}_f(x) = -\int_{x_{i-1}}^{x_i} G_x(x,s) f(s)\, ds,$$

$$\mathrm{U}_{\widehat{u}}(x) = \frac{1}{h}(x - x_{i-1})\widehat{u}_i + \frac{1}{h}(x_i - x)\widehat{u}_{i-1} \qquad \mathrm{U}_f(x) = \int_{x_{i-1}}^{x_i} G(x,s) f(s)\, ds.$$

The function $\widehat{u}$ is the solution of

$$\frac{1}{h}(-\widehat{u}_{i-1} + 2\widehat{u}_i - \widehat{u}_{i+1}) = -\widehat{\mathbf{Q}}_f(x_i^+) + \widehat{\mathbf{Q}}_f(x_i^-) \qquad \text{for } i = 1, \ldots, I-1,$$

$$\widehat{u}(x_i) = u_D(x_i) \qquad \text{for } i = 0, I.$$

The main idea.

A second approach: Rewriting the equations. We use $\overline{\zeta} := (\zeta, 1)_K/|K|$ and $\overline{\widehat{\mathbf{q}} \cdot \mathbf{n}} := \langle \widehat{\mathbf{q}} \cdot \mathbf{n}, 1\rangle_{\partial K}/|K|$.

We can obtain $(\mathbf{q}, u)$ in K in terms of $\widehat{\mathbf{q}} \cdot \mathbf{n}$ on ∂K, $\overline{u}$ and f by solving

$$\begin{aligned} \mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } K,\\ \nabla \cdot \mathbf{q} &= f - \overline{f} + \overline{\widehat{\mathbf{q}} \cdot \mathbf{n}} && \text{in } K,\\ \mathbf{q} \cdot \mathbf{n} &= \widehat{\mathbf{q}} \cdot \mathbf{n} && \text{on } \partial K. \end{aligned}$$

The functions $\widehat{\mathbf{q}} \cdot \mathbf{n}$ and $\overline{u}$ can now be determined as the solution of the equations

$$\begin{aligned} [\![\widehat{u}]\!] &= 0 && \text{for } F \in \mathcal{E}_h^o,\\ \overline{\widehat{\mathbf{q}} \cdot \mathbf{n}} &= \overline{f} && \text{for } K \in \mathcal{T}_h,\\ \widehat{u} &= u_D && \text{for } F \in \mathcal{E}_h^\partial, \end{aligned}$$

where $\widehat{u}$ is the trace of $u = u(\widehat{\mathbf{q}} \cdot \mathbf{n}, \overline{u}, f)$ on ∂K.

The main idea.

A second approach: Characterization of the solution.

We have that $(\mathbf{q}, u) = (\mathbf{Q}_{\widehat{q}}, \mathsf{U}_{\widehat{q}}) + (\mathbf{0}, \overline{u}) + (\mathbf{Q}_f, \mathsf{U}_f)$, where

$$
\begin{aligned}
\mathrm{c}\,\mathbf{Q}_{\widehat{\mathbf{q}}} + \nabla \mathsf{U}_{\widehat{\mathbf{q}}} &= 0 && \text{in } K, \\
\nabla\cdot\mathbf{Q}_{\widehat{\mathbf{q}}} &= \overline{\widehat{\mathbf{q}}\cdot\mathbf{n}} && \text{in } K, \\
\mathbf{Q}_{\widehat{\mathbf{q}}}\cdot\mathbf{n} &= \widehat{\mathbf{q}}\cdot\mathbf{n} && \text{on } \partial K, \\
\overline{\mathsf{U}}_{\widehat{\mathbf{q}}} &= 0,
\end{aligned}
\qquad
\begin{aligned}
\mathrm{c}\,\mathbf{Q}_f + \nabla \mathsf{U}_f &= 0 && \text{in } K, \\
\nabla\cdot\mathbf{Q}_f &= f - \overline{f} && \text{in } K, \\
\mathbf{Q}_f\cdot\mathbf{n} &= 0 && \text{on } \partial K, \\
\overline{\mathsf{U}}_f &= 0.
\end{aligned}
$$

The functions $\widehat{\mathbf{q}}\cdot\mathbf{n}$ and $\overline{u}$ can now be determined as the solution of the equations

$$
\begin{aligned}
-[\![\widehat{\mathsf{U}}_{\widehat{\mathbf{q}}}]\!] - [\![\overline{u}]\!] &= [\![\widehat{\mathsf{U}}_f]\!] && \text{for } F\in\mathcal{E}_h^o, \\
\overline{\widehat{\mathbf{q}}\cdot\mathbf{n}} &= \overline{f} && \text{for } K\in\mathcal{T}_h, \\
\widehat{\mathsf{U}}_{\widehat{\mathbf{q}}} + \overline{u} + \widehat{\mathsf{U}}_f &= u_D && \text{for } F\in\mathcal{E}_h^\partial.
\end{aligned}
$$

The main idea.

A second approach: The one-dimensional case $K = (x_{i-1}, x_i)$ for $i = 1, \ldots, I$, with $\mathrm{c} = 1$.

We have that $(\mathbf{q}, u) = (\mathbf{Q}_{\widehat{u}}, \mathrm{U}_{\widehat{u}}) + (\mathbf{0}, \overline{u}) + (\mathbf{Q}_f, \mathrm{U}_f)$, where

$$
\begin{aligned}
\mathbf{Q}_{\widehat{\mathbf{q}}} + \frac{d}{dx}\mathrm{U}_{\widehat{\mathbf{q}}} &= 0 && \text{in } (x_{i-1}, x_i), & \mathbf{Q}_f + \frac{d}{dx}\mathrm{U}_f &= 0 && \text{in } (x_{i-1}, x_i),\\
\frac{d}{dx}\mathbf{Q}_{\widehat{\mathbf{q}}} &= \overline{\widehat{\mathbf{q}} \cdot \mathbf{n}} && \text{in } (x_{i-1}, x_i), & \frac{d}{dx}\mathbf{Q}_f &= f - \overline{f} && \text{in } (x_{i-1}, x_i),\\
\mathbf{Q}_{\widehat{\mathbf{q}}} \cdot \mathbf{n} &= \widehat{\mathbf{q}} \cdot \mathbf{n} && \text{on } \{x_{i-1}, x_i\}, & \mathbf{Q}_f \cdot \mathbf{n} &= 0 && \text{on } \{x_{i-1}, x_i\},\\
\overline{\mathrm{U}}_{\widehat{\mathbf{q}}} &= 0 && \text{on } \{x_{i-1}, x_i\}, & \overline{\mathrm{U}}_f &= 0 && \text{on } \{x_{i-1}, x_i\}.
\end{aligned}
$$

The functions $\widehat{\mathbf{q}}$ and $\overline{u}$ are the solution of

$$
\begin{aligned}
\widehat{\mathbf{U}}_{\widehat{\mathbf{q}}}(x_i^+) - \widehat{\mathbf{U}}_{\widehat{\mathbf{q}}}(x_i^-) + \overline{u}_{i+1/2} - \overline{u}_{i-1/2} &= -\widehat{\mathbf{U}}_f(x_i^+) + \widehat{\mathbf{U}}_f(x_i^-) && \text{for } i = 1, \ldots, I-1,\\
\widehat{\mathbf{q}}_i - \widehat{\mathbf{q}}_{i-1} &= h\,\overline{f}_{i-1/2} && \text{for } i = 1, \ldots, I-1,\\
\widehat{\mathbf{U}}_{\widehat{\mathbf{q}}}(x_0^+) + \overline{u}_{1/2} + \widehat{\mathbf{U}}_f(x_0^+) &= u_D(x_0),\\
\widehat{\mathbf{U}}_{\widehat{\mathbf{q}}}(x_I^-) + \overline{u}_{I-1/2} + \widehat{\mathbf{U}}_f(x_I^-) &= u_D(x_I).
\end{aligned}
$$

The main idea.

A second approach: The one-dimensional case $K = (x_{i-1}, x_i)$ for $i = 1, \ldots, I$, with $\mathrm{c} = 1$.

We have that $(\mathbf{q}, u) = (\mathbf{Q}_{\widehat{u}}, \mathrm{U}_{\widehat{u}}) + (\mathbf{0}, \overline{u}) + (\mathbf{Q}_f, \mathrm{U}_f)$, where, for $x \in (x_{i-1}, x_i)$,

$$\mathbf{Q}_{\widehat{\mathbf{q}}}(x) = \frac{1}{h}(x - x_{i-1})\widehat{\mathbf{q}}_i + \frac{1}{h}(x_i - x)\widehat{\mathbf{q}}_{i-1}, \qquad \mathbf{Q}_f(x) = -\int_{x_{i-1}}^{x_i} \mathrm{G}_x(x,s)(f - \overline{f})(s)\, ds,$$

$$\mathrm{U}_{\widehat{u}}(x) = \frac{1}{6h}(h^2 - 3(x - x_{i-1})^2)\widehat{\mathbf{q}}_i - \frac{1}{6h}(h^2 - 3(x_i - x)^2)\widehat{\mathbf{q}}_{i-1}, \qquad \mathrm{U}_f(x) = \int_{x_{i-1}}^{x_i} \mathrm{G}(x,s)(f - \overline{f})(s)\, ds.$$

The functions $\widehat{\mathbf{q}}$ and $\overline{u}$ are the solution of

$$\frac{h}{6}(\widehat{\mathbf{q}}_{i-1} + 4\,\widehat{\mathbf{q}}_i + \widehat{\mathbf{q}}_{i+1}) + \overline{u}_{i+1/2} - \overline{u}_{i-1/2} = -\widehat{\mathbf{U}}_f(x_i^+) + \widehat{\mathbf{U}}_f(x_i^-) \quad \text{for } i = 1, \ldots, I-1,$$

$$\widehat{\mathbf{q}}_i - \widehat{\mathbf{q}}_{i-1} = h\,\overline{f}_{i-1/2} \quad \text{for } i = 1, \ldots, I-1,$$

$$\frac{h}{6}(2\widehat{\mathbf{q}}_0 + \widehat{\mathbf{q}}_1) + \overline{u}_{1/2} + \widehat{\mathbf{U}}_f(x_0^+) = u_D(x_0),$$

$$-\frac{h}{6}(\widehat{\mathbf{q}}_{I-1} + 2\widehat{\mathbf{q}}_I) + \overline{u}_{I-1/2} + \widehat{\mathbf{U}}_f(x_I^-) = u_D(x_I).$$

The main idea.

Summary.

- The HDG methods are obtained by constructing discrete versions of the above characterizations of the exact solution.
- In this way, the globally coupled degrees of freedom will be those of the corresponding global formulations.

A first approach. (B.C., J.Gopalakrishnan and R.Lazarov, SINUM, 2009.)

The local solvers: A weak formulation on each element.

On the element $K \in \Omega_h$, given $\widehat{u}$ on ∂K and f, we have that $(\mathbf{q}, u)$ satisfies the equations

$$\begin{aligned}(\mathrm{c}\,\mathbf{q}, \mathbf{v})_K - (u, \nabla \cdot \mathbf{v})_K + \langle \widehat{u}, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} &= 0, \\ -(\mathbf{q}, \nabla w)_K + \langle \widehat{\mathbf{q}} \cdot \mathbf{n}, w \rangle_{\partial K} &= (f, w)_K,\end{aligned}$$

for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$, where

$$\widehat{\mathbf{q}} \cdot \mathbf{n} = \mathbf{q} \cdot \mathbf{n} \qquad \text{on } \partial K.$$

The first approach.

The local solvers: Definition.

On the element $K \in \Omega_h$, we define $(\mathbf{q}_h, u_h)$ terms of $(\widehat{u}_h, f)$ as the element of $\mathbf{V}(K) \times W(K)$ such that

$$\begin{aligned} (\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_K - (u_h, \nabla\cdot\mathbf{v})_K + \langle \widehat{u}_h, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial K} &= 0, \\ -(\mathbf{q}_h, \nabla w)_K + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial K} &= (f, w)_K, \end{aligned}$$

for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$, where

$$\widehat{\mathbf{q}}_h\cdot\mathbf{n} = \mathbf{q}_h\cdot\mathbf{n} + \tau(u_h - \widehat{u}_h) \qquad \text{on } \partial K.$$

If we want that, at any given point x of ∂K at which the normal $\mathbf{n}$ is well defined,

- The numerical trace $\widehat{\mathbf{q}}_h(x) \cdot \mathbf{n}$ only depends on $\mathbf{q}_h(x) \cdot \mathbf{n}$, $u_h(x)$ and the numerical trace $\widehat{u}_h(\mathrm{x})$.
- The dependence is linear.
- The numerical trace $\widehat{\mathbf{q}}_h(x) \cdot \mathbf{n}$ is consistent, that is, $\widehat{\mathbf{q}}_h(x) \cdot \mathbf{n} = \mathbf{q}_h(x) \cdot \mathbf{n}$ whenever $u_h(x) = \widehat{u}_h(x)$,

we must have that $\widehat{\mathbf{q}}_h \cdot \mathbf{n} = \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h)$.

The first approach.

The local solvers are well defined.

Theorem

The local solver on K is well defined if

- $\tau > 0$ *on* ∂K,
- $\nabla W(K) \subset \mathbf{V}(K)$.

The first approach.

Proof.

The system is square. Set $\widehat{u}_h = 0$ and $f = 0$.
For $(\mathbf{v}, w) := (\mathbf{q}_h, u_h)$, the equations read

$$\begin{aligned}(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K - (u_h, \nabla \cdot \mathbf{q}_h)_K &= 0,\\ -(\mathbf{q}_h, \nabla u_h)_K + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, u_h\rangle_{\partial K} &= 0.\end{aligned}$$

Hence

$$(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K + \langle (\widehat{\mathbf{q}}_h - \mathbf{q}_h) \cdot \mathbf{n}, u_h\rangle_{\partial K} = 0,$$

and since $\widehat{\mathbf{q}}_h \cdot \mathbf{n} = \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h)$, we get

$$(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K + \langle \tau\,(u_h), u_h\rangle_{\partial K} = 0.$$

This implies that $\mathbf{q}_h = 0$ on K, and that $u_h = 0$ on ∂K.

The first approach.

Proof.

Now, the first equation defining the local solvers reads

$$-(u_h, \nabla \cdot \mathbf{v})_K = 0,$$

for all $\mathbf{v} \in \mathbf{V}(K)$. Hence

$$(\nabla u_h, \mathbf{v})_K = 0,$$

and so $\nabla u_h = 0$. This proves the result.

The first approach

The local solvers: Examples of the stabilization function τ.

- The simple multiplication stabilization function $\tau(\phi) := \tau \cdot \phi$.

- The Bassi-Rebay stabilization function:

$$\tau(\phi)|_F := \tau\, \mathbf{r}_F(\phi) \cdot \mathbf{n}, \quad \mathbf{r}_F \in \mathbf{V}(K) : \quad (\mathbf{r}_F(\phi), \mathbf{v})_K = \langle \phi, \mathbf{v} \cdot \mathbf{n} \rangle_F$$

- The Lehrenfeld stabilization function:

$$\tau(\phi) := \tau \cdot L^2(\partial K)\text{-projection of } \phi \text{ into } M(\partial K)$$

The first approach.

The global problem: The weak formulation for $\widehat{u}_h$.

For each face $F \in \mathcal{E}_h^o$, we take $\widehat{u}_h|_F$ in the space $M(F)$. We determine $\widehat{u}_h$ by requiring that,

$$\langle \mu, [\![\widehat{\mathbf{q}}_h]\!]\rangle_F = 0 \quad \forall\, \mu \in\, M(F) \quad \text{if } F \in \mathcal{E}_h^o,$$
$$\widehat{u}_h = u_D \quad \text{if } F \in \mathcal{E}_h^\partial.$$

The first approach.

The transmission condition.

Suppose that the transmission condition implies that $[\![\widehat{\mathbf{q}}_h]\!] = 0$ on a face $F \in \mathcal{E}_h^o$. Then, on that face, we have that

$$[\![\mathbf{q}_h]\!] + \tau^+({u_h}^+ - \widehat{u}_h) + \tau^-({u_h}^- - \widehat{u}_h) = 0,$$

which holds if

$$\widehat{u}_h = \frac{\tau^+ {u_h}^+ + \tau^- {u_h}^-}{\tau^+ + \tau^-} + \frac{1}{\tau^+ + \tau^-}[\![\mathbf{q}_h]\!],$$

$$\widehat{\mathbf{q}}_h = \frac{\tau^- {\mathbf{q}_h}^+ + \tau^+ {\mathbf{q}_h}^-}{\tau^+ + \tau^-} + \frac{\tau^+\tau^-}{\tau^+ + \tau^-}[\![u_h]\!]$$

provided $\tau^+ + \tau^- > 0$.

The first approach.

The numerical trace $\widehat{u}_h$ is well defined.

Theorem

The numerical trace $\widehat{u}_h$ is well defined if, for each $K \in \partial\Omega_h$,

- $\tau > 0$ *on* ∂K,
- $\nabla W(K) \subset \mathbf{V}(K)$.

The first approach.

Proof.

The system is square. Set $u_D = 0$ and $f = 0$. For $\mu := \widehat{u}_h$, the equation reads

$$0 = \sum_{F \in \mathcal{E}_h^o} \langle \widehat{u}_h, [\![\widehat{\mathbf{q}}_h]\!] \rangle_F = \sum_{K \in \Omega_h} \langle \widehat{u}_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial K} =: \langle \widehat{u}_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h}.$$

Note that

$$\begin{aligned}
-\langle \widehat{u}_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} &= - \langle \widehat{u}_h, \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h) \rangle_{\partial \Omega_h} \\
&= - \langle \widehat{u}_h, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} - \langle u_h, \tau(u_h - \widehat{u}_h) \rangle_{\partial \Omega_h} \\
&\quad + \langle (u_h - \widehat{u}_h), \tau(u_h - \widehat{u}_h) \rangle_{\partial \Omega_h} \\
&= - \langle \widehat{u}_h, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} - \langle u_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} + \langle u_h, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} \\
&\quad + \langle (u_h - \widehat{u}_h), \tau(u_h - \widehat{u}_h) \rangle_{\partial \Omega_h}
\end{aligned}$$

For $(\mathbf{v}, w) := (\mathbf{q}_h, u_h)$, the equations of the local solvers read

$$
\begin{aligned}
(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K - (u_h, \nabla\cdot\mathbf{q}_h)_K + \langle \widehat{u}_h, \mathbf{q}_h\cdot\mathbf{n}\rangle_{\partial K} &= 0,\\
-(\mathbf{q}_h, \nabla u_h)_K + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, u_h\rangle_{\partial K} &= 0.
\end{aligned}
$$

Then

$$
-\langle \widehat{u}_h, \widehat{\mathbf{q}}_h\cdot\mathbf{n}\rangle_{\partial\Omega_h} = (\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_{\Omega_h} + \langle (u_h - \widehat{u}_h), \tau(u_h - \widehat{u}_h)\rangle_{\partial\Omega_h}.
$$

As a consequence, $\langle \widehat{u}_h, \widehat{\mathbf{q}}_h\cdot\mathbf{n}\rangle_{\partial\Omega_h} = 0$ implies $\mathbf{q}_h = 0$ on Ω_h and $u_h = \widehat{u}_h$ on $\partial\Omega_h$.

The first approach.

Proof.

Now, the first equation definign the local solvers reads

$$-(u_h, \nabla \cdot \mathbf{v})_K + \langle u_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} = 0,$$

for all $\mathbf{v} \in \mathbf{V}(K)$. Hence

$$(\nabla u_h, \mathbf{v})_K = 0,$$

and so $\nabla u_h = 0$.

This shows that u_h is a constant and, since $u_h = \widehat{u}_h = 0$ on $\partial\Omega$, we can conclude that $u_h = 0$ on Ω_h. We now have that $\widehat{u}_h = u_h = 0$ on $\partial\Omega_h$. This proves the result.

The first approach.

First characterization of the approximate solution.

We have that $(\mathbf{q}_h, u_h) = (\mathbf{Q}_{\widehat{u}_h}, \mathsf{U}_{\widehat{u}_h}) + (\mathbf{Q}_f, \mathsf{U}_f)$ where

$$(\mathbf{Q}_{\widehat{u}_h}, \mathsf{U}_{\widehat{u}_h}) := (\mathbf{Q}(\widehat{u}_h, 0), \mathsf{U}(\widehat{u}_h, 0)), \quad (\mathbf{Q}_f, \mathsf{U}_f) := (\mathbf{Q}(0, f), \mathsf{U}(0, f)).$$

where $(\mathbf{Q}(\widehat{u}_h, f), \mathsf{U}(\widehat{u}_h, f))$ is the linear mapping that associates $(\widehat{u}_h, f)$ to $(\mathbf{q}_h, u_h)$, and where the numerical trace $\widehat{u}_h$ is the element of the space

$$M_h(u_D) := \{\mu \in L^2(\mathcal{E}_h) : \quad \mu|_F \in M(F) \ \forall\, F \in \mathcal{E}_h, \quad u_h|_{\partial\Omega} := P_\partial u_D\},$$

satisfying the equations

$$a_h(\widehat{u}_h, \mu) = \ell_h(\mu) \quad \forall\, \mu \in M_h(0),$$

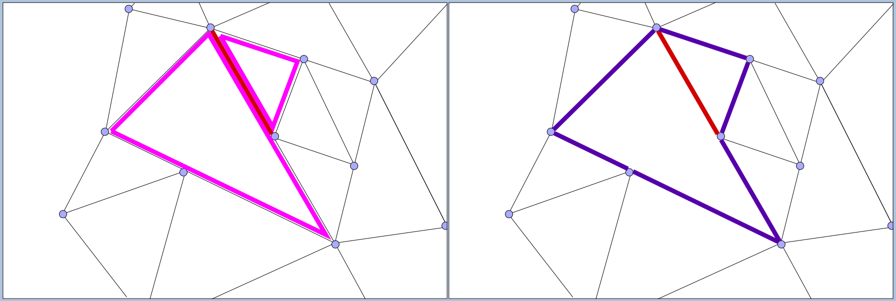
where $a_h(\mu, \lambda) := -\langle \mu, \widehat{\mathbf{Q}}_\lambda \cdot \mathbf{n} \rangle_{\partial\Omega_h}$, and $\ell_h(\mu) := \langle \mu, \widehat{\mathbf{Q}}_f \cdot \mathbf{n} \rangle_{\partial\Omega_h}$.

The first approach.

Sparsity of the stiffness matrix.

The stiffness matrix is sparse by blocks:

$$a_h(\mu, \eta) = -\langle \mu, \widehat{\mathbf{Q}}_\eta \cdot \mathbf{n} \rangle_{\partial \Omega_h} \neq 0.$$



Theorem

We have that

$$a_h(\mu,\lambda) = (\mathrm{c}\mathbf{Q}_\mu, \mathbf{Q}_\lambda)_{\partial\Omega_h} + \langle \tau(\mathsf{U}_\mu - \mu), (\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h}.$$

Moreover, $a_h(\cdot,\cdot)$ is positive definite on $M_h(0) \times M_h(0)$.

The numerical trace $\widehat{u}_h$ minimizes the quadratic functional

$$J_h(\eta) := \frac{1}{2} a_h(\eta,\eta) - \ell_h(\eta),$$

over the functions η in $M_h(u_D)$.

The first approach.

Proof.

$$
\begin{aligned}
a_h(\mu,\lambda) &= -\langle \mu, \widehat{\mathbf{Q}}_\lambda \cdot \mathbf{n}\rangle_{\partial\Omega_h} \\
&= -\langle \mu, \mathbf{Q}_\lambda \cdot \mathbf{n} + \tau(\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h} \\
&= -\langle \mu, \mathbf{Q}_\lambda \cdot \mathbf{n}\rangle_{\partial\Omega_h} - \langle \mathsf{U}_\mu, \tau(\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h} \\
&\quad + \langle \mathsf{U}_\mu - \mu, \tau(\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h} \\
&= -\langle \mu, \mathbf{Q}_\lambda \cdot \mathbf{n}\rangle_{\partial\Omega_h} - \langle \mathsf{U}_\mu, \widehat{\mathbf{Q}}_\lambda \cdot \mathbf{n}\rangle_{\partial\Omega_h} + \langle \mathsf{U}_\mu, \mathbf{Q}_\lambda \cdot \mathbf{n}\rangle_{\partial\Omega_h} \\
&\quad + \langle \mathsf{U}_\mu - \mu, \tau(\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h}
\end{aligned}
$$

The first approach.

Proof.

For $(\mathbf{v}, w) := (\mathbf{Q}_\lambda, \mathsf{U}_\mu)$, the equations of the local solvers read

$$(\mathrm{c}\,\mathbf{Q}_\mu, \mathbf{Q}_\lambda)_K - (\mathsf{U}_\mu, \nabla\cdot\mathbf{Q}_\lambda)_K + \langle \mu, \mathbf{Q}_\lambda\cdot\mathbf{n}\rangle_{\partial K} = 0,$$
$$-(\mathbf{Q}_\lambda, \nabla \mathsf{U}_\mu)_K + \langle \widehat{\mathbf{Q}}_\lambda\cdot\mathbf{n}, \mathsf{U}_\mu\rangle_{\partial K} = 0.$$

Then

$$a_h(\mu,\lambda) = (\mathrm{c}\,\mathbf{Q}_\mu, \mathbf{Q}_\lambda)_K + \langle \mathsf{U}_\mu - \mu, \tau(\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h}.$$

This completes the proof.

The first approach.

A second characterization of the method.

The approximate solution $(\mathbf{q}_h, u_h, \widehat{u}_h)$ is the element of the space $\mathbf{V}_h \times W_h \times M_h(u_D)$ satisfying the equations

$$\begin{aligned}(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_{\Omega_h} - (u_h, \nabla\cdot\mathbf{v})_{\Omega_h} + \langle \widehat{u}_h, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\ -(\mathbf{q}_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h},\\ \langle \mu, \widehat{\mathbf{q}}_h\cdot\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\end{aligned}$$

for all $(\mathbf{v}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h(0)$, where

$$\widehat{\mathbf{q}}_h\cdot\mathbf{n} = \mathbf{q}_h\cdot\mathbf{n} + \tau(u_h - \widehat{u}_h) \qquad \text{on } \partial\Omega_h.$$

The first approach.

A third characterization of the approximate solution

For any $(w, \mu) \in W_h \times M_h$, define $\mathbf{q}_{w,\mu} \in \mathbf{V}_h$ as the solution of

$$(\mathrm{c}\, \mathbf{q}_{w,\mu}, \mathbf{v})_{\Omega_h} - (w, \nabla \cdot \mathbf{v})_{\Omega_h} + \langle \mu, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial \Omega_h} = 0,$$

for all $\mathbf{v} \in V_h$.
The approximate solution is $(\mathbf{q}_{u_h,\widehat{u}_h}, u_h, \widehat{u}_h)$ where $(u_h, \widehat{u}_h)$ is the element of $W_h \times M_h(u_D)$ satisfying the equations

$$\begin{aligned}(\nabla \cdot \mathbf{q}_{u_h,\widehat{u}_h}, w)_{\Omega_h} + \langle \tau(u_h - \widehat{u}_h), w \rangle_{\partial \Omega_h} &= (f, w)_{\Omega_h}, \\ \langle \mu, \mathbf{q}_{u_h,\widehat{u}_h} \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h) \rangle_{\partial \Omega_h} &= 0,\end{aligned}$$

for all $(w, \mu) \in W_h \times M_h(0)$.

The first approach.

A third characterization of the approximate solution

For any $(w, \mu) \in W_h \times M_h$, define $\mathbf{q}_{w,\mu} \in \mathbf{V}_h$ as the solution of

$$(\mathrm{c}\,\mathbf{q}_{w,\mu}, \mathbf{v})_{\Omega_h} - (w, \nabla\cdot\mathbf{v})_{\Omega_h} + \langle \mu, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial\Omega_h} = 0,$$

for all $\mathbf{v} \in V_h$.
The approximate solution is $(\mathbf{q}_{u_h,\widehat{u}_h}, u_h, \widehat{u}_h)$ where $(u_h, \widehat{u}_h)$ is the element of $W_h \times M_h(u_D)$ satisfying the equations

$$\begin{aligned}(\mathrm{c}\,\mathbf{q}_{u_h,\widehat{u}_h}, \mathbf{q}_{w,\mu})_{\Omega_h} + \langle \mu, \mathbf{q}_{w,\mu}\cdot\mathbf{n}\rangle_{\partial\Omega_h} + \langle \tau(u_h - \widehat{u}_h), w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h},\\ \langle \mu, \mathbf{q}_{u_h,\widehat{u}_h}\cdot\mathbf{n} + \tau(u_h - \widehat{u}_h)\rangle_{\partial\Omega_h} &= 0,\end{aligned}$$

for all $(w, \mu) \in W_h \times M_h(0)$.

The first approach.

A third characterization of the approximate solution

For any $(w, \mu) \in W_h \times M_h$, define $\mathbf{q}_{w,\mu} \in \mathbf{V}_h$ as the solution of

$$(\mathrm{c}\,\mathbf{q}_{w,\mu}, \mathbf{v})_{\Omega_h} - (w, \nabla \cdot \mathbf{v})_{\Omega_h} + \langle \mu, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial\Omega_h} = 0,$$

for all $\mathbf{v} \in V_h$.

The approximate solution is $(\mathbf{q}_{u_h,\widehat{u}_h}, u_h, \widehat{u}_h)$ where $(u_h, \widehat{u}_h)$ is the element of $W_h \times M_h(u_D)$ satisfying the equations

$$(\mathrm{c}\,\mathbf{q}_{u_h,\widehat{u}_h}, \mathbf{q}_{w,\mu})_{\Omega_h} + \langle \tau(u_h - \widehat{u}_h), w - \mu \rangle_{\partial\Omega_h} = (f, w)_{\Omega_h},$$

for all $(w, \mu) \in W_h \times M_h(0)$.

The first approach.

The associated minimization property. (H. Kabbaria, A. Lew, and B.C.)

The function $(u_h, \widehat{u}_h)$ minimizes the quadratic functional

$$J_h(w,\mu) := \frac{1}{2}(\mathrm{c}\,\mathbf{q}_{w,\mu}, \mathbf{q}_{w,\mu})_{\Omega_h} + \frac{1}{2}\langle \tau(w-\mu),(w-\mu)\rangle_{\partial\Omega_h} - (f,w)_{\Omega_h},$$

over the functions $(w,\mu) \in W_h \times M_h(u_D)$.

This is the Weak Galerkin method.

The first approach.

The jumps $u_h - \widehat{u}_h$ stabilize the method.

The energy identity for the exact solution is

$$(\mathrm{c}\,\mathbf{q}, \mathbf{q})_\Omega = (f, u)_\Omega - \langle u_D, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial\Omega},$$

and for the approximate solution,

$$(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_\Omega + \Theta_\tau(u_h - \widehat{u}_h) = (f, u_h)_\Omega - \langle u_D, \widehat{\mathbf{q}}_h\cdot\mathbf{n}\rangle_{\partial\Omega}.$$

where $\Theta_\tau(u_h - \widehat{u}_h) := \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h\rangle_{\partial\Omega_h}$.

$\Theta_\tau(u_h - \widehat{u}_h)$ is a dissipative term of the same form of that of the original DG method, when the stabilization function τ is positive.

The first approach.

The jumps $u_h - \widehat{u}_h$ control the four residuals.

The Galerkin formulation on the element K defining the local solver reads

$$\begin{aligned}(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_K - (u_h, \nabla\cdot\mathbf{v})_K + \langle \widehat{u}_h, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial K} &= 0,\\ -(\mathbf{q}_h, \nabla w)_K + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial K} &= (f, w)_K,\end{aligned}$$

for all $(\mathbf{v}, w) \in \mathbf{V}(K)\times W(K)$, or, equivalently,

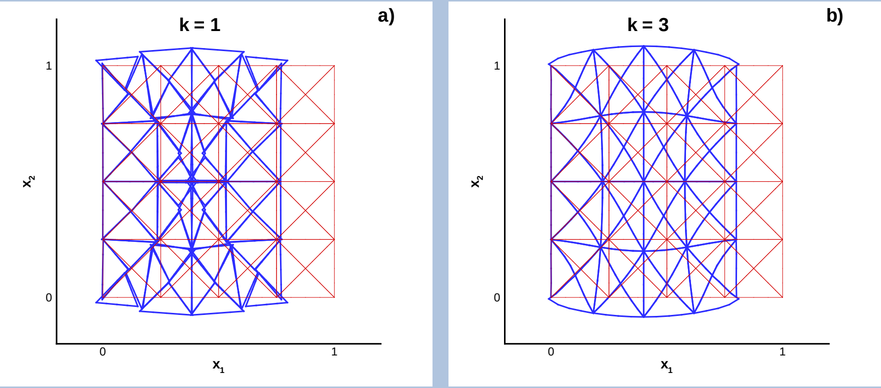
$$\begin{aligned}(\mathbf{R}^u_K, \mathbf{v})_K &= \langle R^u_{\partial K}, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial K} \quad \forall\ \mathbf{v}\in\mathbf{V}(K),\\ (R^{\mathbf{q}}_K, w)_K &= \langle R^{\mathbf{q}}_{\partial K}, w\rangle_{\partial K} \qquad \forall\ w\in W(K),\end{aligned}$$

where

$$\begin{aligned}\mathbf{R}^u_K &:= \mathrm{c}\mathbf{q}_h + \nabla u_h & R^u_{\partial K} &:= u_h - \widehat{u}_h\\ R^{\mathbf{q}}_K &:= \nabla\cdot\mathbf{q}_h - f & R^{\mathbf{q}}_{\partial K} &:= (\mathbf{q}_h - \widehat{\mathbf{q}}_h)\cdot\mathbf{n} = -\tau\,(u_h - \widehat{u}_h).\end{aligned}$$

The first approach.

An illustration: An HDG method for nonlinear elasticity.

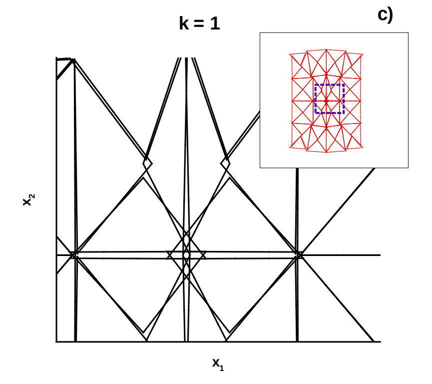


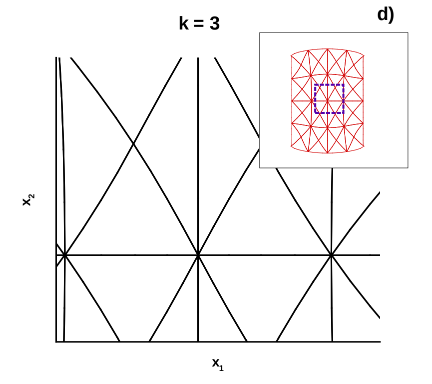
a) deformed shape using $\mathcal{P}^1$, b) deformed shape using $\mathcal{P}^3$.

(S.-C. Soon, U.of M. Ph.D. Thesis, 2008.)

The first approach.

An Illustration: An HDG method for nonlinear elasticity.





c) closeup view of Figure a), d) closeup view of Figure b).

(S.-C. Soon, U.of M. Ph.D. Thesis, 2008.)

The first approach.

An interpretation of the role of τ.

Since

$$\tau = -\frac{R^{\mathbf{q}}_{\partial K}}{R^{u}_{\partial K}} \approx \frac{R^{\mathbf{q}}_{K}}{\mathbf{R}^{u}_{K}}.$$

where

$$\begin{aligned} \mathbf{R}^{u}_{K} &:= \mathrm{c}\mathbf{q}_h + \nabla u_h & \qquad R^{u}_{\partial K} &:= u_h - \widehat{u}_h \\ R^{\mathbf{q}}_{K} &:= \nabla \cdot \mathbf{q}_h - f & \qquad R^{\mathbf{q}}_{\partial K} &:= (\mathbf{q}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n}. \end{aligned}$$

we see that τ forces a ratio between the residuals.

The first approach.

The effect of the local spaces and τ on the accuracy of the method on simplexes.

Method	$\mathbf{V}(K)$	$W(K)$	$M(F)$	k
RT	$\boldsymbol{\mathcal{P}}_k(K) + \mathbf{x}\,\mathcal{P}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$	≥ 0
BDM	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_{k-1}(K)$	$\mathcal{P}_k(F)$	≥ 1
HDG	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$	≥ 0
CG	$\boldsymbol{\mathcal{P}}_{k-1}(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$	≥ 1

The first approach.

The effect of the locals paces and τ on the accuracy of the method on simplexes.

Method	$R^u_{\partial K}$	$R^{\mathbf{q}}_{\partial K}$	$\tau = -R^{\mathbf{q}}_{\partial K}/R^u_{\partial K}$	$\mathbf{q}_h$	u_h	$\overline{u}_h$	k
RT	–	0	0	$k+1$	$k+1$	$k+2$	≥ 0
BDM	–	0	0	$k+1$	k	$k+2$	≥ 2
HDG	–	–	$\mathcal{O}(h)$	$k+1$	k	$k+2$	≥ 1
HDG	–	–	$\mathcal{O}(1)$	$k+1$	$k+1$	$k+2$	≥ 1
HDG	–	–	$\mathcal{O}(1)$	1	1	1	$= 0$
HDG	–	–	$\mathcal{O}(1/h)$	k	$k+1$	$k+1$	≥ 1
CG	0	–	∞	k	$k+1$	$k+1$	≥ 1

On the element $K \in \Omega_h$, given $\widehat{\mathbf{q}} \cdot \mathbf{n}$ on ∂K, $\overline{u}$ and f, we have that $(\mathbf{q}, u)$ satisfies

$$
\begin{aligned}
(\mathrm{c}\,\mathbf{q}, \mathbf{v})_K - (u, \nabla \cdot \mathbf{v})_K + \langle \widehat{u}, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} &= 0, \\
-(\mathbf{q}, \nabla w)_K + \langle \widehat{\mathbf{q}} \cdot \mathbf{n}, w \rangle_{\partial K} &= (f - \overline{f} + \overline{\widehat{\mathbf{q}} \cdot \mathbf{n}}, w)_K, \\
(u, 1)_K &= (\overline{u}, 1)_K
\end{aligned}
$$

for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$, where

$$
\widehat{u} = u \qquad \text{on } \partial K.
$$

On the element $K \in \Omega_h$, we define $(\mathbf{q}_h, u_h)$ in terms of $(\widehat{\mathbf{q}}_h, \overline{u}_h, f)$ as the element of $\mathbf{V}(K) \times W(K)$ such that

$$
\begin{aligned}
(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_K - (u_h, \nabla \cdot \mathbf{v})_K + \langle \widehat{u}_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} &= 0, \\
-(\mathbf{q}_h, \nabla w)_K + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, w \rangle_{\partial K} &= (f - \overline{f} + \overline{\widehat{\mathbf{q}}_h \cdot \mathbf{n}}, w)_K, \\
(u_h, 1)_K &= (\overline{u}_h, 1)_K,
\end{aligned}
$$

for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$, where

$$
\widehat{u}_h = u_h + s(\mathbf{q}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n} \qquad \text{on } \partial K.
$$

The second approach.

The local solvers are well defined.

Theorem

The local solver on K is well defined if

- $s \geq 0$ *on* ∂K,
- $\nabla W(K) \subset \mathbf{V}(K)$.

The second approach.

The global problem: The weak formulation for $\widehat{\mathbf{q}}_h$ and $\overline{u}_h$.

For each face $F \in \mathcal{E}_h$, we take $\widehat{\mathbf{q}}_h|_F$ in the space $\mathbf{N}(F)$. Of course, if $F \in \mathcal{E}_h^o$ we impose the condition that $[\![\widehat{\mathbf{q}}_h]\!] = 0$.

We determine the numerical trace $\widehat{\mathbf{q}}_h$ and the local average $\overline{u}_h$ by requiring that, for each face $F \in \mathcal{E}_h$,

$$\begin{aligned} \langle \boldsymbol{\eta}, [\![\widehat{u}_h]\!] \rangle_F &= 0 && \forall\, \boldsymbol{\eta} \in \mathbf{N}(F) \quad \text{if } F \in \mathcal{E}_h^o, \\ \langle \widehat{u}_h, \boldsymbol{\eta} \cdot \mathbf{n} \rangle_F &= \langle u_D, \boldsymbol{\eta} \cdot \mathbf{n} \rangle_F && \forall\, \boldsymbol{\eta} \in \mathbf{N}(F) \quad \text{if } F \in \mathcal{E}_h^\partial, \end{aligned}$$

and by requiring that, for each element $K \in \Omega_h$,

$$\langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, 1 \rangle_{\partial K} = (f, 1)_K.$$

Suppose that the transmission condition implies that $[\![\widehat{u}_h]\!] = 0$ on a face $F \in \mathcal{E}_h^o$. Then, on that face, we have that

$$[\![u_h]\!] + s^+(\mathbf{q}_h{}^+ - \widehat{\mathbf{q}}_h) + s^-(\mathbf{q}_h{}^- - \widehat{\mathbf{q}}_h) = 0,$$

which holds if

$$\widehat{u}_h = \frac{s^- u_h{}^+ + s^+ u_h{}^-}{s^+ + s^-} + \frac{s^+ s^-}{s^+ + s^-} [\![\mathbf{q}_h]\!],$$

$$\widehat{\mathbf{q}}_h = \frac{s^+ \mathbf{q}_h{}^+ + s^- \mathbf{q}_h{}^-}{s^+ + s^-} + \frac{1}{s^+ + s^-} [\![u_h]\!]$$

provided $s^+ + s^- > 0$.

The second approach.

The numerical trace $\widehat{\mathbf{q}}_h$ and the local average $\overline{u}$ are well defined.

Theorem

The numerical trace $\widehat{\mathbf{q}}_h$ and the local average $\overline{u}$ are well defined if, for each $K \in \partial\Omega_h$,

- $s \geq 0$ *on* ∂K,
- $\nabla W(K) \subset \mathbf{V}(K)$.

The second approach.

A first characterization of the approximate solution.

We have that $(\mathbf{q}_h, u_h) = (\mathbf{Q}_{\widehat{\mathbf{q}}_h}, \mathrm{U}_{\widehat{\mathbf{q}}_h}) + (0, \overline{u}_h) + (\mathbf{Q}_f, \mathrm{U}_f)$ where

$$(\mathbf{Q}_{\widehat{\mathbf{q}}_h}, \mathrm{U}_{\widehat{\mathbf{q}}_h}) := (\mathbf{Q}(\widehat{u}_h, 0), \mathrm{U}(\widehat{\mathbf{q}}_h, 0)), \quad (\mathbf{Q}_f, \mathrm{U}_f) := (\mathbf{Q}(0, f), \mathrm{U}(0, f)).$$

where $(\mathbf{Q}(\widehat{\mathbf{q}}_h, f), \mathrm{U}(\widehat{\mathbf{q}}_h, f))$ is the linear mapping that associates $(\widehat{\mathbf{q}}_h, f)$ to $(\mathbf{q}_h, u_h)$.

Here, we take $(\widehat{\mathbf{q}}_h, \overline{u}_h) \in \mathbf{N}_h \times W_h^0$, where

$$\mathbf{N}_h := \{\boldsymbol{\eta} \in \mathbf{L}^2(\mathcal{E}_h) : \quad \boldsymbol{\eta}|_F \in \mathbf{N}(F) \;\; \forall\, F \in \mathcal{E}_h \;\; [\![\boldsymbol{\eta}]\!] = 0 \text{ on } \mathcal{E}_h^o\},$$
$$W_h^0 := \{\bar{w} \in L^2(\Omega) : \bar{w}|_K \text{ is a constant } \forall K \in \Omega_h\}.$$

The second approach.

A first characterization of the approximate solution.

The function $(\widehat{\mathbf{q}}_h, \overline{u}_h)$ satisfies the equations

$$\begin{aligned} a_h(\widehat{\mathbf{q}}_h, \boldsymbol{\eta}) + b_h(\overline{u}_h, \boldsymbol{\eta}) &= \ell_{1,h}(\boldsymbol{\eta}) && \forall\, \boldsymbol{\eta} \in \mathbf{N}_h, \\ b_h(\bar{w}, \widehat{\mathbf{q}}_h) &= \ell_{2,h}(\bar{w}) \\ \langle \boldsymbol{\eta}\cdot\mathbf{n}, \widehat{u}_h\rangle_{\partial\Omega} &= \langle \boldsymbol{\eta}\cdot\mathbf{n}, u_D\rangle_{\partial\Omega} && \forall\, \boldsymbol{\eta} \in \mathbf{N}_h, \end{aligned}$$

where

$$\begin{aligned} a_h(\boldsymbol{\eta}, \boldsymbol{\zeta}) &:= -\,\langle \boldsymbol{\eta}\cdot\mathbf{n}, \widehat{\mathbf{U}}_{\boldsymbol{\zeta}}\rangle_{\partial\Omega_h}, \\ b_h(\bar{w}, \boldsymbol{\eta}) &:= -\,\langle \bar{w}, \boldsymbol{\eta}\cdot\mathbf{n}\rangle_{\partial\Omega}, \\ \ell_{1,h}(\boldsymbol{\eta}) &:= \langle \boldsymbol{\eta}\cdot\mathbf{n}, \widehat{\mathbf{U}}_f\rangle_{\partial\Omega_h}, \\ \ell_{2,h}(\bar{w}) &:= (f, \bar{w})_{\Omega_h}. \end{aligned}$$

The second approach.

The matrix associated with the form a_h.

Theorem

We have that

$$a_h(\boldsymbol{\eta}, \boldsymbol{\zeta}) = (\mathrm{c}\mathbf{Q}_{\boldsymbol{\eta}}, \mathbf{Q}_{\boldsymbol{\zeta}})_{\partial\Omega_h} + \langle s(\mathbf{Q}_{\boldsymbol{\eta}} - \boldsymbol{\eta}) \cdot \mathbf{n}, (\mathbf{Q}_{\boldsymbol{\zeta}} - \boldsymbol{\zeta}) \cdot \mathbf{n}) \rangle_{\partial\Omega_h}.$$

The second approach.

A second form of the method.

The approximate solution $(\mathbf{q}_h, u_h, \widehat{\mathbf{q}}_h)$ is the element of the space $\mathbf{V}_h \times W_h \times \mathbf{N}_h$ satisfying the equations

$$
\begin{aligned}
(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_{\Omega_h} - (u_h, \nabla\cdot\mathbf{v})_{\Omega_h} + \langle \widehat{u}_h, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
-(\mathbf{q}_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h},\\
\langle \boldsymbol{\eta}\cdot\mathbf{n}, \widehat{u}_h\rangle_{\partial\Omega_h} &= \langle \boldsymbol{\eta}\cdot\mathbf{n}, u_D\rangle_{\partial\Omega},
\end{aligned}
$$

for all $(\mathbf{v}, w, \boldsymbol{\eta}) \in \mathbf{V}_h \times W_h \times \mathbf{N}_h$, where

$$\widehat{u}_h = u_h + s(\mathbf{q}_h - \widehat{\mathbf{q}}_h)\cdot\mathbf{n} \qquad \text{on } \partial\Omega_h.$$

Note that this method is the same as the method obtained with the first approach with $\tau = 1/s$.

Examples. (B.C., J.Gopalakrishnan and R.Lazarov, SINUM, 2009.)

Local spaces for simplexes K.

Method	$\mathbf{V}(K)$	$W(K)$	$M(F)$
RT-H	$\boldsymbol{\mathcal{P}}_k(K) + \mathbf{x}\,\mathcal{P}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$
BDM-H	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_{k-1}(K)$	$\mathcal{P}_k(F)$
LDG-H	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_{k-1}(K)$	$\mathcal{P}_k(F)$
LDG-H	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$
LDG-H	$\boldsymbol{\mathcal{P}}_{k-1}(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$
IP-H	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$

Examples.

Numerical traces for simplexes K.

Method	$\widehat{\mathbf{q}}_h$
RT-H	$\mathbf{q}_h$
BDM-H	$\mathbf{q}_h$
LDG-H	$\mathbf{q}_h + \tau(u_h - \widehat{u}_h) \cdot \mathbf{n}$
IP-H	$-a\nabla u_h + \tau(u_h - \widehat{u}_h) \cdot \mathbf{n}$

Examples.

The bilinear form a_h.

Method	$a_h(\eta,\mu)$
RT-H	$(\mathrm{c}\,\mathbf{Q}\eta,\mathbf{Q}\mu)_{\Omega_h}$
BDM-H	$(\mathrm{c}\,\mathbf{Q}\eta,\mathbf{Q}\mu)_{\Omega_h}$
LDG-H	$(\mathrm{c}\,\mathbf{Q}\eta,\mathbf{Q}\mu)_{\Omega_h}+\langle\tau(\mathsf{U}\mu-\mu),\mathsf{U}\eta-\eta\rangle_{\partial\Omega_h}$
IP-H†	$(\mathrm{c}\nabla\mathsf{U}\mu,\nabla\mathsf{U}\eta)_{\Omega_h}+\langle\tau(\mathsf{U}\mu-\mu),\mathsf{U}\eta-\eta\rangle_{\partial\Omega_h}$
	$\langle(\eta-\mathsf{U}\eta),\mathrm{c}\nabla\mathsf{U}\mu\rangle_{\partial\Omega_h}+\langle\mu-\mathsf{U}\mu,\mathrm{c}\nabla\mathsf{U}\eta\rangle_{\partial\Omega_h}.$

†We assume that c is a constant on each element.

Examples

Some remarks.

- The RT-H method is the hybridized version of the original RT method.
- The BDM-H method is the hybridized version of the original BDM method.
- The LDG-H method is not the hybridized version of the LDG method.
- The IP-H method is not the hybridized version of the IP method.
- The bilinear forms a_h of the RT-H, BDM-H and SF-H methods are the same on simplexes. (For these three methods, $\tau^* = 0$.)
- The LDG-H method is defined for any $\tau > 0$.
- The IP-H method is defined only for $\tau \approx h^{-1}$.
- The LDG-H and IP-H can be applied on any polyhedral element K.

General polyhedral elements.

Convergence properties

If we use the HDG method on general polyhedral elements with $\mathbf{V}(K) := \boldsymbol{\mathcal{P}}_k(K)$, $W(K) := \mathcal{P}_k(K)$ and $M(F) := \mathcal{P}_k(F)$, we have that

- For τ of order one, $\mathbf{q}_h$ converges with order $k + 1/2$ and u_h with order $k + 1$, for any $k \geq 0$.
- For τ of order $1/h$, $\mathbf{q}_h$ converges with order k and u_h with order $k + 1$, for any $k \geq 0$.

Proven in (P.Castillo, B.C., I.Perugia and D.Shotzau, SINUM, 2000.)

We seek HDG methods for which the local averages of the error $u - u_h$, converge faster than the errors $u - u_h$ and $\mathbf{q} - \mathbf{q}_h$.

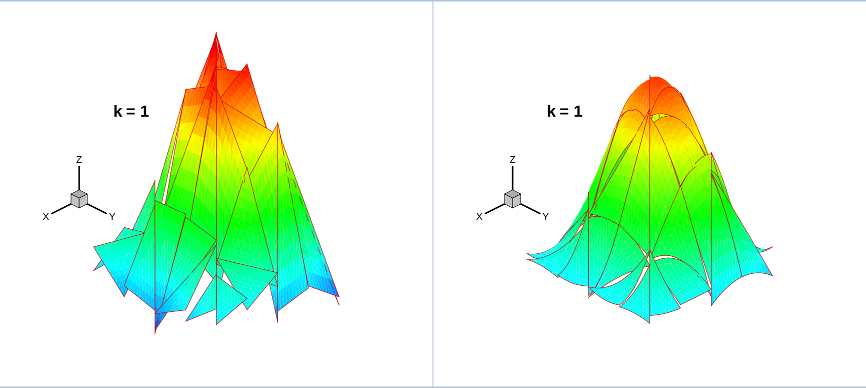
If this property holds, we introduce a new approximation $u_h^\star$. On each element K it lies in the space $W^*(K)$ and defined by

$$\begin{aligned}(\nabla {u_h}^\star, \nabla w)_K &= -\,(\mathrm{c}\mathbf{q}_h, \nabla w)_K \qquad \text{for all } w \in W^*(K),\\ ({u_h}^\star, 1)_K &= (u_h, 1)_K\,,\end{aligned}$$

Then $u - u_h^\star$ will converge faster than $u - u_h$. This does happen for mixed methods!

Illustration of the postprocessing.

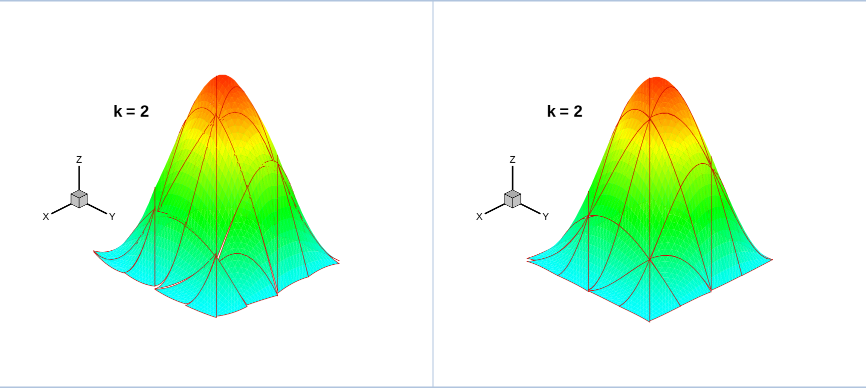
An HDG method for linear elasticity.(S.-C. Soon, B.C. and H. Stolarski, 2008.)



Comparison between the approximate solution (left) and the post-processed solution (right) for linear polynomial approximations.

Illustration of the postprocessing.

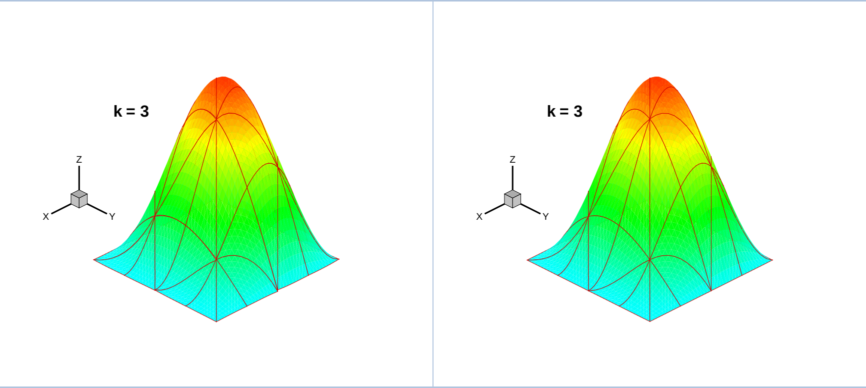
An HDG method for linear elasticity.(S.-C. Soon, B.C. and H. Stolarski, 2008.)



Comparison between the approximate solution (left) and the post-processed solution (right) for quadratic polynomial approximations.

Illustration of the postprocessing.

An HDG method for linear elasticity.(S.-C. Soon, B.C. and H. Stolarski, 2008.)



Comparison between the approximate solution (left) and the post-processed solution (right) for cubic polynomial approximations.

Superconvergent DG methods.

Are there superconvergent DG methods?

The numerical traces of the LDG method are:

$$\widehat{u}_h = \{\!\{u_h\}\!\} + \mathbf{C}_{21} \cdot [\![u_h]\!] + C_{22}\,[\![\mathbf{q}_h]\!],$$
$$\widehat{\mathbf{q}}_h = \{\!\{\mathbf{q}_h\}\!\} + \mathbf{C}_{12}\,[\![\mathbf{q}_h]\!] + C_{11}\,[\![u_h]\!],$$

where $\mathbf{C}_{21} + \mathbf{C}_{12} = 0$ and $C_{22} = 0$.

The numerical traces of the LDG-H method are:

$$\widehat{u}_h = \frac{\tau^+ {u_h}^+ + \tau^- {u_h}^-}{\tau^+ + \tau^-} + \frac{1}{\tau^+ + \tau^-}\,[\![\mathbf{q}_h]\!],$$
$$\widehat{\mathbf{q}}_h = \frac{\tau^- {\mathbf{q}_h}^+ + \tau^+ {\mathbf{q}_h}^-}{\tau^+ + \tau^-} + \frac{\tau^+\tau^-}{\tau^+ + \tau^-}\,[\![u_h]\!]$$

Superconvergent DG methods (B.C., J.Guzmán and H.Wang, Math. Comp., 2009.)

Are there superconvergent DG methods?

Consider DG methods on conforming meshes $\partial\Omega_h$ of simplexes K. Assume they use the local spaces $\mathbf{V}(K) := \mathcal{P}_k(K)$ and $W(K) := \mathcal{P}_k(K)$.

Theorem

For very smooth solutions, we have, for $k \geq 1$,

$$\|\mathbf{q} - \mathbf{q}_h\|_\Omega \leq C(h^{k+1} + \|\widehat{\mathbf{q}}_h - \mathbf{q}_h\|_{\partial\Omega_h,h}),$$
$$\|u - u_h^*\|_\Omega \leq C\,h\,(h^{k+1} + \|\widehat{\mathbf{q}}_h - \mathbf{q}_h\|_{\partial\Omega_h,h}),$$

where $\|\widehat{\mathbf{q}}_h - \mathbf{q}_h\|^2_{\partial\Omega_h,h} := \sum_{K\in\Omega_h} h_K \,\|(\widehat{\mathbf{q}}_h - \mathbf{q}_h)\cdot\mathbf{n}\|^2_{\partial K}$*. Moreover,*

$$\|\widehat{\mathbf{q}}_h - \mathbf{q}_h\|_{\partial\Omega_h,h} \leq C \max_{K\in\Omega_h}\{C_{22}, 1/C_{22}, C_{11}, 1/C_{11}\}\, h^{k+1}.$$

Hence, for C_{11} and C_{22} of order one, the DG method superconverges.

Superconvergent DG methods

The effect of τ on the accuracy.

- If $\tau^{\pm}$, C_{11} are of order h^{-1} and $C_{22} = 0$, the LDG and HDG methods have the same convergence properties. The scalar variable converges with order $k+1$ but the vector variable only with order k. They do not converge for $k = 0$.
- If $\tau^{\pm}$, C_{11} and C_{22} are of order one, the DG and HDG methods have the same convergence properties. Both variables converge with order $k+1$ for $k \geq 0$. For $k \geq 1$, the local average of the scalar variable superconverges with order $k+2$.

Superconvergent DG methods

The effect of the size of the jumps on the accuracy.

The energy identity is

$$(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_\Omega + \Theta_\tau(u_h - \widehat{u}_h) = (f, u_h)_\Omega - \langle u_D, \widehat{\mathbf{q}}_h \cdot \mathbf{n}\rangle_{\partial\Omega}.$$

where, for the HDG,

$$\begin{aligned}\Theta_\tau(u_h - \widehat{u}_h) &= \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h\rangle_{\partial\Omega_h} \\ &= \langle \tau(u_h - P_M u_D), u_h - P_M u_D\rangle_{\partial\Omega} + \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h\rangle_{\partial\Omega_h\setminus\partial\Omega} \\ &= \langle \tau(u_h - P_M u_D), u_h - P_M u_D\rangle_{\partial\Omega} \\ &\quad + \langle \frac{\tau^+\tau^-}{\tau^+ + \tau^-} [\![u_h]\!], [\![u_h]\!]\rangle_{\mathcal{E}_h^\circ} + \langle \frac{1}{\tau^+ + \tau^-} [\![\mathbf{q}_h]\!], [\![\mathbf{q}_h]\!]\rangle_{\mathcal{E}_h^\circ}.\end{aligned}$$

For the LDG,

$$\begin{aligned}\Theta_\tau(u_h - \widehat{u}_h) &= \langle \tau(u_h - P_M u_D), u_h - P_M u_D\rangle_{\partial\Omega} \\ &\quad + \langle C_{11}\, [\![u_h]\!], [\![u_h]\!]\rangle_{\mathcal{E}_h^\circ} + \langle C_{22}\, [\![\mathbf{q}_h]\!], [\![\mathbf{q}_h]\!]\rangle_{\mathcal{E}_h^\circ}.\end{aligned}$$

Devising superconvergent HDG methods. (B.C., W.Qiu and K.Shi, Math. Comp.,2012 + SINUM, 2012. B.C.)

The conditions on the local spaces

We decompose the local spaces $\mathbf{V}(K)$ and $W(K)$ as follows:

$$\mathbf{V}(K) = \widetilde{\mathbf{V}}(K) \oplus \widetilde{\mathbf{V}}^{\perp}(K),$$
$$W(K) = \widetilde{W}(K) \oplus \widetilde{W}^{\perp}(K),$$

and assume that the following inclusions hold:

$$\boldsymbol{\mathcal{P}}_0(K) \subset \nabla W(K) \subset \widetilde{\mathbf{V}}(K),$$
$$\mathcal{P}_0(K) \subset \nabla \cdot \mathbf{V}(K) \subset \widetilde{W}(K),$$
$$\mathbf{V}(K) \cdot \mathbf{n} + W(K) \subset M(\partial K).$$

Moreover, we assume that we have that:

$$\widetilde{\mathbf{V}^{\perp}} \cdot \mathbf{n} \oplus \widetilde{W}^{\perp} = M(\partial K),$$

where $M(\partial K) = \{\mu \in L^2(\partial K) : \mu|_F \in M(F) \ \ \forall F \in \mathcal{F}(K)\}$.

Then, the function $\Pi_h(\mathbf{q}, u) := (\boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q}, \Pi_W u)$ is the element of $\mathbf{V}(K) \times W(K)$ satisfying the equations

$$\begin{aligned}
(\boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q}, \widetilde{\mathbf{v}})_K &= (\mathbf{q}, \widetilde{\mathbf{v}})_K && \forall\, \widetilde{\mathbf{v}} \in \widetilde{\mathbf{V}}(K),\\
(\Pi_W u, \widetilde{w})_K &= (u, \widetilde{w})_K && \forall\, \widetilde{w} \in \widetilde{W}(K),\\
\langle \boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q} \cdot \mathbf{n} + \tau(\Pi_W u), \mu\rangle_F &= \langle \mathbf{q} \cdot \mathbf{n} + \tau(P_M u), \mu\rangle_F && \forall\, \mu \in M(F),
\end{aligned}$$

for all faces F of the element K, is well defined provided $\tau > 0$ on ∂K. (This condition on τ can be relaxed!)

Devising superconvergent HDG methods

Estimate of the projection of the errors.

Theorem

We have

$$\|\boldsymbol{\Pi_V q} - \mathbf{q}_h\|_{c,\Omega} \le \|\mathbf{q} - \boldsymbol{\Pi_V q}\|_{c,\Omega},$$
$$\|\Pi_W u - u_h\|_\Omega \le C\, h \|\mathbf{q} - \boldsymbol{\Pi_V q}\|_\Omega,$$
$$\|u - u_h^*\|_\Omega \le \|\Pi_W(u - u_h)\|_\Omega + C\, h\,(\|\mathbf{q}_h - \mathbf{q}\|_\Omega + \inf_{w \in W_h^*} \|\nabla(u - w)\|_\Omega).$$

Construction of a superconvergent HDG method

Methods for which $M(F) = P^k(F), k \geq 1$, and K is a simplex.

method	$\mathbf{V}(K)$	$W(K)$	$\widetilde{\mathbf{V}}(K)$	$\widetilde{W}(K)$
$\mathbf{BDFM}_{k+1}$	$\{\mathbf{q} \in \mathbf{P}^{k+1}(K) :$ $\mathbf{q} \cdot \mathbf{n}\|_{\partial K} \in \mathcal{R}^k(\partial K)\}$	$P^k(K)$	$\nabla P^k(K) \oplus \mathbf{\Phi}_{k+1}(K)$	$P^k(K)$
$\mathbf{RT}_k$	$\mathbf{P}^k(K) \oplus \mathbf{x}\widetilde{P}^k(K)$	$P^k(K)$	$\mathbf{P}^{k-1}(K)$	$P^k(K)$
$\mathbf{HDG}_k$	$\mathbf{P}^k(K)$	$P^k(K)$	$\mathbf{P}^{k-1}(K)$	$P^{k-1}(K)$
$\mathbf{BDM}_k$ $k \geq 2$	$\mathbf{P}^k(K)$	$P^{k-1}(K)$	$\nabla P^{k-1}(K) \oplus \mathbf{\Phi}_k(K)$	$P^{k-1}(K)$

Examples of superconvergent methods. (B.C., W.Qiu and K.Shi, Math. Comp., 2012 + SINUM, 2012.)

Methods for which $M(F) = P^k(F), k \geq 1$, and K is a simplex.

method	τ	$\|\mathbf{q} - \mathbf{q}_h\|_\Omega$	$\|\Pi_W u - u_h\|_\Omega$	$\|u - u_h^\star\|_\Omega$
$\mathbf{BDFM}_{k+1}$	0	$k+1$	$k+2$	$k+2$
$\mathbf{RT}_k$	0	$k+1$	$k+2$	$k+2$
$\mathbf{HDG}_k$	$\mathcal{O}(1), > 0$	$k+1$	$k+2$	$k+2$
$\mathbf{BDM}_k$ $k\geq 2$	0	$k+1$	$k+2$	$k+2$

Examples of superconvergent methods

Methods for which $M(F) = P^k(F), k \geq 1$, and K is a square.

method	$\mathbf{V}(K)$	$W(K)$
$\mathbf{BDFM}_{[k+1]}$	$P^{k+1}(K)\backslash\{y^{k+1}\}$ $\times(P^{k+1}(K)\backslash\{x^{k+1}\})$	$P^k(K)$
$\mathbf{HDG}^P_{[k]}$	$\mathbf{P}^k(K)$ $\oplus\nabla\times(xy\widetilde{P}^k(K))$	$P^k(K)$
$\mathbf{BDM}_{[k]}$ $k \geq 2$	$\mathbf{P}^k(K)$ $\oplus\nabla\times(xy\,x^k)$ $\oplus\nabla\times(xy\,y^k)$	$P^{k-1}(K)$

Examples of superconvergent methods

Methods for which $M(F) = P^k(F), k \geq 1$, and K is a cube.

method	$\mathbf{V}(K)$	$W(K)$
$\mathbf{BDFM}_{[k+1]}$	$P^{k+1}(K)\backslash\widetilde{P}^{k+1}(y,z)$ $\times P^{k+1}(K)\backslash\widetilde{P}^{k+1}(x,z)$ $\times P^{k+1}(K)\backslash\widetilde{P}^{k+1}(x,y)$	$P^k(K)$
$\mathbf{HDG}^P_{[k]}$	$\mathbf{P}^k(K)$ $\oplus\nabla\times(yz\widetilde{P}^k(K),0,0)$ $\oplus\nabla\times(0,zx\widetilde{P}^k(K),0)$	$P^k(K)$
$\mathbf{BDM}_{[k]}$ $k \geq 2$	$\mathbf{P}^k(K)$ $\oplus\nabla\times(0,0,xy\widetilde{P}^k(y,z))$ $\oplus\nabla\times(0,xz\widetilde{P}^k(x,y),0)$ $\oplus\nabla\times(yz\widetilde{P}^k(x,z),0,0)$	$P^{k-1}(K)$

Examples of superconvergent methods

Methods for which $M(F) = P^k(F), k \geq 1$, and K is a square or a cube.

method	τ	$\|\mathbf{q} - \mathbf{q}_h\|_\Omega$	$\|\Pi_W u - u_h\|_\Omega$	$\|u - u_h^\star\|_\Omega$
$\mathbf{BDFM}_{[k+1]}$	0	$k+1$	$k+2$	$k+2$
$\mathbf{HDG}_{[k]}^{P}$	$\mathcal{O}(1), > 0$	$k+1$	$k+2$	$k+2$
$\mathbf{BDM}_{[k]}$ $k \geq 2$	0	$k+1$	$k+2$	$k+2$

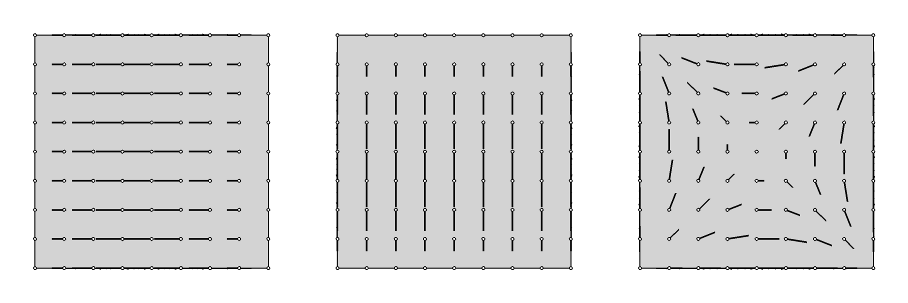
Examples of superconvergent methods

Methods for which $M(F) = Q^k(F), k \geq 1$, and K is a square.

method	$\mathbf{V}(K)$	$W(K)$
$\mathbf{RT}_{[k]}$	$P^{k+1,k}(K) \times P^{k,k+1}(K)$	$Q^k(K)$
$\mathbf{TNT}_{[k]}$	$\mathbf{Q}^k(K) \oplus \mathbf{H}_3^k(K)$	$Q^k(K)$
$\mathbf{HDG}_{[k]}^{Q}$	$\mathbf{Q}^k(K) \oplus \mathbf{H}_2^k(K)$	$Q^k(K)$

Examples of superconvergent methods

The space $\mathbf{H}_3^k(K)$.



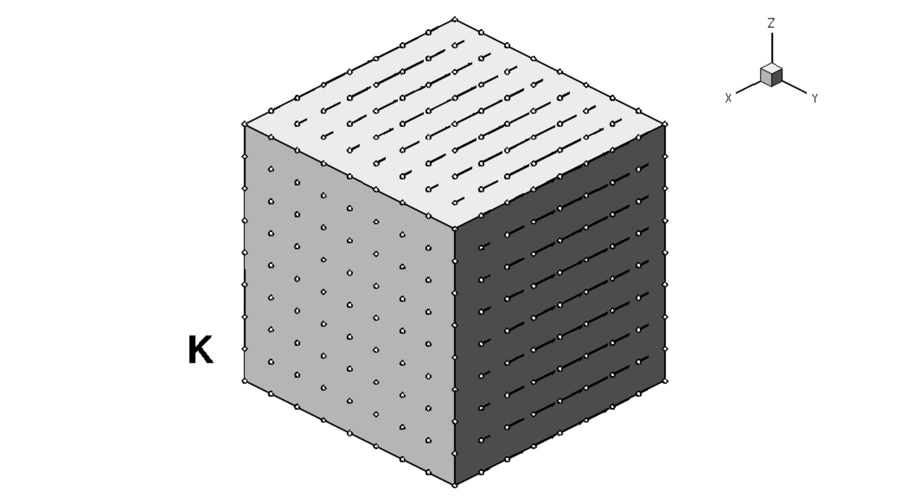
Examples of superconvergent methods

Methods for which $M(F) = Q^k(F), k \geq 1$, and K is a cube.

method	$\mathbf{V}(K)$	$W(K)$
$\mathbf{RT}_{[k]}$	$P^{k+1,k,k}(K) \times P^{k,k+1,k}(K) \times P^{k,k,k+1}(K)$	$Q^k(K)$
$\mathbf{TNT}_{[k]}$	$\mathbf{Q}^k(K) \oplus \mathbf{H}^k_7(K)$	$Q^k(K)$
$\mathbf{HDG}^Q_{[k]}$	$\mathbf{Q}^k(K) \oplus \mathbf{H}^k_6(K)$	$Q^k(K)$

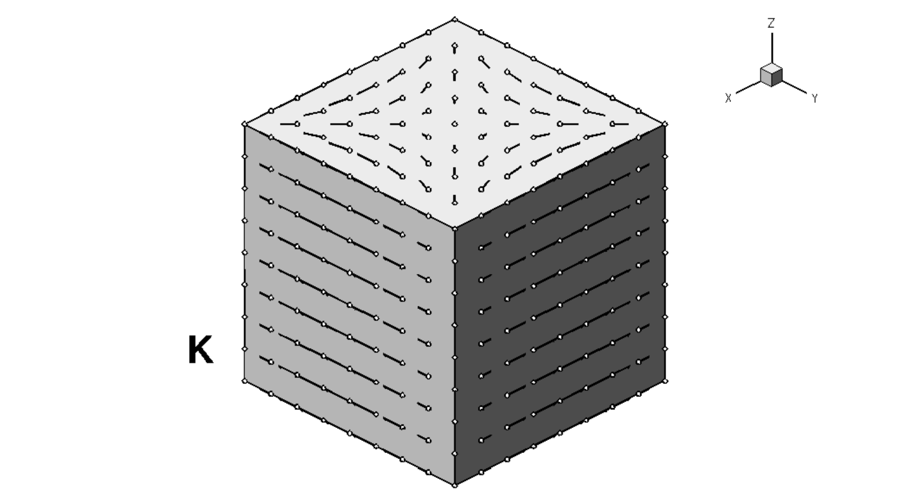
Examples of superconvergent methods

The space $\mathbf{H}^k_7(K)$.



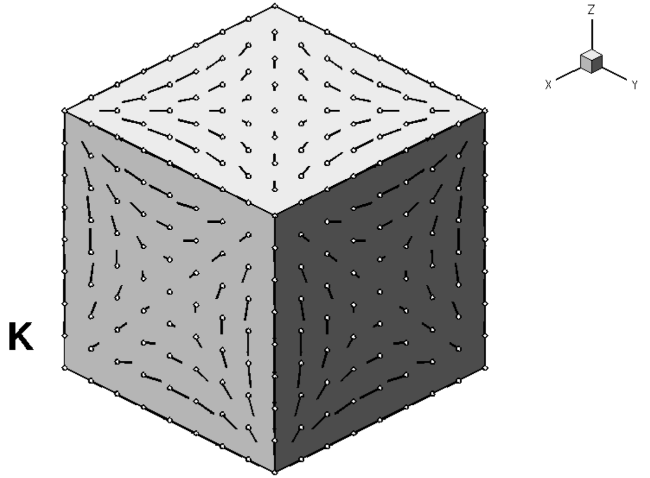
Examples of superconvergent methods

The space $\mathbf{H}_7^k(K)$.



Examples of superconvergent methods

The space $\mathbf{H}_7^k(K)$.



Examples of superconvergent methods

Methods for which $M(F) = Q^k(F), k \geq 1$, and K is a square or a cube.

method	τ	$\|\mathbf{q} - \mathbf{q}_h\|_\Omega$	$\|\Pi_W u - u_h\|_\Omega$	$\|u - u_h^\star\|_\Omega$
$\mathbf{RT}_{[k+1]}$	0	$k+1$	$k+2$	$k+2$
$\mathbf{TNT}_{[k]}$	0	$k+1$	$k+2$	$k+2$
$\mathbf{HDG}_{[k]}^{Q}$	$\mathcal{O}(1) > 0$	$k+1$	$k+2$	$k+2$

Variable-degree HDG methods on nonconforming meshes.

(Y.Chen and B.C., IMA,2012 + Math. Comp.,2014.)

Definition.

$$\begin{aligned}
\mathbf{V}_h &= \{\mathbf{r} \in \mathbf{L}^2(\mathcal{T}_h) : \quad \mathbf{r}|_K \in \mathbf{P}_{k(K)}(K) \quad \forall\, K \in \mathcal{T}_h\},\\
W_h &= \{w \in L^2(\mathcal{T}_h) : \quad w|_K \in P_{k(K)}(K) \quad \forall\, K \in \mathcal{T}_h\},\\
M_h &= \{\mu \in L^2(\mathcal{E}_h) : \quad \mu|_F \in P_{k(F)}(F) \quad \forall\, F \in \mathcal{E}_h\}.
\end{aligned}$$

and

$$\begin{aligned}
k(F) &= k(K) && \text{if } F = \partial K \cap \partial\Omega,\\
k(F) &= \max\{k(K^+), k(K^-)\} && \text{if } F = \partial K^+ \cap \partial K^-.
\end{aligned}$$

Variable-degree HDG methods on nonconforming meshes

Overview of convergence properties

method	conformity of the meshes $\mathcal{T}_h$	order (flux)	order (scalar)
DG pure diffusion	conforming	k	$k+1$
LDG pure diffusion	conforming Cartesian meshes	$k+1/2$	$k+1$
LDG pure diffusion	nonconforming	k	$k+1$
HDG pure diffusion	conforming	$k+1$	$k+1+\min\{k,1\}$ projection of the scalar variable
HDG	nonconforming	$k+1/2$	$k+1$
HDG	nonconforming semimatching	$k+1$	$k+1+\min\{k,1\}$ projection of the scalar variable

Theorem

For any mesh of shape-regular simplexes, we have

$$\|\varepsilon_{\mathbf{q}}\|_c \leq \|\mathbf{\Pi_V q} - \mathbf{q}\|_c + C\,\|(P_M - P_{\mathcal{M}})(\mathbf{q}\cdot\mathbf{n} + \tau\, u)\|_{\partial\Omega_h},$$

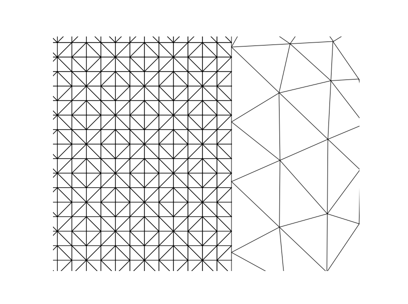
Moreover,

$$\|\varepsilon_u\| \leq C\,h^{1/2}\left(\|\mathbf{\Pi_V q} - \mathbf{q}\| + \|(P_M - P_{\mathcal{M}})(\mathbf{q}\cdot\mathbf{n} + \tau\, u)\|_{\partial\Omega_h}\right).$$

$$\|(P_M - P_{\mathcal{M}})(\mathbf{q}\cdot\mathbf{n} + \tau\, u)\|_{\partial\Omega_h} \leq C\,|S_{P,h}|^{1/2}\,h_P^{k+1}\,D(\mathbf{q}, u),$$

$$D(\mathbf{q}, u) := |\mathbf{q}\cdot\mathbf{n} + \tau\, u|_{W^{k+1,\infty}(S_{P,h})},$$

$$S_{P,h} := \{F:\ P_M \neq P_{\mathcal{M}} \text{ on } F\}.$$



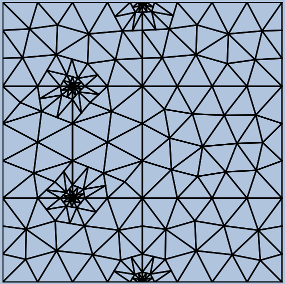


Figure: Examples of sets $S_{P,h}$ of size of order one.

Variable-degree HDG methods on nonconforming meshes

General meshes

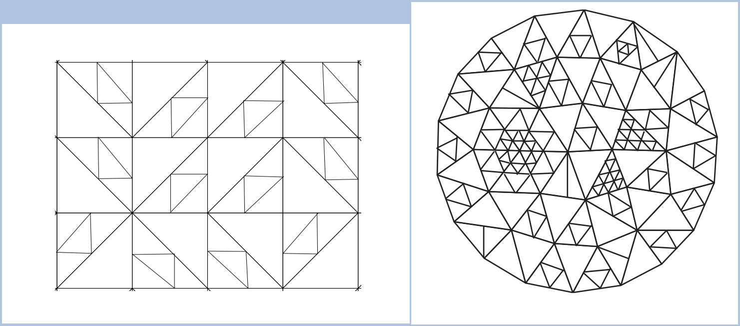


Figure: Examples of sets $S_{P,h}$ of size of order h^{-1}.

Variable-degree HDG methods on nonconforming meshes

The semimatching nonconforming meshes.

For every level index $\ell \geq 1$,

- Shape regularity:

$$\mathsf{T}_h^\ell \text{ is made of simplexes } \mathsf{K} \text{ such that } \frac{h_\mathsf{K}}{\rho_\mathsf{K}} \leq \sigma.$$

- Mandatory refinement:

$$\mathsf{T}_h^{\ell+1} \text{ is a refinement of } \mathsf{T}_h^\ell : \text{ no element of } \mathsf{T}_h^\ell \text{ is unrefined.}$$

- Local Uniformity:

$$\forall\, \mathsf{K} \in \mathsf{T}_h^\ell : \max_{\mathsf{K}' \in \mathsf{T}_h^{\ell+1} : \mathsf{K}' \subset \mathsf{K}} h_{\mathsf{K}'} \leq \kappa \min_{\mathsf{K}' \in \mathsf{T}_h^{\ell+1} : \mathsf{K}' \subset \mathsf{K}} h_{\mathsf{K}'}.$$

- Uniform refinement:

$$\forall\, \mathsf{K} \in \mathsf{T}_h^\ell : \max_{\mathsf{K}' \in \mathsf{T}_h^{\ell+n} : \mathsf{K}' \subset \mathsf{K}} h_{\mathsf{K}'} \leq \mathsf{c}\, \eta^n\, h_\mathsf{K}.$$

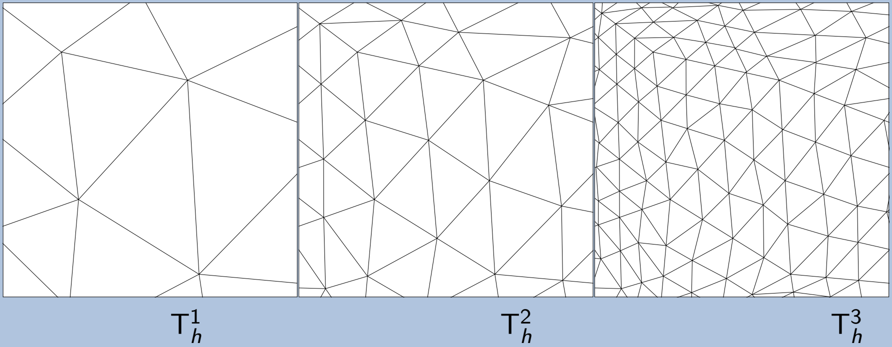


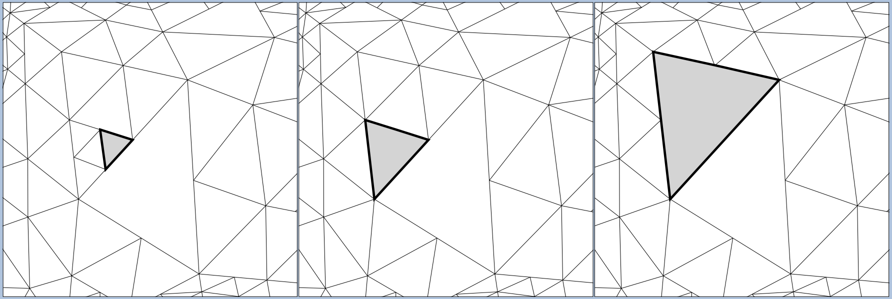
Figure: An example of a family of triangulations $\{\mathrm{T}_h^\ell\}_{\ell \geq 1}$ for which $\eta = 1/2$.

$\mathcal{T}_h = \{K\}$ is a semimatching nonconforming mesh if, for each element $K \in \mathcal{T}_h$ there is a set $\{\mathsf{K}_K^\ell\}_{\ell=1}^{\ell_K}$ such that:

- $\mathsf{K}_K^\ell \in \mathsf{T}_h^\ell$, for $\ell = 1, \cdots, \ell_K$.
- $\mathsf{K}_K^\ell \supset K$, for $\ell = 1, \cdots, \ell_K$.
- $\mathsf{K}_K^{\ell_K} = K$.

Variable-degree HDG methods on nonconforming meshes

The semimatching meshes.



$K = \mathsf{K}^3_K \in \mathsf{T}^3_h, \ell_K = 3$ $\qquad$ $\mathsf{K}^2_K \in \mathsf{T}^2_h$ $\qquad$ $\mathsf{K}^1_K \in \mathcal{T}^1_h$

Figure: A nonconforming mesh $\mathcal{T}_h$ (left) and of the set $\{\mathsf{K}^\ell_K\}_{\ell=1}^{\ell_K}$ (in gray).

We *further* require that

$$k(K^+) \geq k(K^-) \text{ whenever } \ell_{K^+} \geq \ell_{K^-}.$$

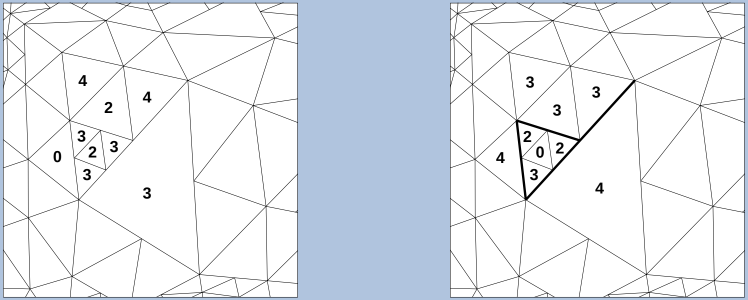


Figure: Illustration of the last condition: Yes (left), no (right).

Theorem

For any semimatching mesh, we have

$$\|\varepsilon_{\mathbf{q}}\| \leq C\,(\|\mathbf{q}-\mathbf{\Pi_V q}\| + \|(P_M - P_{\mathcal{M}})(\mathbf{q}\cdot\mathbf{n}+\tau\,u)\|_{\partial\Omega_h,h}),$$

Moreover, if the standard elliptic regularity holds,

$$\|\epsilon_u\| \leq C\,h^{\min_{K\in\mathcal{T}_h}\{1,k(K)\}}\,(\|\mathbf{q}-\mathbf{\Pi_V q}\| + \|(P_M - P_{\mathcal{M}})(\mathbf{q}\cdot\mathbf{n}+\tau\,u)\|_{\partial\Omega_h,h}).$$

$$\|\mathbf{q}\cdot\mathbf{n}+\tau\,u\|_{\partial K,h} \leq C h_K^{k(K)+1}\,\mathcal{D}_K(\mathbf{q},u)$$
$$\mathcal{D}_K(\mathbf{q},u) := |\mathbf{q}|_{H^{k+1}(K)} + \|\tau\|_{L^\infty(\partial K)}\,|u|_{H^{k+1}(K)}.$$

Variable-degree HDG methods on nonconforming meshes

Conclusions.

For uniform-degree methods on simplexes,

- HDG as well as DG methods always converge with order $k+1$ in the scalar variable.
- HDG methods can converge in the flux with order $k+1$ on some general nonconforming meshes. In this case, they superconverge with order $k+3/2$ for $k \geq 1$ in the scalar variable.
- For general meshes, they might lose $1/2$ an order of convergence in the flux and might not exhibit superconvergence of the scalar variable.
- HDG methods superconverge with order $k+2$ on semimatching meshes for $k \geq 1$.

A posteriori error estimation (B.C., R.Nochetto and W. Zhang, Math. Comp., in revision.)

Setting.

We take meshes $\mathcal{T}_k$ made of simplexes, and set

$$\mathbf{V}(K) := \boldsymbol{\mathcal{P}}_n(K), \quad W(K) := \mathcal{P}_n(K), \quad M(F) := \mathcal{P}_n(K).$$

We assume that, for $K \in \mathcal{T}_k$,

- **A1** The parameter τ_K is a positive constant on ∂K.
- **A2** If $K \supset T \in \mathcal{T}_{k+1}$, then $\tau_T = \tau_K$.
- **A3** $\tau_K h_K \leq C_\tau$ for some constant C_τ.

A posteriori error estimation (B.C., R.Nochetto and W. Zhang, Math. Comp., in revision.)

Contraction of the quasi-error.

We consider the quasi-error

$$E_{\beta,\gamma}(\mathbf{q}_k, f, \mathcal{T}_k)^2 = \| \mathbf{q} - \mathbf{q}_k \|_\Omega^2 + \beta \eta_{div}^2(f, \mathbf{q}_k, \mathcal{T}_k) + \gamma \eta_{curl}^2(\mathbf{q}_k, \mathcal{T}_k),$$

where

$$\eta_{curl}^2(\mathbf{q}_k, K) := h_K^2 \| \nabla \times \mathbf{q}_k \|_K^2 + h_K \| [\![\mathbf{q}_k]\!]_t \|_{\partial K}^2,$$
$$\eta_{div}^2(f, \mathbf{q}_k, K) := \tau_K^2 h_K^2 \| \mathsf{P}_{\mathbf{V}_k}^{\perp} \mathbf{q}_k \|_K^2 + h_K^2 \| \mathsf{P}_{W_k}^{\perp} f \|_K^2.$$

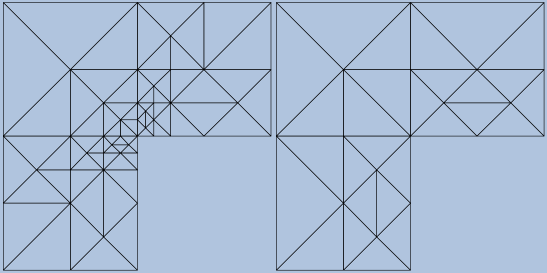
Theorem

If C_τ is small enough, there exist positive constants β, γ, and $\alpha < 1$ such that

$$E_{\beta,\gamma}(\mathbf{q}_{k+1}, f, \mathcal{T}_{k+1}) \le \alpha E_{\beta,\gamma}(\mathbf{q}_k, f, \mathcal{T}_k).$$

A posteriori error estimation (B.C., R.Nochetto and W. Zhang, Math. Comp., in revision.)

Illustration.

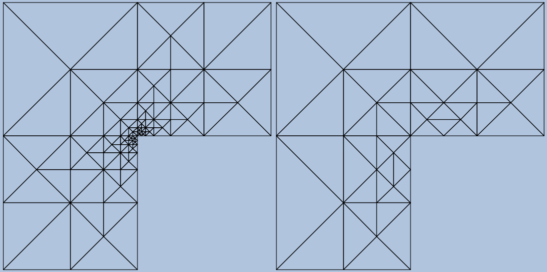


$n = 1$, $\mathcal{T}_3$, 74 elements, error= 0.052 $n = 2$, $\mathcal{T}_1$, 28 elements, error= 0.047

Figure: Adapted meshes for $n = 1$ (left) and for $n = 2$ (right).

A posteriori error estimation (B.C., R.Nochetto and W. Zhang, Math. Comp., in revision.)

Illustration.



$n = 1$, $\mathcal{T}_5$, 146 elements, error= 0.033 $n = 2$, $\mathcal{T}_2$, 48 elements, error= 0.030

Figure: Adapted meshes for $n = 1$ (left) and for $n = 2$ (right).

A posteriori error estimation (B.C., R.Nochetto and W. Zhang, Math. Comp., in revision.)

The adaptive refinement.

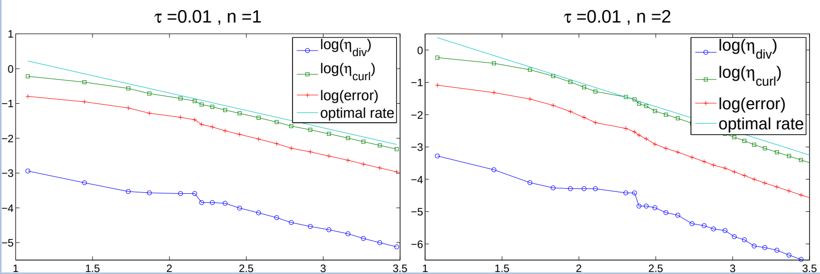


Figure: History of convergence of the adaptive method

A posteriori error estimation (B.C., R.Nochetto and W. Zhang, Math. Comp., in revision.)

The adaptive refinement.

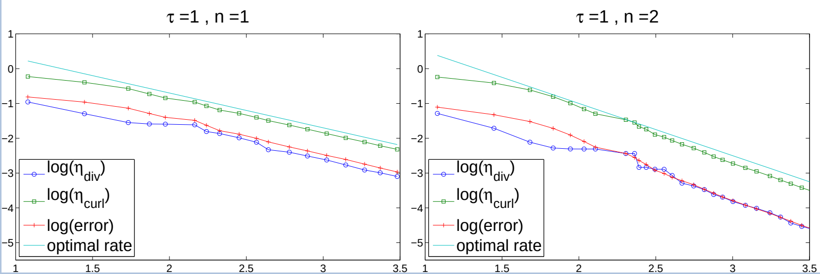


Figure: History of convergence of the adaptive method

A posteriori error estimation(B.C., R.Nochetto and W. Zhang, Math. Comp., in revision.)

The adaptive refinement.

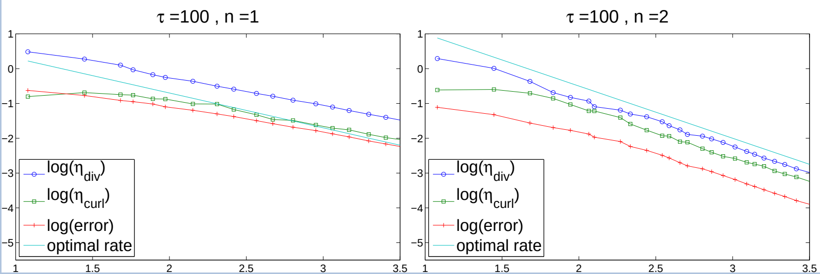


Figure: History of convergence of the adaptive method

HDG methods for the heat equation. (B.Chabaud and B.C., Math. Comp., 2012.)

The model problem.

Consider the model problem:

$$
\begin{aligned}
\mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } \Omega \times (0, T),\\
u_t + \nabla \cdot \mathbf{q} &= f && \text{in } \Omega \times (0, T),\\
\widehat{u} &= u_D && \text{on } \partial\Omega \times (0, T),\\
u &= u_0 && \text{on } \Omega \times \{0\}.
\end{aligned}
$$

Here c is a matrix-valued function which is symmetric and uniformly positive definite on Ω.

HDG methods for the heat equation.

The approach.

We can obtain $(\mathbf{q}, u)$ in $K \times (0, T)$ in terms of $\widehat{u}$ on $\partial K \times (0, T)$, f and u_0 by solving

$$
\begin{aligned}
\mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } K \times (0, T),\\
u_t + \nabla \cdot \mathbf{q} &= f && \text{in } K \times (0, T),\\
u &= \widehat{u} && \text{on } \partial K \times (0, T),\\
u &= u_0 && \text{on } K \times \{0\}.
\end{aligned}
$$

The function $\widehat{u}$ can now be determined as the solution on each $F \times (0, T)$, $F \in \mathcal{E}_h$, of the equations

$$
\begin{aligned}
[\![\widehat{\mathbf{q}}]\!] &= 0 && \text{if } F \in \mathcal{E}_h^o,\\
\widehat{u} &= u_D && \text{if } F \in \mathcal{E}_h^\partial,
\end{aligned}
$$

where $\widehat{\mathbf{q}}$ is the trace of $\mathbf{q} = \mathbf{q}(\widehat{u}, f, u_0)$ on ∂K.

At any time, the approximate solution $(\mathbf{q}_h, u_h, \widehat{u}_h)$ is an element of the space $\mathbf{V}_h \times W_h \times M_h$. It satisfies the equations

$$
\begin{aligned}
(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_{\Omega_h} - (u_h, \nabla\cdot\mathbf{v})_{\Omega_h} + \langle \widehat{u}_h, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
((u_h)_t, \nabla w)_{\Omega_h} - (\mathbf{q}_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h},\\
\langle \mu, \widehat{\mathbf{q}}_h\cdot\mathbf{n}\rangle_{\partial\Omega_h\setminus\partial\Omega} &= 0,\\
\langle \mu, \widehat{u}_h\rangle_{\partial\Omega} &= \langle \mu, u_D\rangle_{\partial\Omega},
\end{aligned}
$$

for all $(\mathbf{v}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h$, where

$$
\widehat{\mathbf{q}}_h\cdot\mathbf{n} = \mathbf{q}_h\cdot\mathbf{n} + \tau(u_h - \widehat{u}_h) \qquad \text{on } \partial\Omega_h.
$$

The HDG method retains all the convergence and superconvergence, uniformly in time, of the HDG method for the steady-state case provided the initial condition is properly defined.

To approximate the time derivative at time $t^n := n\Delta t$, we could use the BDF approximation

$$(u_h)_t^n \approx \Big(\sum_{j=0}^{\ell} \gamma_j {u_h}^{n-j}\Big)/\Delta t,$$

and set

$$\tilde{f}^n = f^n - \Big(\sum_{j=1}^{\ell} \gamma_j {u_h}^{n-j}\Big)/\Delta t,$$

Then, at any time $t^n = n\,\Delta t$, the approximate solution $(\mathbf{q}_h, u_h, \widehat{u}_h)$ is an element of the space $\mathbf{V}_h \times W_h \times M_h$. It satisfies the equations

$$\begin{aligned}
(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_{\Omega_h} - (u_h, \nabla\cdot\mathbf{v})_{\Omega_h} + \langle \widehat{u}_h, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
\frac{\gamma_0}{\Delta t}(u_h, \nabla w)_{\Omega_h} - (\mathbf{q}_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial\Omega_h} &= (\tilde{f}, w)_{\Omega_h},\\
\langle \mu, \widehat{\mathbf{q}}_h\cdot\mathbf{n}\rangle_{\partial\Omega_h\setminus\partial\Omega} &= 0,\\
\langle \mu, \widehat{u}_h\rangle_{\partial\Omega} &= \langle \mu, u_D\rangle_{\partial\Omega},
\end{aligned}$$

for all $(\mathbf{v}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h$, where

$$\widehat{\mathbf{q}}_h\cdot\mathbf{n} = \mathbf{q}_h\cdot\mathbf{n} + \tau(u_h - \widehat{u}_h) \qquad \text{on } \partial\Omega_h.$$

HDG methods for the wave equation. (N.C.Nguyen, J. Peraire and B.C., Math. Comp., JCP, 2011.)

The model problem.

Consider the model problem:

$$
\begin{aligned}
u_{tt} + \nabla \cdot (\mathrm{c} \nabla u) &= f && \text{in } \Omega \times (0, T), \\
\widehat{u} &= (u_D) && \text{on } \partial\Omega \times (0, T), \\
u &= u_0 && \text{on } \Omega \times \{0\}, \\
u_t &= u_1 && \text{on } \Omega \times \{0\}.
\end{aligned}
$$

Here c is a matrix-valued function which is symmetric and uniformly positive definite on Ω.

We rewrite it in terms of $(\mathbf{q}, v) := (-\mathrm{c}\nabla u, u_t)$ as follows:

$$
\begin{aligned}
\mathrm{c}\,\mathbf{q}_t + \nabla v &= 0 && \text{in } \Omega \times (0,T),\\
v_t + \nabla \cdot \mathbf{q} &= f && \text{in } \Omega \times (0,T),\\
v &= (u_D)_t && \text{on } \partial\Omega \times (0,T),\\
\mathrm{c}\,\mathbf{q} &= -\nabla u_0 && \text{on } \Omega \times \{0\},\\
v &= u_1 && \text{on } \Omega \times \{0\}.
\end{aligned}
$$

Here c is a matrix-valued function which is symmetric and uniformly positive definite on Ω.

HDG methods for the wave equation.

The approach.

We can obtain $(\mathbf{q}, v)$ in $K \times (0, T)$ in terms of $\widehat{v}$ on $\partial K \times (0, T)$, f, u_0 and u_1 by solving

$$\begin{aligned} \mathrm{c}\,\mathbf{q}_t + \nabla u &= 0 && \text{in } K \times (0, T),\\ v_t + \nabla \cdot \mathbf{q} &= f && \text{in } K \times (0, T),\\ \mathrm{c}\,\mathbf{q} &= -\nabla u_0 && \text{on } \Omega \times \{0\},\\ v &= u_1 && \text{on } \Omega \times \{0\}. \end{aligned}$$

The function $\widehat{v}$ can now be determined as the solution on each $F \times (0, T)$, $F \in \mathcal{E}_h$, of the equations

$$\begin{aligned} [\![\widehat{\mathbf{q}}]\!] &= 0 && \text{if } F \in \mathcal{E}_h^o,\\ \widehat{v} &= (u_D)_t && \text{if } F \in \mathcal{E}_h^\partial, \end{aligned}$$

where $\widehat{\mathbf{q}}$ is the trace of $\mathbf{q} = \mathbf{q}(\widehat{u}, f, u_0, u_1)$ on ∂K.

At any time, the approximate solution $(\mathbf{q}_h, v_h, \widehat{v}_h)$ is an element of the space $\mathbf{V}_h \times W_h \times M_h$. It satisfies the equations

$$\begin{aligned}
(\mathrm{c}\,(\mathbf{q}_h)_t, \mathbf{r})_{\Omega_h} - (v_h, \nabla\cdot\mathbf{r})_{\Omega_h} + \langle \widehat{v}_h, \mathbf{r}\cdot\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
((v_h)_t, \nabla w)_{\Omega_h} - (\mathbf{q}_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h},\\
\langle \mu, \widehat{\mathbf{q}}_h\cdot\mathbf{n}\rangle_{\partial\Omega_h\setminus\partial\Omega} &= 0,\\
\langle \mu, \widehat{v}_h\rangle_{\partial\Omega} &= \langle \mu, (u_D)_t\rangle_{\partial\Omega},
\end{aligned}$$

for all $(\mathbf{r}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h$, where

$$\widehat{\mathbf{q}}_h\cdot\mathbf{n} = \mathbf{q}_h\cdot\mathbf{n} + \tau(v_h - \widehat{v}_h) \qquad \text{on } \partial\Omega_h.$$

HDG methods for the wave equation. (B.C. and V.Queneville-Bélair, Math. Comp., 2nd. revision.)

The semidiscrete method.

For simplexes, $\mathbf{V}(K) := \boldsymbol{\mathcal{P}}_k(K)$ and $W(K) := \mathcal{P}_k(K)$:

- The HDG method converges in $\mathbf{q}_h$ and v_h with the optimal order of $k+1$, for $k \geq 0$, in the $L^\infty(0, T; L^2(\Omega))$-norm.
- The variable $\int_0^t v_h$ superconverges with order $k+2$, for $k \geq 1$, in the $L^\infty(0, T; L^2(\Omega))$-norm provided the initial conditions are suitably defined.
- In this case, the postprocessed solution u_h^* superconverges with order $k+2$, for $k \geq 1$, in the $L^\infty(0, T; L^2(\Omega))$-norm.

Recall that, on each element K, u_h^* lies in the space $\mathcal{P}_{k+1}(K)$ and is defined by

$$(\nabla {u_h}^\star, \nabla w)_K = -\,(\mathrm{c}\mathbf{q}_h, \nabla w)_K \qquad \text{for all } w \in \mathcal{P}_{k+1}(K),$$

$$({u_h}^\star, 1)_K = (u_h, 1)_K = (\int_0^t v_h + u_h(0), 1)_K .$$

HDG methods for convection-diffusion equations. (N.C.Nguyen, J. Peraire and B.C., JCP, 2009.

The model problem.

Consider the model problem:

$$\begin{aligned} \mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } \Omega \times (0, T), \\ \nabla \cdot (\mathbf{q} + \mathrm{v}\, u) &= f && \text{in } \Omega \times (0, T), \\ \widehat{u} &= u_D && \text{on } \partial\Omega \times (0, T). \end{aligned}$$

Here c is a matrix-valued function which is symmetric and uniformly positive definite on Ω.

HDG methods for convection-diffusion equations.

The approach.

We can obtain $(\mathbf{q}, u)$ in $K \times (0, T)$ in terms of $\widehat{u}$ on $\partial K \times (0, T)$, f and u_0 by solving

$$\begin{aligned} \mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } K \times (0, T), \\ \nabla \cdot (\mathbf{q} + \mathrm{v}\, u) &= f && \text{in } K \times (0, T), \\ u &= \widehat{u} && \text{on } \partial K \times (0, T). \end{aligned}$$

The function $\widehat{u}$ can now be determined as the solution on each $F \times (0, T)$, $F \in \mathcal{E}_h$, of the equations

$$\begin{aligned} [\![\widehat{\mathbf{q}} + \mathrm{v}\, \widehat{u}]\!] &= 0 && \text{if } F \in \mathcal{E}_h^o, \\ \widehat{u} &= u_D && \text{if } F \in \mathcal{E}_h^\partial, \end{aligned}$$

where $\widehat{\mathbf{q}}$ is the trace of $\mathbf{q} = \mathbf{q}(\widehat{u}, f, u_0)$ on ∂K.

The HDG method defines the approximation $(\mathbf{q}_h, u_h, \widehat{u}_h)$ in $\mathbf{V}_h \times W_h \times M_h$ by requiring that

$$
\begin{aligned}
(\mathrm{c}\,\mathbf{q}_h, \mathbf{r})_{\Omega_h} - (u_h, \nabla\cdot\mathbf{r})_{\Omega_h} + \langle \widehat{u}_h, \mathbf{r}\cdot\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
-(\mathbf{q}_h + u_h\mathbf{v}, \nabla w)_{\Omega_h} + \langle (\widehat{\mathbf{q}}_h + \widehat{u}_h\,\mathbf{v})\cdot\mathbf{n}, w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h},\\
\langle \mu, \widehat{u}_h\rangle_{\partial\Omega} &= \langle \mu, g\rangle_{\partial\Omega},\\
\langle \mu, (\widehat{\mathbf{q}}_h + \widehat{u}_h\,\mathbf{v})\cdot\mathbf{n}\rangle_{\partial\Omega_h\setminus\partial\Omega} &= 0,
\end{aligned}
$$

hold for all $(\mathbf{r}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h$, where

$$
\widehat{\mathbf{q}}_h + \widehat{u}_h\,\mathbf{v} = \mathbf{q}_h + \widehat{u}_h\,\mathbf{v} + \tau(u_h - \widehat{u}_h)\mathbf{n} \quad \text{on } \partial\Omega_h.
$$

Theorem

The method is well defined if

A1 *There is a constant* $\gamma_0 > 0$*:* $\min(\tau - \frac{1}{2}\mathbf{v}\cdot\mathbf{n})|_{\partial K} \geq \gamma_0 \ \forall\, K \in \mathcal{T}_h$*.*

A2 *On any face* $F \in \mathcal{E}_h$*,* τ *is a constant.*

The following practical choices of stabilization functions τ do satisfy these two conditions:

$$\tau^+ = \tau^- = |\mathbf{v}\cdot\mathbf{n}| + \frac{\kappa}{\ell},$$

$$(\tau^+,\tau^-) = \begin{cases} (|\mathbf{v}\cdot\mathbf{n}| + \frac{\kappa}{\ell}, 0) & \text{when } \mathbf{v}\cdot\mathbf{n}^- \leq 0,\\ (0, |\mathbf{v}\cdot\mathbf{n}| + \frac{\kappa}{\ell}) & \text{when } \mathbf{v}\cdot\mathbf{n}^- > 0.\end{cases}$$

Here κ is a scalar proportional to some norm of the diffusivity matrix c^{-1} and ℓ denotes a representative length scale.

HDG methods for convection-diffusion

The numerical traces.

For the first choice of τ, we have

$$\widehat{u}_h = \{\!\{u_h\}\!\} + \frac{1}{2\,\tau}[\![\mathbf{q}_h \cdot \mathbf{n}]\!],$$

$$\widehat{u}_h\,\mathbf{v} + \widehat{\mathbf{q}_h} = \{\!\{u_h\}\!\}\mathbf{v} + \{\!\{\mathbf{q}_h\}\!\} + \frac{1}{2\,\tau}[\![\mathbf{q}_h \cdot \mathbf{n}]\!]\mathbf{v} + \frac{\tau}{2}[\![u_h\mathbf{n}]\!],$$

whereas for the second choice for τ,

$$\begin{cases} \widehat{u}_h & = {u_h}^{+} + \frac{1}{\tau^{+}}[\![\mathbf{q}_h \cdot \mathbf{n}]\!], \\ \widehat{u}_h\,\mathbf{v} + \widehat{\mathbf{q}_h} & = u_h^{+}\mathbf{v} + {\mathbf{q}_h}^{-} + \frac{1}{\tau^{+}}[\![\mathbf{q}_h \cdot \mathbf{n}]\!]\,\mathbf{v} \end{cases} \qquad \text{if } \mathbf{v}\cdot\mathbf{n}^{-} \leq 0,$$

and

$$\begin{cases} \widehat{u}_h & = {u_h}^{-} + \frac{1}{\tau^{-}}[\![\mathbf{q}_h \cdot \mathbf{n}]\!], \\ \widehat{u}_h\,\mathbf{v} + \widehat{\mathbf{q}_h} & = u_h^{-}\mathbf{v} + {\mathbf{q}_h}^{+} + \frac{1}{\tau^{-}}[\![\mathbf{q}_h \cdot \mathbf{n}]\!]\mathbf{v}, \end{cases} \qquad \text{if } \mathbf{v}\cdot\mathbf{n}^{-} > 0.$$

The auxiliary projection.

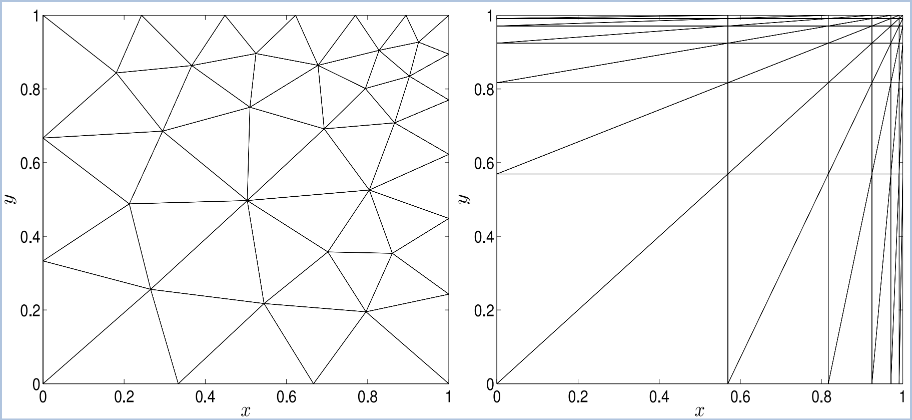
On any simplex K, the projection $(\boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q}, \Pi_W u)$ is the element of $\boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$ which solves the equations

$$\begin{aligned} ((\boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q} - \mathbf{q}) + \mathbf{v}\,(\Pi_W u - u), \mathbf{r})_K &= 0 \;\; \forall\, \mathbf{r} \in \boldsymbol{\mathcal{P}}_{k-1}(K), \\ (\Pi_W u - u, w)_K &= 0 \;\; \forall\, w \in \mathcal{P}_{k-1}(K), \\ \langle ((\boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q} - \mathbf{q}) + \mathbf{v}\,(P_M u - u)) \cdot \mathbf{n} + \tau(\Pi_W u - u), \mu\rangle_F &= 0 \;\; \forall\, \mu \in \mathcal{P}_k(F), \end{aligned}$$

for all faces F of the simplex K.

HDG methods for convection-diffusion. (N.C.Nguyen, J. Peraire and B.C., JCP, 2009.)

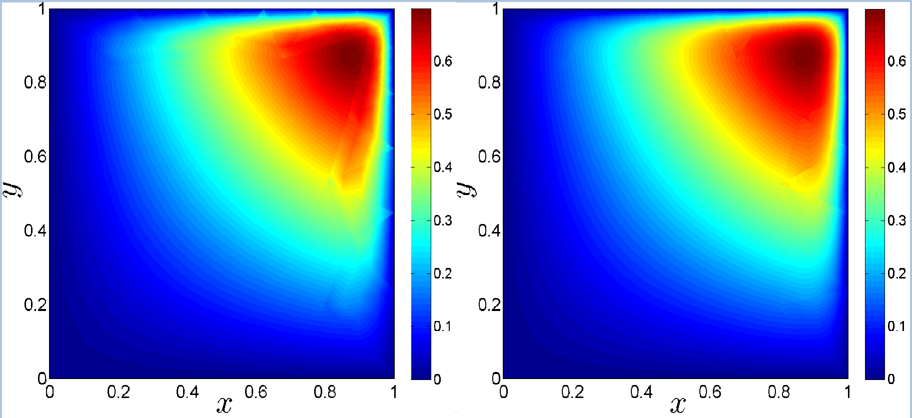
Numerical examples.



Unstructured and anisotropic meshes.

HDG methods for convection-diffusion. (N.C.Nguyen, J. Peraire and B.C., JCP, 2009.)

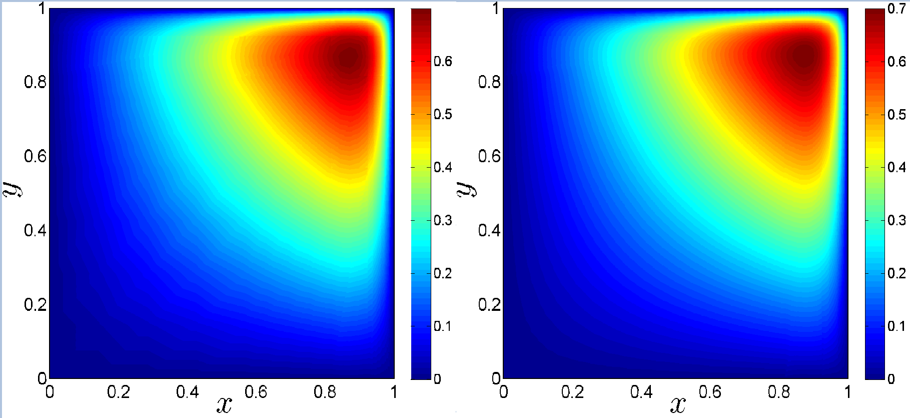
Numerical examples.



HDG approximation with quadratic polynomials on the unstructured triangulation.

HDG methods for convection-diffusion. (N.C.Nguyen, J. Peraire and B.C., JCP, 2009.)

Numerical examples.



HDG approximation with quadratic polynomials on the unstructured triangulation.

Consider the following problem:

$$\begin{aligned}
&\sigma_{ij,j} + b_i = 0 && \text{in } \Omega,\\
&\epsilon_{ij} - \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = 0 && \text{in } \Omega,\\
&\sigma_{ij} - D_{ijkl}\,\epsilon_{kl} = 0 && \text{in } \Omega,\\
&\widehat{u}_i = u_i && \text{on } \partial\Omega_D,\\
&\widehat{\sigma}_{ij}\, n_j = t_i && \text{on } \partial\Omega_N.
\end{aligned}$$

Linear elasticity.

A characterization of the solution.

We can obtain (σ, u) in K in terms of $\widehat{u}$ by solving

$$\begin{aligned}
\sigma_{ij,j} + b_i &= 0 && \text{in } K,\\
\epsilon_{ij} - \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) &= 0 && \text{in } K,\\
\sigma_{ij} - D_{ijkl}\, \epsilon_{kl} &= 0 && \text{in } K,\\
\widehat{u}_i &= \widehat{u}_i && \text{on } \partial K.
\end{aligned}$$

The function $\widehat{u}$ can now be determined as the solution of the transmission condition

$$\begin{aligned}
[\![\widehat{\sigma}_{ij}\, n_j]\!] &= 0 && \text{on } \mathcal{E}_h^o,\\
\widehat{u}_i &= u_i && \text{on } \partial\Omega_D,\\
\widehat{\sigma}_{ij}\, n_j &= t_i && \text{on } \partial\Omega_N.
\end{aligned}$$

The approximation $(\mathbf{u}^h, \underline{\boldsymbol{\sigma}}^h, \underline{\boldsymbol{\epsilon}}^h, \widehat{\mathbf{u}}^h)$ is taken in the finite dimensional space $\mathbf{V}^h \times \underline{\mathbf{W}}^h \times \underline{\mathbf{Z}}^h \times \mathbf{M}^h$ where

$$\begin{aligned}
\mathbf{V}^h &= \{\mathbf{v} \in \mathbf{L}^2(\Omega) \;:\; v_i\,|_K \in \mathcal{P}_k(K) \quad \forall K \in \Omega_h, \quad i = 1,2,3\},\\
\underline{\mathbf{W}}^h &= \{\underline{\mathbf{w}} \in \underline{\mathbf{L}}^2(\Omega) \;:\; w_{ij}\,|_K \in \mathcal{P}_k(K) \quad \forall K \in \Omega_h, \quad i,j = 1,2,3\},\\
\underline{\mathbf{Z}}^h &= \{\underline{\mathbf{z}} \in \underline{\mathbf{L}}^2(\Omega) \;:\; z_{ij}\,|_K \in \mathcal{P}_k(K) \quad \forall K \in \Omega_h, \quad i,j = 1,2,3\},\\
\mathbf{M}^h &= \{\boldsymbol{\mu} \in \mathbf{L}^2(\mathcal{E}_h) \;:\; \mu_i\,|_F \in \mathcal{P}_k(F) \quad \forall F \in \mathcal{E}_h, \quad i = 1,2,3\}.
\end{aligned}$$

Linear elasticity .

An HDG method.

On the element K, $(\mathbf{u}^h, \underline{\boldsymbol{\sigma}}^h, \underline{\boldsymbol{\epsilon}}^h)$ is obtained in terms of $\widehat{u}^h$ by solving

$$\left(v_{i,j}, \sigma^h_{ij}\right)_K - \langle v_i, \widehat{\sigma}^h_{ij} n_j \rangle_{\partial K} - \left(v_i, b_i\right)_K = 0,$$
$$\left(w_{ij}, \epsilon^h_{ij}\right)_K - \frac{1}{2}\langle w_{ij}, (\widehat{u}^h_i n_j + \widehat{u}^h_j n_i)\rangle_{\partial K} + \frac{1}{2}\left(w_{ij,j}, u^h_i\right)_K + \frac{1}{2}\left(w_{ij,i}, u^h_j\right)_K = 0,$$
$$\left(z_{ij}, \sigma^h_{ij}\right)_K - \left(z_{ij}, D_{ijkl}\epsilon^h_{kl}\right)_K = 0,$$

for all $(\mathbf{v}, \underline{\mathbf{w}}, \underline{\mathbf{z}}, \boldsymbol{\mu}) \in \mathcal{P}_k(K) \times \underline{\mathcal{P}_k(K)} \times \underline{\mathcal{P}_k(K)} \times \mathcal{P}_k(K)$, where

$$\widehat{\sigma}^h_{ij} = \sigma^h_{ij} - \tau_{ijkl}\left(u^h_k - \widehat{u}^h_k\right) n_l \qquad \text{on } \partial\Omega_h.$$

The function $\widehat{u}^h$ is now determined as the element of $\mathbf{M}_h$ satisfying

$$\left\langle \mu_i, \widehat{\sigma}^h_{ij} n_j \right\rangle_{\partial\Omega_h \setminus \partial\Omega_D} = \left\langle \mu_i, t_i \right\rangle_{\partial\Omega_N},$$
$$\left\langle \mu_i, \widehat{u}^h_i \right\rangle_{\partial\Omega_D} = \left\langle \mu_i, u_i \right\rangle_{\partial\Omega_D}.$$

for all $\boldsymbol{\mu} \in \mathbf{M}_h$.

In compact form, the methods an be written as follows:

$$\left(v_{i,j}, \sigma^h_{ij}\right)_{\Omega_h} - \langle v_i, \widehat{\sigma}^h_{ij} n_j\rangle_{\partial\Omega_h} - \left(v_i, b_i\right)_{\Omega_h} = 0,$$

$$\left(w_{ij}, \epsilon^h_{ij}\right)_{\Omega_h} - \frac{1}{2}\langle w_{ij}, (\widehat{u}^h_i n_j + \widehat{u}^h_j n_i)\rangle_{\partial\Omega_h} + \frac{1}{2}\left(w_{ij,j}, u^h_i\right)_{\Omega_h} + \frac{1}{2}\left(w_{ij,i}, u^h_j\right)_{\Omega_h} = 0,$$

$$\left(z_{ij}, \sigma^h_{ij}\right)_{\Omega_h} - \left(z_{ij}, D_{ijkl}\epsilon^h_{kl}\right)_{\Omega_h} = 0,$$

$$\left\langle \mu_i, \widehat{\sigma}^h_{ij} n_j\right\rangle_{\partial\Omega_h\setminus\partial\Omega_D} = \langle \mu_i, t_i\rangle_{\partial\Omega_N},$$

$$\left\langle \mu_i, \widehat{u}^h_i\right\rangle_{\partial\Omega_D} = \langle \mu_i, u_i\rangle_{\partial\Omega_D},$$

for all $(\mathbf{v}, \underline{\mathbf{w}}, \underline{\mathbf{z}}, \boldsymbol{\mu}) \in \mathbf{V}^h \times \underline{\mathbf{W}}^h \times \underline{\mathbf{Z}}^h \times \mathbf{M}^h$, where

$$\widehat{\sigma}^h_{ij} = \sigma^h_{ij} - \tau_{ijkl}\left(u^h_k - \widehat{u}^h_k\right) n_l \qquad \text{on } \partial\Omega_h.$$

In compact form:

$$
\begin{aligned}
&\left(v_{i,j}, \sigma_{ij}^h\right)_{\Omega_h} - \langle v_i, \widehat{\sigma}_{ij}^h n_j\rangle_{\partial\Omega_h} - \left(v_i, b_i\right)_{\Omega_h} = 0,\\
&\left(w_{ij}, \epsilon_{ij}^h\right)_{\Omega_h} - \frac{1}{2}\langle w_{ij}, \left(\widehat{u}_i^h n_j + \widehat{u}_j^h n_i\right)\rangle_{\partial\Omega_h} + \frac{1}{2}\left(w_{ij,j}, u_i^h\right)_{\Omega_h} + \frac{1}{2}\left(w_{ij,i}, u_j^h\right)_{\Omega_h} = 0,\\
&\left(z_{ij}, \sigma_{ij}^h\right)_{\Omega_h} - \left(z_{ij}, D_{ijkl}\epsilon_{kl}^h\right)_{\Omega_h} = 0,\\
&\left\langle \mu_i, \widehat{\sigma}_{ij}^h n_j\right\rangle_{\partial\Omega_h\setminus\partial\Omega_D} = \left\langle \mu_i, t_i\right\rangle_{\partial\Omega_N},\\
&\left\langle \mu_i, \widehat{u}_i^h\right\rangle_{\partial\Omega_D} = \left\langle \mu_i, u_i\right\rangle_{\partial\Omega_D},
\end{aligned}
$$

for all $(\mathbf{v}, \underline{\mathbf{w}}, \underline{\mathbf{z}}, \boldsymbol{\mu}) \in \mathbf{V}^h \times \underline{\mathbf{W}}^h \times \underline{\mathbf{Z}}^h \times \mathbf{M}^h$, where

$$\widehat{\sigma}_{ij}^h = \sigma_{ij}^h - \tau_{ijkl}\left(u_k^h - \widehat{u}_k^h\right) n_l \qquad \text{on } \partial\Omega_h.$$

Theorem

The approximate solution

$$(\mathbf{u}^h, \underline{\boldsymbol{\sigma}}^h, \underline{\boldsymbol{\epsilon}}^h) = (\mathtt{U}^{(\widehat{\mathbf{u}}^h)}, \mathtt{S}^{(\widehat{\mathbf{u}}^h)}, \mathtt{E}^{(\widehat{\mathbf{u}}^h)}) + (\mathtt{U}^{(\mathbf{u})}, \mathtt{S}^{(\mathbf{u})}, \mathtt{E}^{(\mathbf{u})}),$$

is well defined if we take $\tau_{ijkl}\, n_j\, n_l$ *positive definite on* $\partial\Omega_h$*. Moreover, the function* $\boldsymbol{\lambda}^h := \widehat{\mathbf{u}}^h - \mathbf{u}$*, is the only element of* $\mathbf{M}^h$ *satisfying*

$$a^h\left(\boldsymbol{\mu}, \boldsymbol{\lambda}^h\right) = b^h\left(\boldsymbol{\mu}\right) \qquad \forall \boldsymbol{\mu} \in \mathbf{M}^h(\mathbf{0}),$$

where

$$a^h\left(\boldsymbol{\zeta}, \boldsymbol{\eta}\right) = \left(D_{ijkl}\, \mathtt{E}^{(\zeta)}_{ij}, \mathtt{E}^{(\eta)}{}_{kl}\right)_{\Omega_h} + \left\langle \left(\mathtt{U}^{(\eta)}_i - \eta_i\right), \tau_{ijkl}\, n_j\, n_l \left(\mathtt{U}^{(\zeta)}_k - \zeta_k\right)\right\rangle_{\partial\Omega_h},$$

$$b^h\left(\boldsymbol{\zeta}\right) = \left\langle \zeta_i, t_i \right\rangle_{\partial\Omega_N} - \left\langle \widehat{\mathtt{S}}^{(\zeta)}_{ij} n_j, u_i \right\rangle_{\partial\Omega_D} + \left(\mathtt{U}^{(\zeta)}_i, b_i\right)_{\Omega_h},$$

for all $\boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathbf{L}^2(\mathscr{E}^h)$.

Linear elasticity.

Numerical experiments.

- For $k \geq 0$ all unknowns converge with order $k+1$.
- For $k \geq 2$ the local average of the displacement superconverges with order $k+2$. A local postprocessing can be devised that provides another approximate displacement converging with order $k+2$.
- Analysis for general polyhedral elements: Convergence of order $k+1/2$ for the stress and $k+1$ for the displacement. The estimates are sharp.

(G. Fu, B.C. and H.Stolarski, submitted.)

HDG methods for the Stokes flow. (N.C Nguyen, J. Peraire and B.C., JCP+CMAME, 2010.)

The model problem.

Consider the model problem:

$$\begin{aligned} -\nu \Delta u + \nabla p &= f && \text{in } \Omega, \\ \nabla \cdot u &= 0 && \text{on } \Omega, \\ \widehat{u} &= u_D && \text{on } \partial\Omega, \end{aligned}$$

where $\langle u_D \cdot \mathbf{n}, 1 \rangle_{\partial\Omega} = 0$ and $(p, 1)_\Omega = 0$.

We begin by rewriting it as follows:

$$\begin{aligned} \boldsymbol{\omega} - \nabla \times \mathbf{u} &= 0 && \text{in } \Omega, \\ \nu \nabla \times \boldsymbol{\omega} + \nabla p &= f && \text{in } \Omega, \\ \nabla \cdot u &= 0 && \text{on } \Omega, \\ \widehat{u} &= u_D && \text{on } \partial\Omega, \end{aligned}$$

where $\langle u_D \cdot \mathbf{n}, 1 \rangle_{\partial\Omega} = 0$ and $(p, 1)_\Omega = 0$.

HDG methods for the Stokes flow

Using the vorticity.

We can express $(\boldsymbol{\omega}, \mathbf{u}, p)$ in K in terms of $\widehat{\mathbf{u}}$ on ∂K and $\overline{p} := (p, 1)_K/|K|$ by solving

$$\begin{aligned} \boldsymbol{\omega} - \nabla \times \mathbf{u} &= 0, \qquad \nu \nabla \times \boldsymbol{\omega} + \nabla p = \mathbf{f} && \text{in } K,\\ \nabla \cdot \mathbf{u} &= \frac{1}{|K|} \langle \widehat{\mathbf{u}} \cdot \mathbf{n}, 1 \rangle_{\partial K} && \text{in } K,\\ \mathbf{u} &= \widehat{\mathbf{u}} && \text{on } \partial K. \end{aligned}$$

The functions $\widehat{\mathbf{u}}$ and $\overline{p}$ are the solution of

$$\begin{aligned} [\![-\nu \widehat{\boldsymbol{\omega}} \times \mathbf{n} + \widehat{p}\, \mathbf{n}]\!] &= 0 && \text{for all } F \in \mathcal{E}_h^o,\\ \langle \widehat{\mathbf{u}} \cdot \mathbf{n}, 1 \rangle_{\partial K} &= 0 && \text{for all } K \in \Omega_h,\\ \widehat{\mathbf{u}} &= \mathbf{u}_D && \text{on } \partial\Omega,\\ (\overline{p}, 1)_\Omega &= 0. \end{aligned}$$

HDG methods for the Stokes flow.

Using the velocity gradient.

We begin by rewriting it as follows:

$$
\begin{aligned}
\mathrm{L} - \nabla \mathbf{u} &= 0 && \text{in } \Omega,\\
-\nu \nabla \cdot \mathrm{L} + \nabla p &= f && \text{in } \Omega,\\
\nabla \cdot u &= 0 && \text{on } \Omega,\\
\widehat{u} &= u_D && \text{on } \partial\Omega,
\end{aligned}
$$

where $\langle u_D \cdot \mathbf{n}, 1\rangle_{\partial\Omega} = 0$ and $(p, 1)_\Omega = 0$.

We can express $(\mathrm{L}, \mathbf{u}, p)$ in K in terms of $\widehat{\mathbf{u}}$ on ∂K and $\overline{p} := (p, 1)_K/|K|$ by solving

$$
\begin{aligned}
\mathrm{L} - \nabla \mathbf{u} &= 0, \qquad -\nu\,\nabla\cdot\mathrm{L} + \nabla p = \mathbf{f} && \text{in } K,\\
\nabla\cdot\mathbf{u} &= \frac{1}{|K|}\langle \widehat{\mathbf{u}}\cdot\mathbf{n}, 1\rangle_{\partial K} && \text{in } K,\\
\mathbf{u} &= \widehat{\mathbf{u}} && \text{on } \partial K.
\end{aligned}
$$

The functions $\widehat{\mathbf{u}}$ and $\overline{p}$ are the solution of

$$
\begin{aligned}
[\![-\nu\widehat{\mathrm{L}}\mathbf{n} + \widehat{p}\,\mathbf{n}]\!] &= 0 && \text{for all } F \in \mathcal{E}_h^o,\\
\langle \widehat{\mathbf{u}}\cdot\mathbf{n}, 1\rangle_{\partial K} &= 0 && \text{for all } K \in \Omega_h,\\
\widehat{\mathbf{u}} &= \mathbf{u}_D && \text{on } \partial\Omega,\\
(\overline{p}, 1)_\Omega &= 0.
\end{aligned}
$$

The HDG methods for the Stokes flow

Which approach should we use?

- Both approaches give rise to saddle-point problems of the same sparsiy structure.
- In both approaches, the only globally coupled degrees of freedom are those of the velocity trace $\widehat{u}$ and the average of the pressure on each element $\overline{p}$.
- The local solvers for the vorticity formulation have less degrees of freedom. However, there is no superconvergence of the velocity.
- The local solvers for the velocity gradient formulation have more degrees of freedom. However, there is superconvergence of the velocity.

On the element $K \in \Omega_h$, we define $(\mathrm{L}_h, \mathbf{u}_h, p_h)$ in terms of $(\widehat{\mathbf{u}}_h, \overline{p}_h, f)$ as the element of $\mathrm{G}(K) \times \mathbf{V}(K) \times Q(K)$ solving

$$\begin{aligned}
(\mathrm{L}_h, \mathrm{G})_K + (\mathbf{u}_h, \nabla \cdot \mathrm{G})_K - \langle \widehat{\mathbf{u}}_h, \mathrm{G}\mathbf{n} \rangle_{\partial K} &= 0, \\
(\nu \mathrm{L}_h, \nabla \mathbf{v})_K - (p_h, \nabla \cdot \mathbf{v})_K - \langle \nu \widehat{\mathrm{L}}_h \mathbf{n} - \widehat{p}_h \mathbf{n}, \mathbf{v} \rangle_{\partial K} &= (\mathbf{f}, \mathbf{v})_K, \\
-(\mathbf{u}_h, \nabla q)_{\Omega_h} + \langle \widehat{\mathbf{u}}_h \cdot \mathbf{n}, q - \overline{q} \rangle_{\partial K} &= 0,
\end{aligned}$$

for all $(\mathrm{G}, \mathbf{v}, q) \in \mathrm{G}(K) \times \mathbf{V}(K) \times Q(K)$, where

$$-\nu \widehat{\mathrm{L}}_h \mathbf{n} + \widehat{p}_h \mathbf{n} = -\nu \mathrm{L}_h \mathbf{n} + p_h \mathbf{n} + \nu\, \tau\, (\mathbf{u}_h - \widehat{\mathbf{u}}_h) \quad \text{on } \partial K,$$

and $(p_h, 1)_K / |K| = \overline{p}_h$.

We take $\widehat{\mathbf{u}}_h|_F$ in $\mathbf{M}(F)$ and $\overline{p}_h|_K$ in $\mathcal{P}_0(K)$ and determine them by requiring

$$
\begin{aligned}
\langle [\![-\nu \widehat{\mathrm{L}}_h \mathbf{n} + \widehat{p}_h \,\mathbf{n}]\!], \boldsymbol{\mu} \rangle_F &= 0 && \forall \boldsymbol{\mu} \in \mathbf{M}(F) \;\; \forall\, F \in \mathcal{E}_h^o, \\
\langle \widehat{\mathbf{u}}_h \cdot \mathbf{n}, 1 \rangle_{\partial K} &= 0 && \forall\, K \in \Omega_h, \\
\langle \widehat{\mathbf{u}}_h, \boldsymbol{\mu} \rangle_F &= \langle \mathbf{u}_D, \boldsymbol{\mu} \rangle_F && \forall \boldsymbol{\mu} \in \mathbf{M}(F) \;\; \forall\, F \in \mathcal{E}_h^\partial, \\
(\overline{p}_h, 1)_\Omega &= 0.
\end{aligned}
$$

Theorem

The HDG methods are well defined if

- $\tau > 0$ *on* $\partial\Omega_h$,
- $\nabla \mathbf{V}(K) \in \mathrm{G}(\mathrm{K}) \;\; \forall K \in \Omega_h$,
- $\nabla Q(K) \in \mathbf{V}(K) \;\; \forall K \in \Omega_h$.

We denote by $(\mathrm{L}, \mathbf{U}, \mathrm{P})$ the linear mapping that associates $(\widehat{\mathbf{u}}_h, \overline{p}_h, f)$ to $(\mathrm{L}_h, \mathbf{u}_h, p_h)$, and set

$$
\begin{aligned}
(\mathrm{L}^{\widehat{\mathbf{u}}_h}, \mathbf{U}^{\widehat{\mathbf{u}}_h}, \mathrm{P}^{\widehat{\mathbf{u}}_h}) &:= (\mathrm{L}, \mathbf{U}, \mathrm{P})(\widehat{\mathbf{u}}_h, 0, 0),\\
(\mathrm{L}^{\overline{p}_h}, \mathbf{U}^{\overline{p}_h}, \mathrm{P}^{\overline{p}_h}) &:= (\mathrm{L}, \mathbf{U}, \mathrm{P})(0, \overline{p}_h, 0),\\
(\mathrm{L}^{f}, \mathbf{U}^{f}, \mathrm{P}^{f}) &:= (\mathrm{L}, \mathbf{U}, \mathrm{P})(0, 0, f).
\end{aligned}
$$

Then we have that

$$
(\mathrm{L}_h, \mathbf{u}_h, p_h) = (\mathrm{L}^{\widehat{\mathbf{u}}_h}, \mathbf{U}^{\widehat{\mathbf{u}}_h}, \mathrm{P}^{\widehat{\mathbf{u}}_h}) + (\mathrm{L}^{\overline{p}_h}, \mathbf{U}^{\overline{p}_h}, \mathrm{P}^{\overline{p}_h}) + (\mathrm{L}^{f}, \mathbf{U}^{f}, \mathrm{P}^{f}).
$$

The function $(\widehat{\mathbf{u}}_h, \overline{p}_h)$ is the only element in $\mathbf{M}_h \times \overline{P}_h$ such that

$$
\begin{aligned}
a_h(\widehat{\mathbf{u}}_h, \boldsymbol{\mu}) + b_h(\overline{p}_h, \boldsymbol{\mu}) &= \ell_h(\boldsymbol{\mu}), && \forall\, \boldsymbol{\mu} \in \mathbf{M}_h : \boldsymbol{\mu}|_{\partial\Omega} = \mathbf{0}),\\
b_h(\overline{q}, \widehat{\mathbf{u}}_h) &= 0, && \forall\, \overline{q} \in \overline{P}_h,\\
\widehat{\mathbf{u}}_h &= \mathbf{u}_D,\\
(\overline{p}_h, 1)_\Omega &= 0.
\end{aligned}
$$

where $\mathbf{M}_h := \{\boldsymbol{\mu} \in \mathbf{L}^2(\mathcal{E}_h) : \quad \boldsymbol{\mu}|_F \in \mathbf{M}(F) \;\; \forall\, F \in \mathcal{E}_h^o\}$.

- The bilinear form $a_h(\cdot,\cdot)$ is symmetric and positive definite on $\mathbf{M}_{h,0} \times \mathbf{M}_{h,0}$.

The HDG methods for the Stokes flow.

Compact form of the HDG methods.

$(\mathrm{L}_h, \mathbf{u}_h, p_h, \widehat{\mathbf{u}}_h)$ is the element of $\mathrm{G}_h \times \mathbf{V}_h \times Q_h \times \mathbf{M}_h$ solving

$$\begin{aligned}
(\mathrm{L}_h, \mathrm{G})_{\Omega_h} + (\mathbf{u}_h, \nabla\cdot \mathrm{G})_{\Omega_h} - \langle \widehat{\mathbf{u}}_h, \mathrm{G}\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
(\nu \mathrm{L}_h, \nabla \mathbf{v})_{\Omega_h} - (p_h, \nabla\cdot \mathbf{v})_{\Omega_h} - \langle \nu \widehat{\mathrm{L}}_h\mathbf{n} - \widehat{p}_h\mathbf{n}, \mathbf{v}\rangle_{\partial\Omega_h} &= (\mathbf{f}, \mathbf{v})_{\Omega_h},\\
-(\mathbf{u}_h, \nabla q)_{\Omega_h} + \langle \widehat{\mathbf{u}}_h\cdot \mathbf{n}, q\rangle_{\partial\Omega_h} &= 0,\\
\langle -\nu\widehat{\mathrm{L}}_h\mathbf{n} + \widehat{\mathbf{u}}_h\, \widehat{\mathbf{u}}_h\cdot\mathbf{n} + \widehat{p}_h\,\mathbf{n}, \boldsymbol{\mu}\rangle_{\partial\Omega_h\setminus\partial\Omega} &= 0\\
\langle \widehat{\mathbf{u}}_h, \boldsymbol{\mu}\rangle_{\partial\Omega} &= \langle \mathbf{u}_D, \boldsymbol{\mu}\rangle_{\partial\Omega}\\
(p_h, 1)_{\Omega} &= 0,
\end{aligned}$$

for all $(\mathrm{G}, \mathbf{v}, q, \boldsymbol{\mu}) \in \mathrm{G}_h \times \mathbf{V}_h \times Q_h \times \mathbf{M}_h$, where

$$-\nu\widehat{\mathrm{L}}_h\mathbf{n} + \widehat{p}_h\mathbf{n} = -\nu\mathrm{L}_h\mathbf{n} + p_h\mathbf{n} + \nu\,\tau\,(\mathbf{u}_h - \widehat{\mathbf{u}}_h) \quad \text{on } \partial\Omega_h.$$

The energy identity for the exact solution is

$$(\mathrm{L}, \mathrm{L})_\Omega = (\mathbf{f}, \mathbf{u})_\Omega + \langle -\nu \mathrm{L}\mathbf{n} + p\,\mathbf{n}, \mathbf{u}_D \rangle_{\partial\Omega},$$

and for the approximate solution we have,

$$(\mathrm{L}_h, \mathrm{L}_h)_\Omega + \Theta_\tau(\mathbf{u}_h - \widehat{\mathbf{u}}_h) = (\mathbf{f}, \mathbf{u}_h)_\Omega + \langle (-\nu \widehat{\mathrm{L}}_h + \widehat{p}_h\, \mathrm{I})\mathbf{n}, \mathbf{u}_D \rangle_{\partial\Omega},$$

where $\Theta_\tau(\mathbf{u}_h - \widehat{\mathbf{u}}_h) := \sum_{K \in \Omega_h} \langle \tau(\mathbf{u}_h - \widehat{\mathbf{u}}_h), \mathbf{u}_h - \widehat{\mathbf{u}}_h \rangle_{\partial K}$. We see that the jumps $\mathbf{u}_h - \widehat{\mathbf{u}}_h$ stabilize the method if we require the function τ to be positive on $\partial\Omega_h$.

The Galerkin formulation on the element K reads

$$
\begin{aligned}
(\mathrm{R}_K^{\mathbf{u}}, \mathrm{G})_K &= \langle \mathrm{R}_{\partial K}^{\mathbf{u}}, \mathrm{G}\rangle_{\partial K}\\
(\mathbf{R}_K^{\mathrm{L},p}, \mathbf{v})_K &= \langle R_{\partial K}^{\mathrm{L},p}, \mathbf{v}\rangle_{\partial K},\\
(R_K^{\nabla\cdot\mathbf{u}}, q)_K &= \langle tr\mathrm{R}_{\partial K}^{\mathbf{u}}, q\rangle_{\partial K},
\end{aligned}
$$

for all $(\mathrm{G}, \mathbf{v}, q) \in \mathrm{G}(K) \times \mathbf{V}(K) \times P(K)$ where

$$
\begin{aligned}
\mathrm{R}_K^{\mathbf{u}} &:= \mathrm{L}_h - \nabla \mathbf{u}_h,\\
\mathbf{R}_K^{\mathrm{L},p} &:= \nabla \cdot (-\nu \mathrm{L}_h + p_h\, \mathrm{I}) - \mathbf{f},\\
R_K^{\nabla\cdot\mathbf{u}} &:= \nabla \cdot \mathbf{u}_h,\\
\mathrm{R}_{\partial K}^{\mathbf{u}} &:= (\widehat{\mathbf{u}}_h - \mathbf{u}_h) \otimes \mathbf{n},\\
\mathbf{R}_{\partial K}^{\mathrm{L},p} &:= (-\nu \mathrm{L}_h \mathbf{n} + p_h\, \mathbf{n}) - (-\nu \widehat{\mathrm{L}}_h \mathbf{n} + \widehat{p}_h\, \mathbf{n}) = -\nu\, \tau\, (\mathbf{u}_h - \widehat{\mathbf{u}}_h)
\end{aligned}
$$

The HDG methods for the Stokes flow. (B.C., J. Gopalakrishnan, N.C.Nguyen, J. Peraire and F.-J. Sayas, Math. Comp., 2011.) (B.C. and K. Shi, Math. Comp.,2012 + SINUM, 2012.)

Construction of superconvergent HDG methods.

- Let $\mathbf{V}^D(K)$, $W^D(K)$ and $M^D(F)$ be the local spaces of a superconvergent HDG method for diffusion.

- Set $\mathrm{G}_i(K) := \mathbf{V}^D(K)$, $\mathbf{V}_i(K) := W^D(K)$ and $\mathbf{M}_i(F) := M^D(F)$.

- Take a local space $Q(K)$ such that

$$\nabla \cdot \mathbf{V}(K) \subset Q(K), \quad Q(K)\,\mathrm{I} \subset \mathrm{G}(K).$$

Theorem

We have

$$\| \mathrm{E}^{\mathrm{L}} \|_\Omega \leq C \, \| \Pi \, \mathrm{L} - \mathrm{L} \, \|_\Omega,$$

$$\|\varepsilon^p\|_\Omega \leq C \sqrt{C_\tau} \, \nu \, \|\Pi \; \mathrm{L} - \mathrm{L}\|_\Omega,$$

where $C_\tau := \max_{K\in\Omega_h}\{1, \tau_K \, h_K\}$. *Moreover,*

$$\| \, \varepsilon_u \, \|_\Omega \leq C \, C_\tau \, h^{\min\{k,1\}} \, \| \, \Pi \, \mathrm{L} - \mathrm{L} \, \|_\Omega,$$

provided a standard elliptic regularity result holds.

Note that, by an energy argument, we get

$$(\mathrm{E}^{\mathrm{L}}, \mathrm{E}^{\mathrm{L}})_\Omega + \Theta_\tau(\varepsilon_u - \varepsilon_{\widehat{u}}) = (\Pi \, \mathrm{L} - \mathrm{L}, \mathrm{E}^{\mathrm{L}})_\Omega.$$

A new approximate velocity $\mathbf{u}_h^\star$ can be obtained which has the following properties:

- It is computed in an element-by-element fashion.
- $\mathbf{u}_h^\star \in \mathbf{H}(div, \Omega)$.
- $\nabla\cdot \mathbf{u}_h^\star = 0$ on Ω.
- $\|\mathbf{u}_h^\star - \mathbf{u}\|_\Omega \leq C\, C_\tau\, h^{\min\{k,1\}}\, \|\Pi\,\mathrm{L} - \mathrm{L}\|_\Omega + C\, h^{k+2} |\,\mathbf{u}\,|_{\mathbf{H}^{k+2}(\Omega)}$.

The incompressible Navier-Stokes equations. (N.C. Nguyen, J.Peraire and B.C., Math. Comp., JCP, 2011.)

The model problem.

Consider the model problem:

$$\begin{aligned} -\nu\Delta u + \nabla\cdot(u\otimes u) + \nabla p &= f && \text{in } \Omega,\\ \nabla\cdot u &= 0 && \text{on } \Omega,\\ \widehat{u} &= u_D && \text{on } \partial\Omega, \end{aligned}$$

where $\langle u_D\cdot\mathbf{n}, 1\rangle_{\partial\Omega} = 0$ and $(p, 1)_\Omega = 0$.

The incompressible Navier-Stokes equations.

Compact form of the HDG methods.

$(\mathrm{L}_h, \mathbf{u}_h, p_h, \widehat{\mathbf{u}}_h)$ is the element of $\mathrm{G}_h \times \mathbf{V}_h \times Q_h \times \mathbf{M}_h$ solving

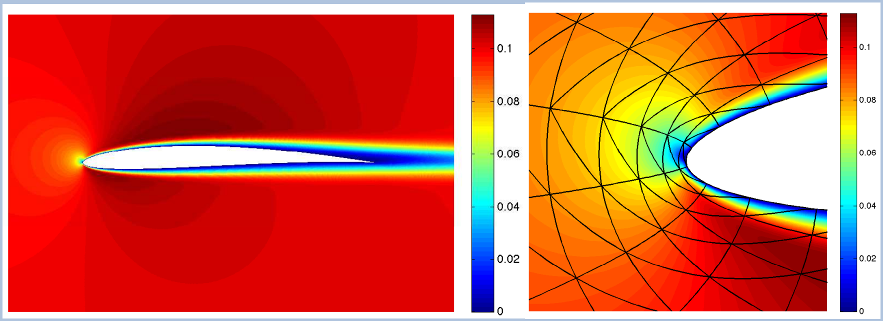
$$
\begin{aligned}
(\mathrm{L}_h, \mathrm{G})_{\Omega_h} + (\mathbf{u}_h, \nabla\cdot \mathrm{G})_{\Omega_h} - \langle \widehat{\mathbf{u}}_h, \mathrm{G}\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
(\nu \mathrm{L}_h, \nabla \mathbf{v})_{\Omega_h} - (\mathbf{u}_h \otimes \mathbf{u}_h, \nabla \mathbf{v})_{\Omega_h} & \\
-(p_h, \nabla\cdot\mathbf{v})_{\Omega_h} - \langle \nu \widehat{\mathrm{L}}_h\mathbf{n} + \widehat{\mathbf{u}}_h\, \widehat{\mathbf{u}}_h\cdot\mathbf{n} - \widehat{p}_h\mathbf{n}, \mathbf{v}\rangle_{\partial\Omega_h} &= (\mathbf{f}, \mathbf{v})_{\Omega_h},\\
-(\mathbf{u}_h, \nabla q)_{\Omega_h} + \langle \widehat{\mathbf{u}}_h\cdot\mathbf{n}, q\rangle_{\partial\Omega_h} &= 0,\\
\langle -\nu\widehat{\mathrm{L}}_h\mathbf{n} + \widehat{\mathbf{u}}_h\, \widehat{\mathbf{u}}_h\cdot\mathbf{n} + \widehat{p}_h\,\mathbf{n}, \boldsymbol{\mu}\rangle_{\partial\Omega_h\setminus\partial\Omega} &= 0\\
\langle \widehat{\mathbf{u}}_h, \boldsymbol{\mu}\rangle_{\partial\Omega} &= \langle \mathbf{u}_D, \boldsymbol{\mu}\rangle_{\partial\Omega}\\
(p_h, 1)_{\Omega} &= 0,
\end{aligned}
$$

for all $(\mathrm{G}, \mathbf{v}, q, \boldsymbol{\mu}) \in \mathrm{G}_h \times \mathbf{V}_h \times Q_h \times \mathbf{M}_h$, where

$$
-\nu\widehat{\mathrm{L}}_h\mathbf{n} + \widehat{p}_h\mathbf{n} = -\nu\mathrm{L}_h\mathbf{n} + p_h\mathbf{n} + \nu\,\tau\,(\mathbf{u}_h - \widehat{\mathbf{u}}_h) \quad \text{on } \partial\Omega_h.
$$

The compressible Navier-Stokes equations.

A numerical example.

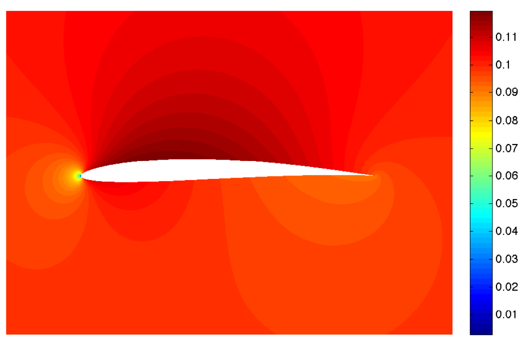


Viscous flow over a Kármán-Trefftz airfoil: $M_\infty = 0.1$, $Re = 4000$ and $\alpha = 0$. Mach number distribution (left) and detail of the mesh and Mach number solution near the leading edge region (right) using fourth order polynomial approximations.

(N.-C. Nguyen, J. Peraire and B.C., 2011.)

The Euler equations of gas dynamics.

A numerical example.



Inviscid flow over a Kármán-Trefftz airfoil: $M_\infty = 0.1$, $\alpha = 0$. Detail of the mesh employed (left) and Mach number contours of the solution using fourth order polynomial approximations (right).

(N.-C. Nguyen, J. Peraire and B.C., 2011.)

Ongoing work and open problems

- Other stabilization functions? Other choices of local spaces?
- Superconvergence for pyramidal, hexahedral elements?
- A posteriori error estimates: Only in terms of $u_h - \widehat{u}_h$ and τ?
- Efficient solvers: Domain decomposition methods?
- Stokes flow: Superconvergence with other formulations?
- Solid mechanics: Optimal convergence for all variables?
- Linear transport: Which unknowns superconverge?
- HDG methods for KdV equations: Superconvergence?
- Nonlinear hyperbolic conservation laws: How to deal with shocks?

The HDG methods.

References: Until 2005.

- Pian, Tong, Kikuchi, 1965: Hybrid methods. (There is a vast literature on these methods. See the book by Roberts and Thomas.)
- De Veubeque, 1965: Introduction of "hybridization" as a technique to implementing FEM for plates.
- Arnold and Brezzi, 1985: Analysis of the "hybridization" technique for mixed methods.
- C. and Gopalakrishnan, 2004: A new characterization of "hybridized" mixed methods.
- C. and Gopalakrishnan, 2005: Error analysis of variable-degree "hybridized" mixed methods for diffusion.
- C. and Gopalakrishnan, 2005. Hybridization of a (vorticity) mixed method for the Stokes problem.

The HDG methods.

References after 2005 (I am aware of!).

- Introduced in (B.C., J.Gopalakrishnan and R.Lazarov, SINUM, 2009.)
- Steady-state diffusion:
  - (B.C., J.Gopalakrishnan and H.Wang, Math. Comp., 2007.) Local conservativity of the CG method.
  - (B.C., B. Dong and J.Guzmán, Math. Comp., 2008.) The first superconvergent HDG method.
  - (B.C., J.Guzmán and H.Wang, Math. Comp., 2009.) Superconvergence of DG methods.
  - (B.C., J.Guzmán, S.-C.Soon and H.Stolarski, SINUM, 2009 .) Analysis of EDG methods.
  - (B.C., J.Gopalakrishnan and F.-J.Sayas, Math. Comp., 2010 .) Analysis of a superconvergent HDG method.
  - (F. Kikuchi and I.Oikawa, JSIAM Lett., 2010.) DG methods of hybrid type.
  - (F. Kikuchi, Japan Journal I.A.M., 2012.) DG methods of hybrid type.
  - (B.C., O.Dubois, J.Gopalakrishnan and S.Tan, Math. Comp., to appear.) Multigrid for HDG methods.
  - (M.Kirby, S.Sherwin and B.C., J. Sci. Comput., 2012.) Implementation of HDG methods.
  - (B.C., J.Guzmán and H.Wang, Math. Comp., 2009.) Superconvergence of DG methods.
  - (B.C., W.Qiu and K.Shi, Math. Comp., 2012 + SINUM, 2012.) Construction of superconvergent HDG methods
  - (Y.Chen and B.C., IMA,2012 + Math. Comp. 2014.) Variable-degree on nonconforming meshes.
  - (B.C. and M.Solano, SISC, 2012.) HDG methods with extensions from subdomains.
  - (B.C., F.-J.Sayas and M.Solano, SISC, 2012.) Coupling at a distance BEM with HDG methods .
  - (B.C. and W.Zhang, JSC, 2012 + SINUM, 2013.) A posteriori error estimates.
  - (B.C., R.Nochetto and W.Zhang, submitted.) Convergence of the adaptive method.

The HDG methods.

References after 2005 (I am aware of!).

- Time-dependent diffusion:
  - (B.Chabaud and B.C., Math. Comp., 2012.) Analysis of the semidiscrete case.
- The wave equation:
  - (N.C.Nguyen, J. Peraire and B.C., Math. Comp., JCP, 2011.) Devising HDG methods.
  - (N.C.Nguyen, J.Peraire and B.C., JCP, 2011.) Maxwell equations.
  - (Y. Griesmaier and P. Monk, J. Sci. Comput., 2012.) Helmholtz equation for low frequencies.
  - (B.C. and V.Queneville-Bélair, Math. Comp., 2nd. revision.) Analysis of the semidiscrete case.
  - (X. Feng and Y.Xing, Math. Comp., 2013.) Helmholtz equation for arbitrary frequencies.
- Convection -diffusion:
  - (B.C., B.Dong, J.Guzmán, R.Sacco and M.Restelli, SISC, 2009.) Devising an HDG method for the linear, steady-state case.
  - (N.C.Nguyen, J. Peraire and B.C., JCP, 2009. Devising another HDG method for linear and nonlinear convection and for the steady-state and time-dependent cases.
  - (H.Egger and J.Schöberl, IMA, 2010.) Using a mixed method for diffusion and DG for convection.
  - (S. Rheberghen and B.C., "80 years of the CFL Condition", 2013.) Devising space-time HDG methods.
  - (I.Oikawa, JSIAM, to appear.) DG methods of hybrid type.

The HDG methods.

References after 2005 (I am aware of!).

- Linear and nonlinear elasticity:
  - (S.-C.Soon, U. of M. PhD Thesis, 2008.) HDG methods for linear/nonlinear elasticity.
  - (S.-C.Soon, B.C. and H.Stolarski, JNME, 2009.) Devising HDG methods for linear elasticity.
  - (F. Kikuchi, K.Ishii and I.Oikawa, Theo. & Appli. Mech. Japan, 2009.) DG methods of hybrid type.
  - (B.C. and K.Shi, IMA, 2013.) Devising and analysis of superconvergent methods for linear elasticity.
  - (N.C.Nguyen and J.Peraire, JCP, 2012.) HDG methods for solid mechanics.
- Stokes flow:
  - (J. Carrero, B.C. and D. Schötzau, Math. Comp., 2006.) Hybridization of a DG method.
  - (B.C. and J. Gopalakrishnan, SINUM, 2009.) Devising HDG methods with vorticity formulation.
  - (N.C Nguyen, J. Peraire and B.C., JCP+CMAME, 2010.) Devising HDG methods with velocity gradient formulation+ comparison with other formulations.
  - (B.C., J. Gopalakrishnan, N.C.Nguyen, J. Peraire and F.-J. Sayas, Math. Comp., 2011.) Analysis of HDG methods.
  - (B.C. and K. Shi, Math. Comp., 2013.) Construction of superconvergent HDG methods.
  - (B.C. and J. Cui, Math. Comp., 2012+ J. Sci. Comput., 2012.) Analysis of HDG methods with a vorticity formulation.
  - (A, Cesmelioglu and B.C., J. Sci. Comput., 2013.) Analysis of HDG methods for Oseen.

The HDG methods.

References after 2005 (I am aware of!).

- Incompressible Navier-Stokes:
  - (N.C. Nguyen, J.Peraire and B.C., Math. Comp., JCP, 2011.) Devising HDG methods.
  - (S. Rheberghen and B.C., JCP, 2013.) Devising space-time HDG methods.
- Timoshenko beams, biharmonic:
  - (F.Celiker, B.C. and K.Shi, J. Sci. Comput. + Math. Comp., 2012.) Devising and analysis of HDG methods for the Timoshenko beam.
  - (B.C., B.Dong and J.Guzmán, J. Sci. Comput., 2009.) Devising and analysis of an HDG method for the biharmonic.
- Nonlinear conservation laws:
  - (N.C.Nguyen, J.Peraire and B.C., AIAA, 2011.) Devising HDG methods for the CNS equations.
  - (N.C.Nguyen, J.Peraire and B.C., AIAA, 2011.) Devising HDG methods for the Euler equations.
  - (N.C.Nguyen and J.Peraire, AIAA, 2012.) Devising HDG methods for the Euler equations with shocks.
  - (D. Moro, N.C.Nguyen and J.Peraire, IJNME, 2012.) Devising HDG methods for scalar conservation laws.