Jacobians of degenerate Riemann surfaces and graphs

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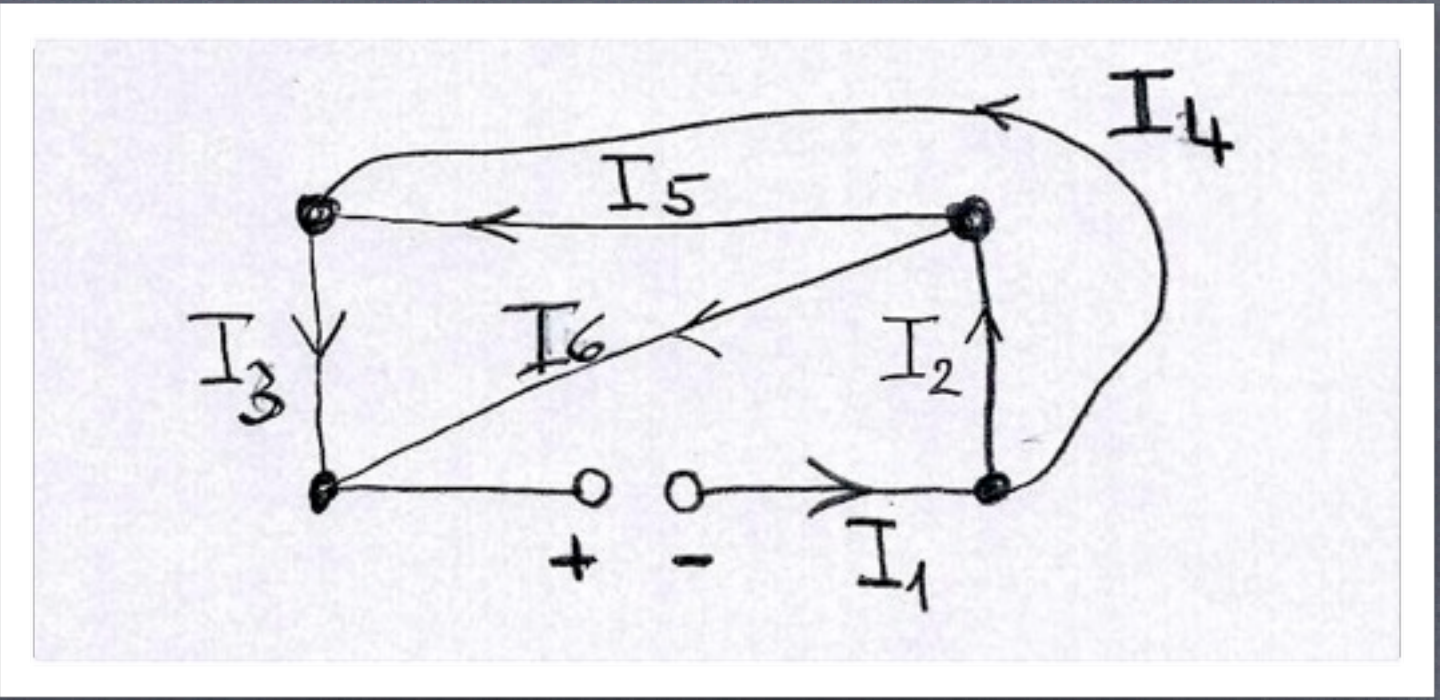
In physics we learned the following rule for electric circuits:



"At any node the sum of all (oriented) currents is zero"

Some history

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1) In 1973 (Springer Lect. Notes 352) Fay studied Jacobians when a sequence of Riemann surfaces converges to a Riemann surface with nodes. The convergence is described in algebraic terms.

2) We wanted to understand this from the point of view of hyperbolic geometry.

3) In 1974 (Ann. of Math. Stud. 79) Linda Keen proved the Collar Lemma: If a geodesic shrinks to zero the width of its collar goes to infinity. This leads to a graph whose edges stand for the collars and whose vertices stand for the remaining thick parts.

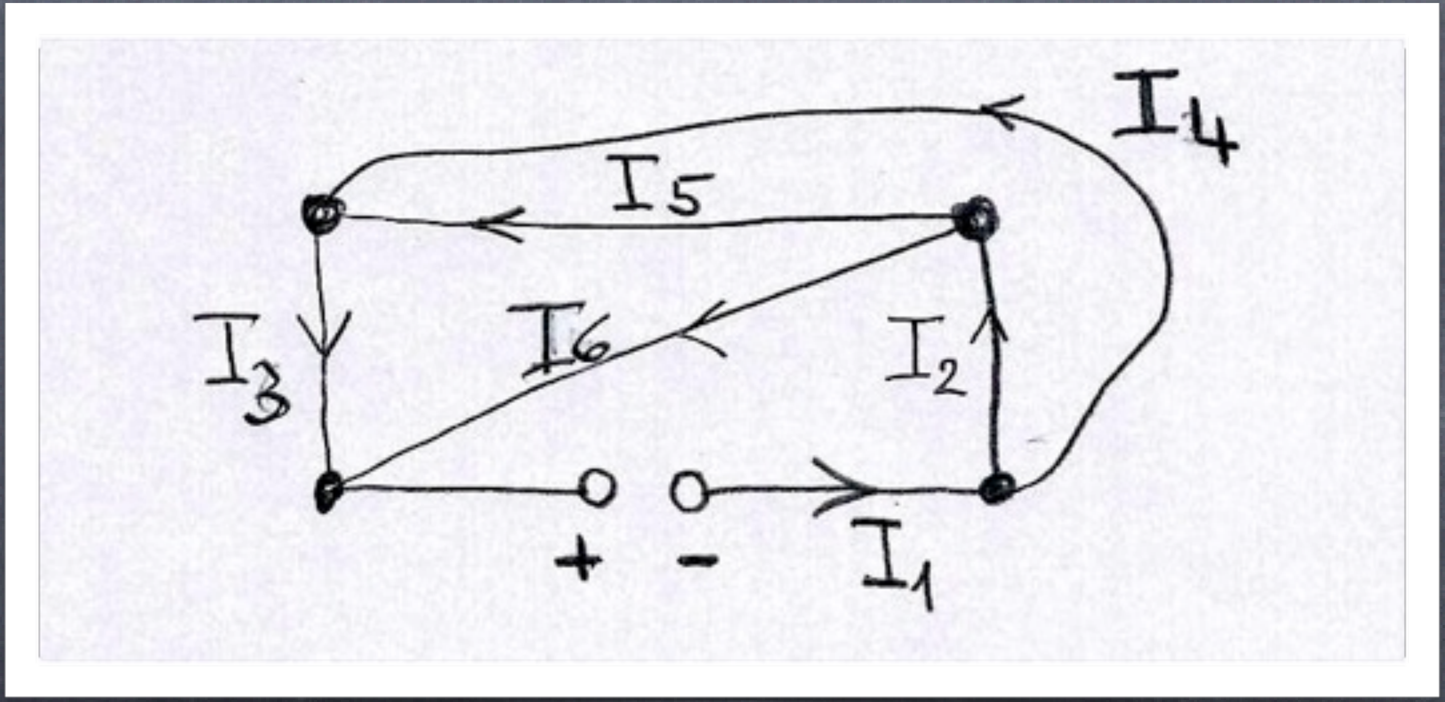
Some history

4) In 1989 (Comment. Math. Helv. 64) Courtois and Colbois showed that if the lengths of a set of pairwise disjoint geodesics go to zero then the small eigenvalues of the Laplace operator of the surface converge (after proper rescaling) to the eigenvalues of the Laplace operator of the corresponding graph.

5) We try something similar for the Jacobians.

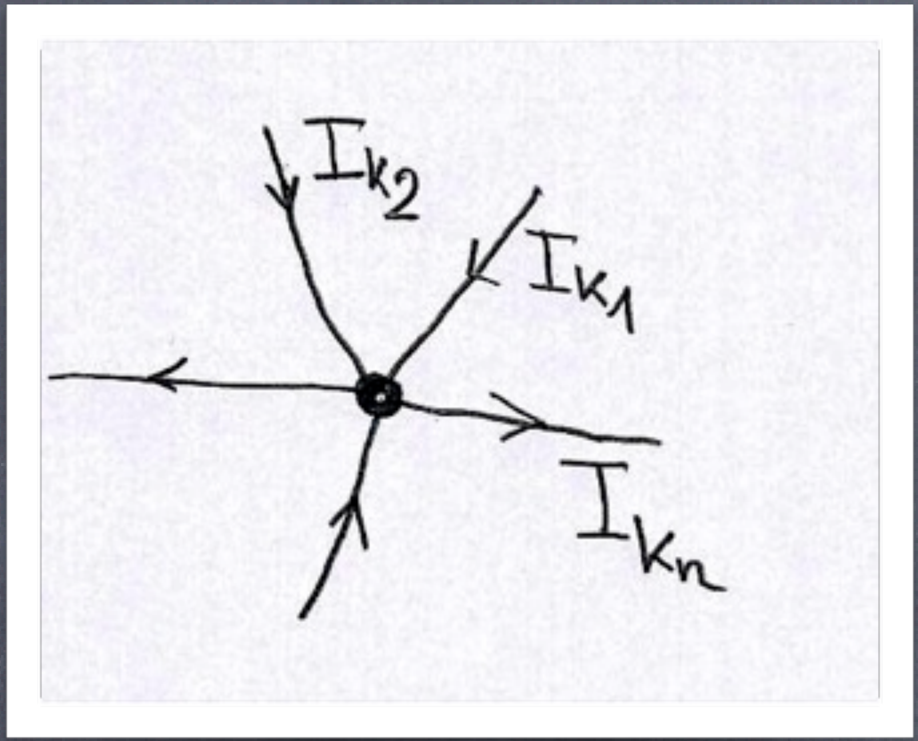
Back to the circuits:

In physics we learned the following rule for electric circuits:



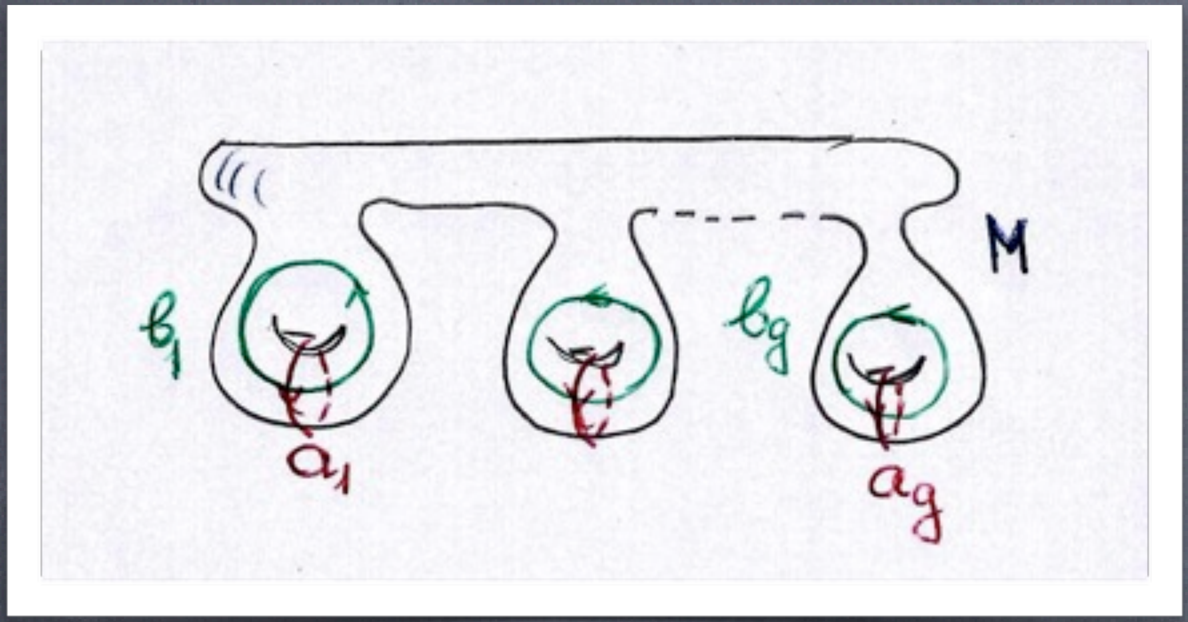
"At any node the sum of all (oriented) currents is zero"

We stumbled over this while studying Jacobians of Riemann surfaces computationally.



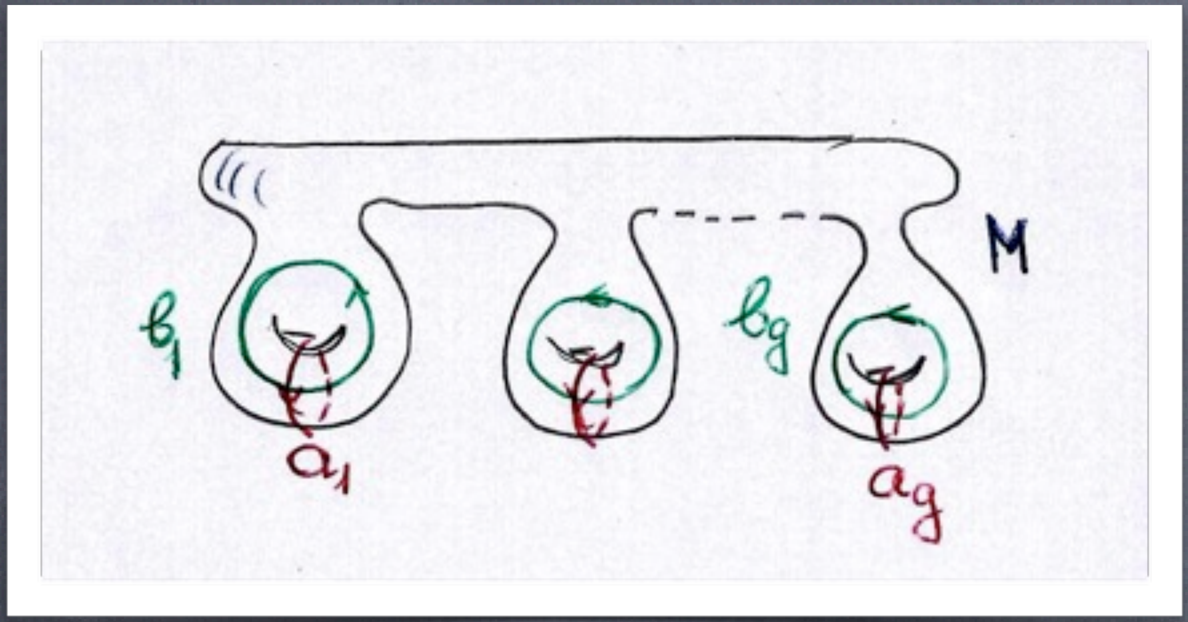
$$\mathbf{I}_{k_1} + \mathbf{I}_{k_2} + \cdots + \mathbf{I}_{k_n} = 0$$

"At any node the sum of all (oriented) currents is zero"



More serious business now.

On a compact Riemann surface M we choose a canonical homology basis



$$a_1, b_1, \ldots, a_g, b_g$$

$$= a_1, a_{g+1}, \ldots, a_g, a_{2g}$$

Then there exists the dual basis of harmonic 1-forms $\sigma_1, \ldots, \sigma_{2g}$ satisfying

$$\int_{a_i} \sigma_j = \delta_{ij},\ i, j = 1, \ldots, 2g.$$

The $2g \times 2g-$ matrix

$$P = \left(p_{kl}\right) \quad \text{with} \quad p_{kl} = \int_M \sigma_k \wedge *\sigma_l$$

sets up a Riemannian metric on the torus $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$.

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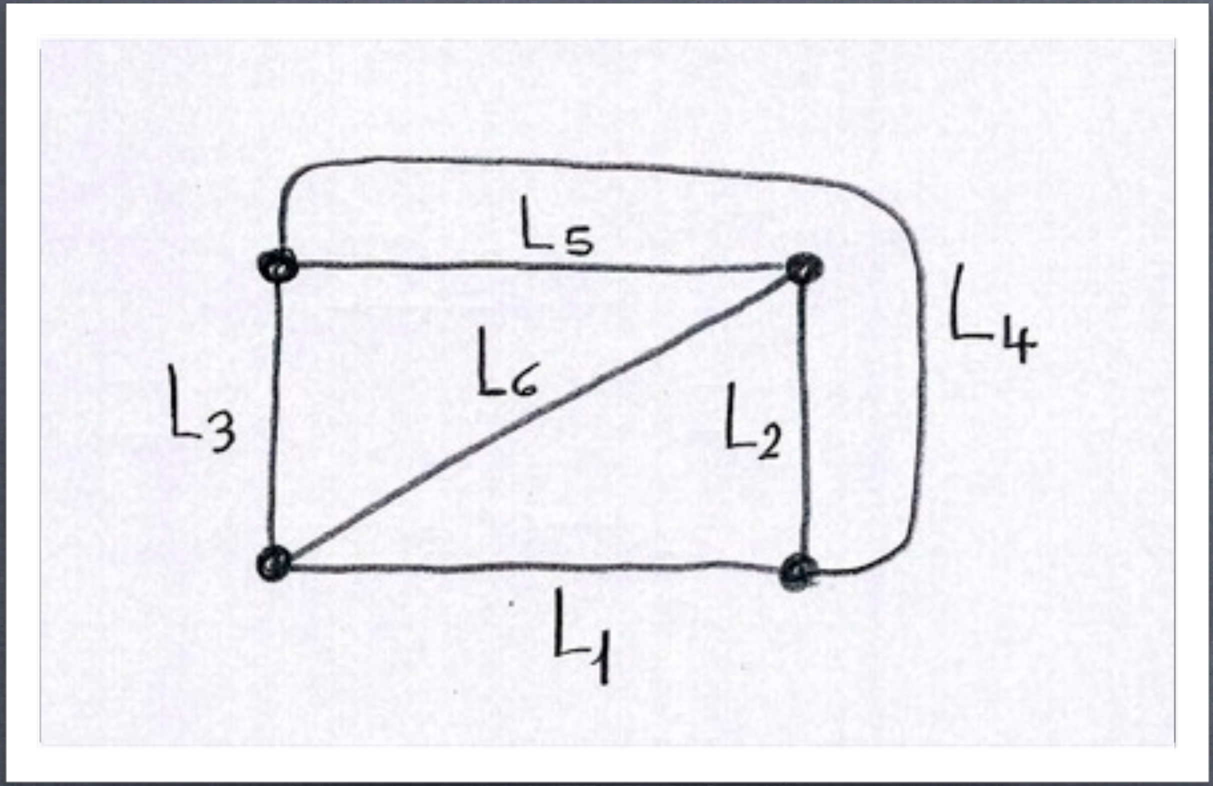
The torus with this metric is the **Jacobian** of M.

Problem: Compute this in the limit when certain geodesics shrink to a point.

A **model** for this problem:

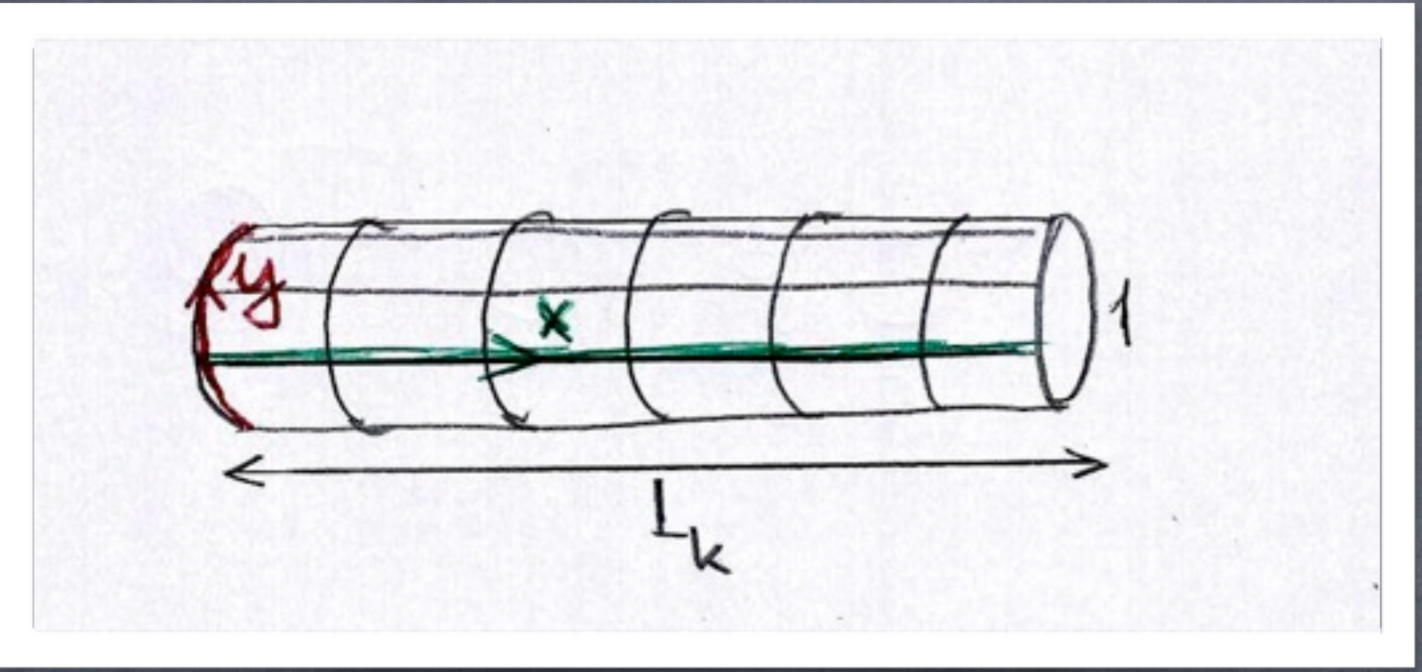
Take a finite graph $G = (V, E)$ with



$v = |V|$ vertices

$e = |E|$ edges

Choose positive real numbers (lengths) $L_1, \ldots, L_e$ and build flat cylinders.



$$C_k = \left\{ (x, y) \middle| 0 \le x \le L_k, y \in \mathbb{R} / \langle x \mapsto x + 1 \rangle \right\}$$

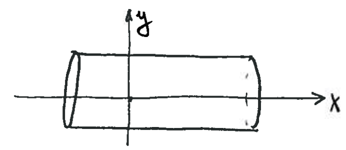
with the
Euclidean metric

$$ds^2 = dx^2 + dy^2.$$

This sheet indicates that the harmonic forms on a cylinder with zero periods may be calculated explicitly, but we do not need the details in this lecture.

Computations for the Jacobian paper

① Harmonic forms on the flat cylinder



$$C = [x_0, x_1] \times \mathbb{R}/[y \mapsto y+2\pi]$$

Any harmonic function $h(x,y)$ on C is a series

$$h(x,y) = a_0 + b_0 x + \sum_{n=1}^{\infty} \cos(ny)\left(a_n e^{-nx} + c_n e^{nx}\right) + \sum_{n=1}^{\infty} \sin(ny)\left(b_n e^{-nx} + d_n e^{nx}\right)$$

Its differential:

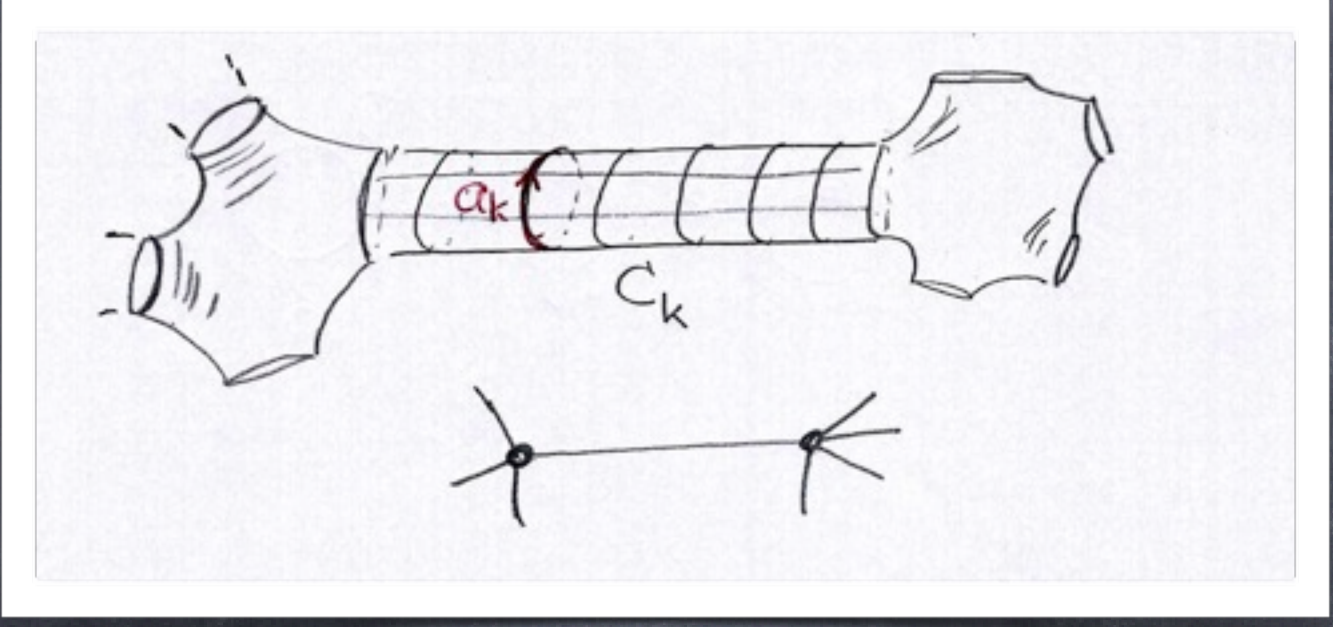
$$dh = \left\{ b_0 + \sum_{n=1}^{\infty} n \cos(ny)\left(-a_n e^{-nx} + c_n e^{nx}\right) + \sum_{n=1}^{\infty} n \sin(ny)\left(-b_n e^{-nx} + d_n e^{nx}\right) \right\} dx$$

$$+ \left\{ \sum_{n=1}^{\infty} n \sin(ny)\left(-a_n e^{-nx} - c_n e^{nx}\right) + \sum_{n=1}^{\infty} n \cos(ny)\left(b_n e^{-nx} + d_n e^{nx}\right) \right\} dy$$

$$= A(x,y)\, dx + B(x,y)\, dy$$

Replace the vertices by genus 0 surfaces with boundaries of length 1, and insert the cylinders according to the graph G (**Lego**). The resulting surface **M** has genus

$$g = e - v + 1.$$

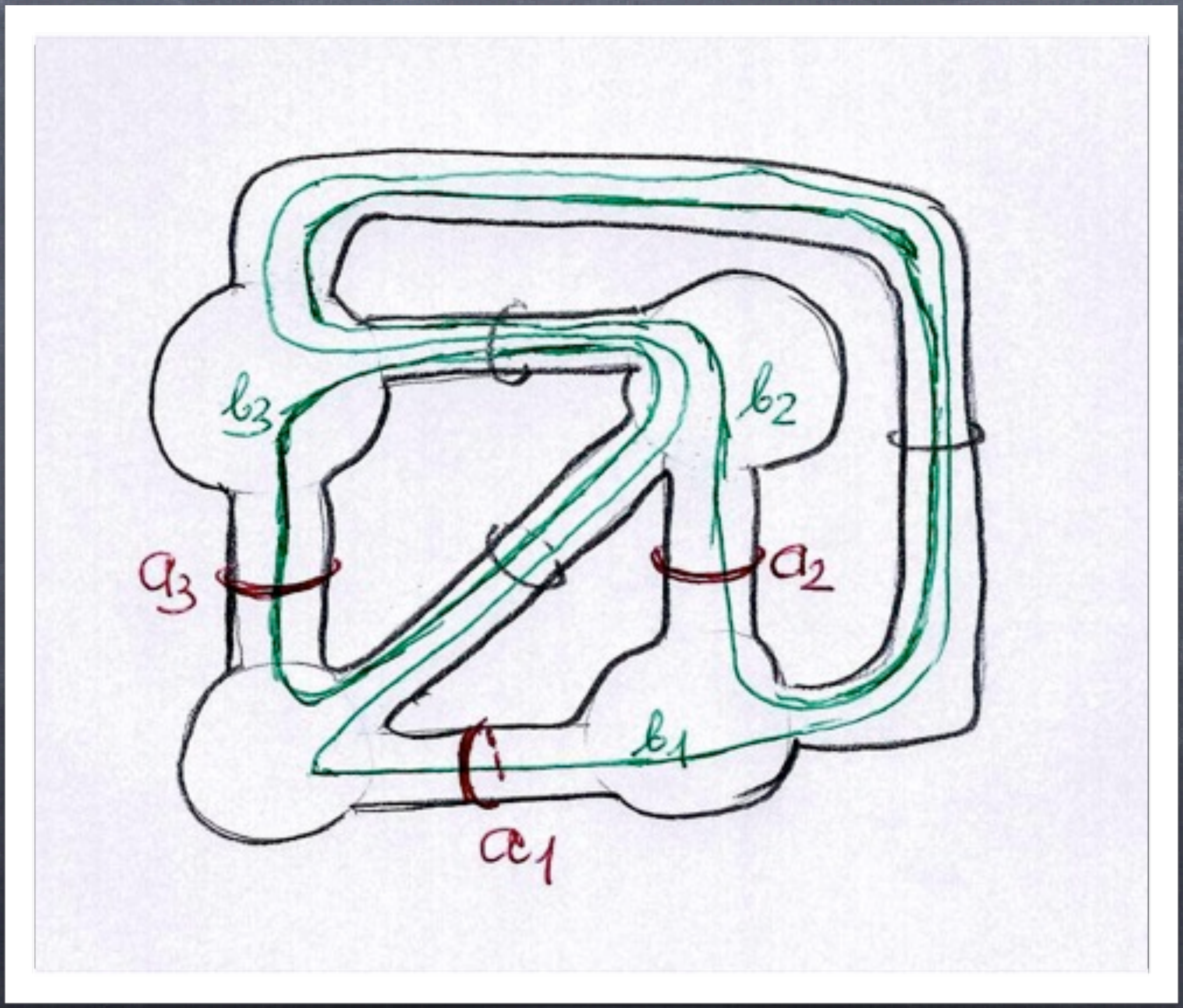


Each C_k is directed according to its x-axis.

We choose simple closed curve a_k oriented according to the y-axis.

We may enumerate the edges in such a way that the first g curves $a_1, \ldots, a_g$ can be completed to a homology basis

$a_1, \ldots, a_g, b_1, \ldots, b_g.$



For $k = 1,\ldots,e$ we define the differential forms

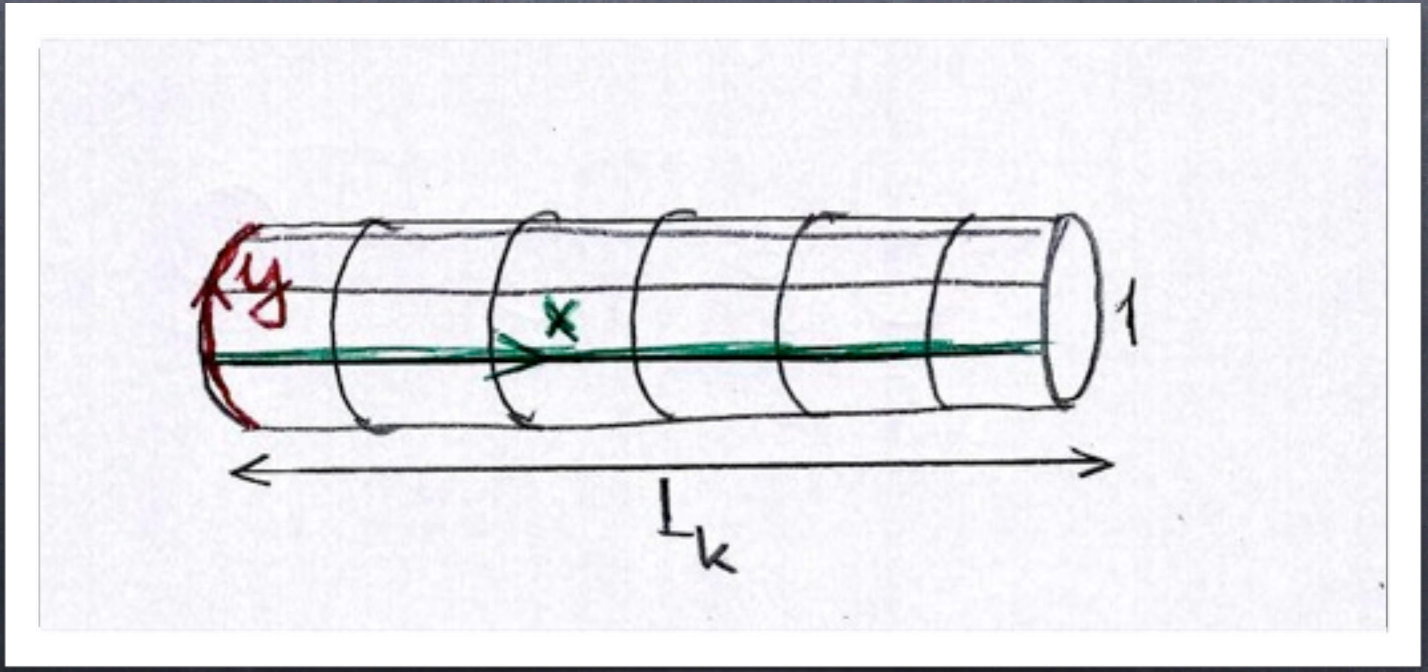
$$s_k = \begin{cases} dx & \text{on } C_k \\ 0 & \text{elsewhere} \end{cases}$$

and look at the linear combination

$$\omega = \sum_{k=1}^{e} \lambda_k s_k .$$

The non-smoothness is not a problem.

(hmmm...)



The differential $\omega = \sum_{k=1}^{e} \lambda_k s_k$ is **harmonic** if it minimizes the energy in its cohomology class.

It turns out that this is the case if and only if, at each vertex,

$$\sum_{j=1}^{n} \frac{\lambda_{k_j}}{L_{k_j}} = 0.$$

This yields a space W_1 of **"mimicking harmonic forms of the first kind".**

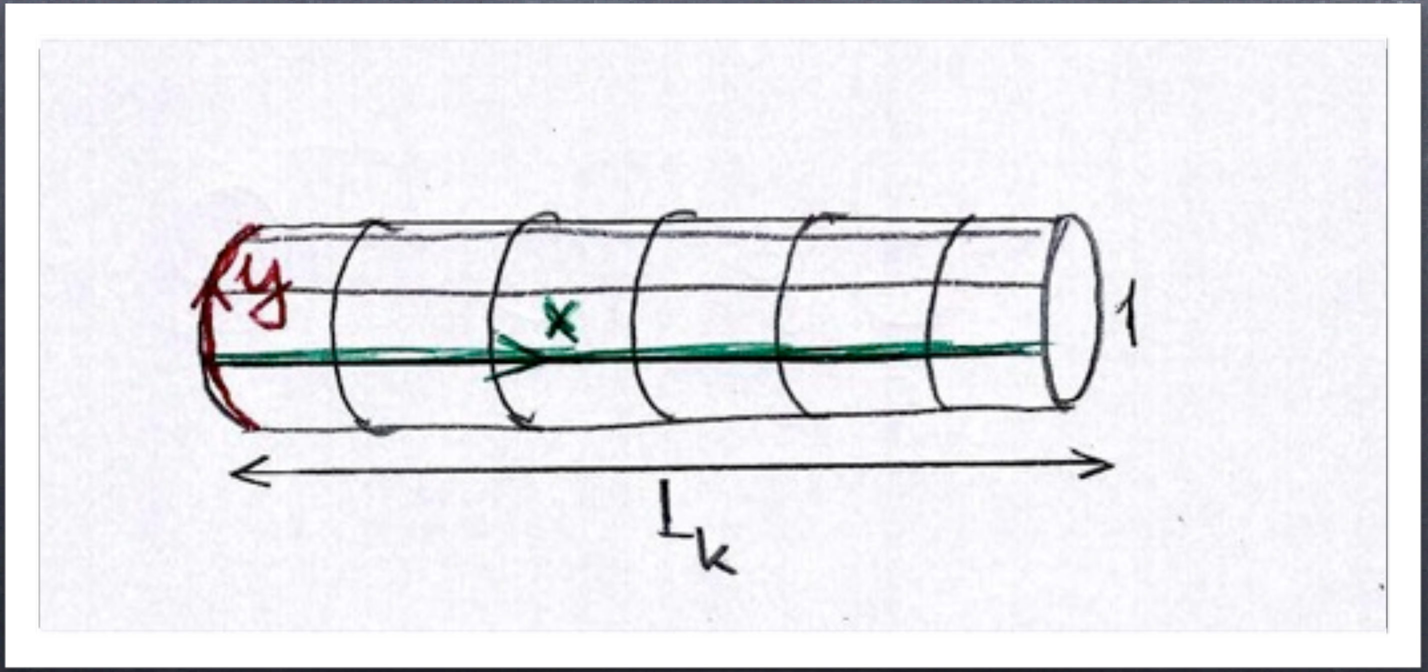
Corollary $\dim W_1 = g.$

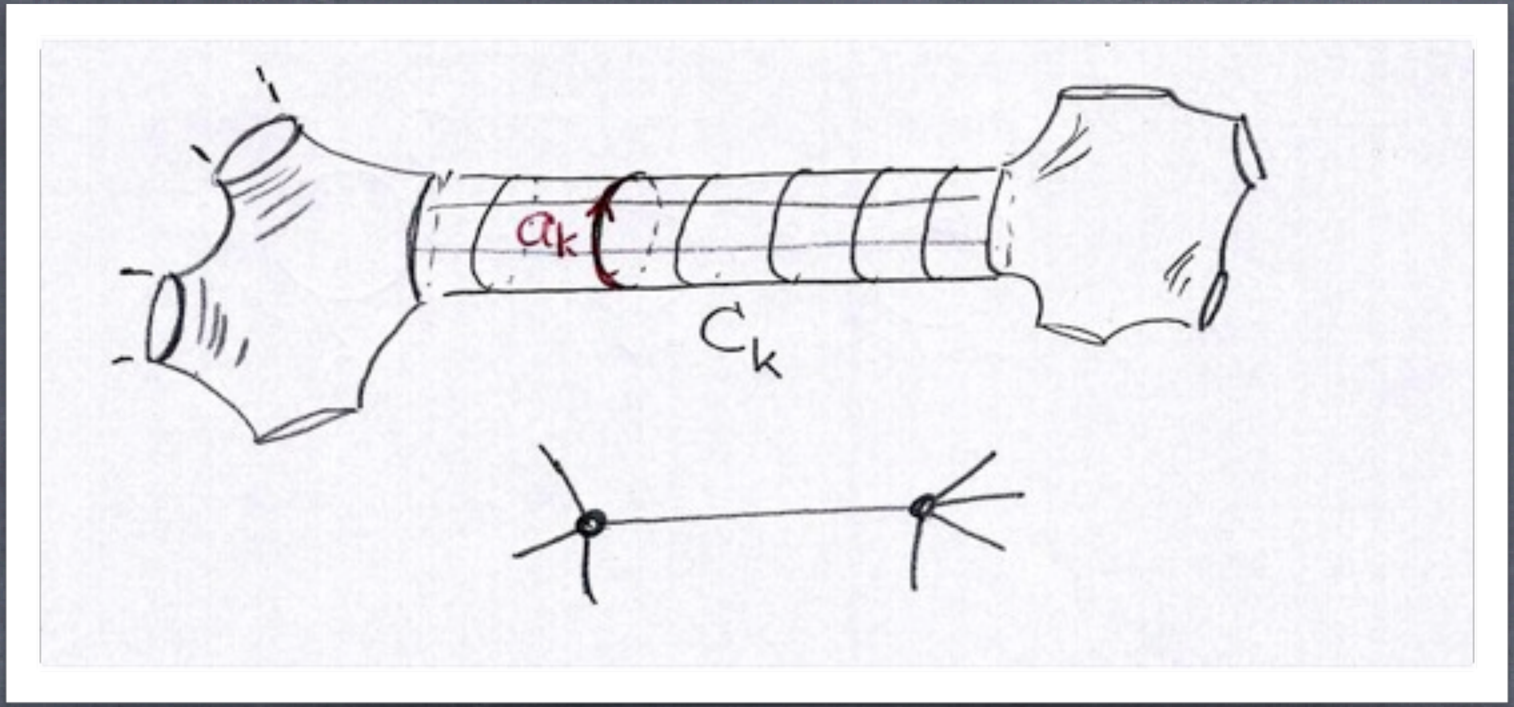
Let us define the space W_2 of

"mimicking harmonic forms of the second kind".

For $k = 1, \dots, e$ let $t_k = \begin{cases} dy & \text{on } C_k \\ 0 & \text{elsewhere} \end{cases}$

and look at the linear combinations $\eta = \sum_{k=1}^{e} \mu_k t_k$.





The defining condition of mimicking **closed** forms is that on each "vertex surface" X we have

$$\int_{\partial X} \eta = 0.$$

In the interiors of the X's we define $\eta = 0$. (hmmm...)

Corollary $\dim W_2 = g.$

Construction done!   Questions:

1. Is span(W_1, W_2) the correct model for limit Riemann surfaces?

2. Should one introduce a Jacobian of dimension 2g for the graph G?

3. The condition $\int_{\partial X} \eta = 0$ for the mimicking forms of the second kind makes one think of Kirchhoff's rules. Is there a connection?

For 3. we made computational experiments:

What is the resistance of the electric circuit G, assuming that each edge has resistance $\frac{1}{L_k}$?

The physics approach is to introduce unknown potentials $U_1, \ldots, U_v$ on the vertices, compute the corresponding currents $I_1, \ldots, I_e$ according to Ohm's law and then solve for the U_k such that Kirchhoff's rules are satisfied. The resistance becomes

$$R = U/I_1,$$

where U is the voltage of the battery.

When we computed numerically the mimic harmonic form σ_{g+1} from the dual basis (which is of the first kind) we found

$$\text{energy}(\sigma_{g+1}) = I_1$$

Hence a relation between Kirchhoff's rules and the harmonic forms of the first kind. For the forms of the second kind we have not stumbled over such a relation as yet.

End

but not the end