Classifying spaces for families of subgroups and their finiteness properties

Stefan Witzel

WWU Münster

LMS Durham Symposium, 13 August 2013

Outline

Actions with prescribed stabilizers

Let a group G act on a CW-complex X (always assume that G_σ fixes $\sigma \in X$ pointwise). Then X is a G-CW-complex.
Let $\mathcal{F}$ be a family of subgroups of G:

- $H \in \mathcal{F} \implies H^g \in \mathcal{F}, g \in G$,
- $H \in \mathcal{F}, K < H \implies K \in \mathcal{F}$.

We say that G **has stabilizers in** $\mathcal{F}$ if $G_\sigma \in \mathcal{F}$ for every $\sigma \in X$.

Example

- $\mathcal{F} = \{1\} \rightsquigarrow$ free actions.
- $\mathcal{F} = \{\text{finite subgroups}\} \rightsquigarrow$ proper actions.
- $\mathcal{F} = \{\text{virtually cyclic subgroups}\} \rightsquigarrow \ldots$

Classifying spaces for families of stabilizers

The category of G-CW-complexes up to equivariant homotopy equivalence has a terminal object $E_{\mathcal{F}}G$.
A model X for $E_{\mathcal{F}}G$ is a **classifying space for G with stabilizers in $\mathcal{F}$** and is characterized by

1. $X^H \cong$ pt. for $H \in \mathcal{F}$,
2. $X^H = \emptyset$ for $H < G, H \notin \mathcal{F}$.

Example

- $\mathcal{F} = \{1\}$: 1 $\Leftrightarrow X \cong$ pt., 2 $\Leftrightarrow$ free action.
  $\rightsquigarrow X$ classifying space (for free actions).
- $\mathcal{F} = \{$finite subgroups$\}$, G acts properly on a CAT(0)-cell complex X. Then 2 by assumption and 1 by CAT(0)-geometry. $\rightsquigarrow X$ is a model for $E_{\mathcal{F}}G$.

Homology

If X is a classifying space (for free actions) then

$$\ldots \to H_2(X^{(2)}, X^{(1)}) \to H_1(X^{(1)}, X^{(0)}) \to H_0(X^{(0)}) \to \mathbb{Z}$$

is a free resolution of the $\mathbb{Z}G$-module $\mathbb{Z}$.

Here $\mathcal{O}_{\mathcal{F}}G$ is the small category with objects $G/H, H \in \mathcal{F}$ and morphisms $G/H \to G/K^g, H \mapsto gK^g, g \in G, H < K$.

A (right) $\mathcal{O}_{\mathcal{F}}G$-module is a (contravariant) functor $\mathcal{O}_{\mathcal{F}}G \to \mathbf{Ab}$.

Example

- $\underline{\mathbb{Z}}$: $G/H \mapsto \mathbb{Z}, (G/H \to G/K) \mapsto (\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z})$
- $\underline{H}_n(X)$: $G/H \mapsto H_n(X^H)$
  $$(G/H \xrightarrow{g} G/K^g) \mapsto H_n(g^{-1}X^K \hookrightarrow X^H), H < K$$

Homology

If X is a classifying space with stabilizers in $\mathcal{F}$ then

$$\ldots \to \underline{H}_2(X^{(2)}, X^{(1)}) \to \underline{H}_1(X^{(1)}, X^{(0)}) \to \underline{H}_0(X^{(0)}) \to \underline{\mathbb{Z}}$$

is a free resolution of the $\mathcal{O}_{\mathcal{F}}G$-module $\underline{\mathbb{Z}}$.

Here $\mathcal{O}_{\mathcal{F}}G$ is the small category with objects $G/H, H \in \mathcal{F}$ and morphisms $G/H \to G/K^g, H \mapsto gK^g, g \in G, H < K$.

A (right) $\mathcal{O}_{\mathcal{F}}G$-module is a (contravariant) functor $\mathcal{O}_{\mathcal{F}}G \to$ **Ab**.

Example

- $\underline{\mathbb{Z}}$: $G/H \mapsto \mathbb{Z}, (G/H \to G/K) \mapsto (\mathrm{id} \colon \mathbb{Z} \to \mathbb{Z})$
- $\underline{H}_n(X)$: $G/H \mapsto H_n(X^H)$

  $(G/H \xrightarrow{g} G/K^g) \mapsto H_n(g^{-1}X^K \hookrightarrow X^H), H < K$

Finiteness properties

G is said to be **of type $F_n^{\mathcal{F}}$** if there is a model for $E_{\mathcal{F}}G$ whose n-skeleton is finite modulo the action of G.

G is said to be **of type $FP_n^{\mathcal{F}}$** if there is a projective resolution (by right $\mathcal{O}_{\mathcal{F}}G$-modules) of $\underline{\mathbb{Z}}$ whose first n terms are finitely generated.

Last slide: $F_n^{\mathcal{F}} \implies FP_n^{\mathcal{F}}$.

Example

1. G is $F_1^{\{1\}} \Leftrightarrow G$ is $FP_1^{\{1\}} \Leftrightarrow G$ is finitely generated.
2. G is $F_2^{\{1\}} \Leftrightarrow G$ is finitely presented.
3. G is $F_0^{\{finites\}} \Leftrightarrow G$ is $FP_0^{\{finites\}} \Leftrightarrow G$ has finitely many conjugacy classes of finite subgroups.

Essential trivialness

A directed system of $\mathcal{O}_{\mathcal{F}}G$-modules $(M_\alpha)_{\alpha\in D}$ is **essentially trivial** if

$$\varinjlim_{\alpha} \prod_{H\in\mathcal{F}} \prod_{I_H} M_\alpha(G/H) = 0$$

for any family of index sets I_H.

Observation

That $(M_\alpha)_\alpha$ be essentially trivial is equivalent to either of

- $\forall\alpha\ \exists\beta \geq \alpha$ *such that* $M_\alpha \to M_\beta$ *is trivial.*
- $\forall\alpha\ \exists\beta \geq \alpha\ \forall H \in \mathcal{F}\ M_\alpha(G/H) \to M_\beta(G/H)$ *is trivial.*

Brown's criterion

Theorem (Brown '87 $\mathcal{F} = \{1\}$, Fluch–W.)

Let G be a group and let $\mathcal{F}$ a family of subgroups. Assume that G acts on X such that

- $\underline{\tilde{H}}_i(X) = 0, 0 \leq i \leq n-1$,
- G_σ *is of type* $FP_{n-p}^{\mathcal{F}\cap G}$ *for every p-cell of* X.

Let $(X_\alpha)_{\alpha\in D}$ *be a cocompact filtration of* X*. Then* G *is of type* $FP_n^{\mathcal{F}}$ *if and only if* $(\underline{\tilde{H}}_i(X_\alpha))_\alpha$ *is essentially trivial for* $0 \leq i < n$.

Note: $(\underline{\tilde{H}}_i(X_\alpha))_\alpha$ is essentially trivial if

$$\forall \alpha \in D \quad \exists \beta \geq \alpha \quad \forall H \in \mathcal{F} \quad \tilde{H}_i(X_\alpha^H \to X_\beta^H) = 0$$

Proof sketch 1

Proposition (Bieri–Eckmann '74 $\mathcal{F} = \{1\}$, Martínez-Pérez–Nucinkis '11)

Let G be a group and $\mathcal{F}$ a family of subgroups. These are equivalent:

1. *G is of type $FP_n^{\mathcal{F}}$,*
2. *for any index sets $I_H, H \in \mathcal{F}$ the morphism*

$$H_i^{\mathcal{F}}(G, \prod_{H\in\mathcal{F}} \prod_{I_H} \mathbb{Z}[G/H, -]) \to \prod_{H\in\mathcal{F}} \prod_{I_H} H_i^{\mathcal{F}}(G, \mathbb{Z}[G/H, -])$$

   is an isomorphism for $i < n$ and an epimorphism for $i = n$.
3. *LHS=0 in 2 for $0 < i < n$ and 2 for $i = 0$.*

Proof sketch 2

$i > 0$:

$$\overbrace{H_i^{\mathcal{F}}(G, \prod \mathbb{Z}[G/H, -]) \cong H_i^{\mathcal{F}}(X, \prod \mathbb{Z}[G/H, -])}^{\underline{\tilde{H}}_j(X)=0,\, j<n}$$

$$\cong$$

$$\varinjlim \underbrace{\prod \underline{H}_i(X_\alpha)(G/H) \cong \varinjlim H_i^{\mathcal{F}}(X_\alpha, \prod}_{X_\alpha \text{ cocompact, } G_\sigma \text{ of type } FP_{n-p}^{\mathcal{F}\cap G_\sigma}} \mathbb{Z}[G/H, -])$$

$= 0$ if and only if $\underline{H}_i(X_\alpha)$ essentially trivial.

Generalizations of Abels's groups

Let $v, w \in \mathbb{Z}^{n+1}$ be such that

- $(v_i)_i$ and $(w_i)_i$ are monotonically decreasing,
- $\sum_i v_i > 0$ and $\sum_i w_i \leq 0$.

Consider

$$G = \left\{ \begin{pmatrix} d_1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & d_{n+1} \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{Z}[1/p]) \;\middle|\; \begin{matrix} \prod_i d_i^{v_i} = 1 \\ \prod_i d_i^{w_i} = 1 \end{matrix} \right\} .$$

Finiteness properties of generalized Abels's groups

- The group G acts on a Bruhat–Tits building.
- Applying Brown's criterion:
  G is of type F_{n-1} but not of type F_n.
- Applying the generalization of Brown's criterion:
  For any chosen $0 < m \leq n$ on can arrange v, w so that G to has type $F_{m-1}^{\{\text{finites}\}}$ but not type $F_m^{\{\text{finites}\}}$.