Difference Equations on Graphs

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LMS Symposium on Graph Theory

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First LMS Durham Symposium Schedule

ON GLOBAL RIEMANNIAN GEOMETRY

11th - 22nd July 1974

P R O G R A M M E

Tuesday 16th

9.30 - 10.45	Dr. W. Schmid	"Moduli of Algebraic manifolds" (continued)
11.15 - 12.30	Professor M.F.Atiyah	"Eigenvalues of the Laplacian - Introduction and Historical Survey"
5.00 - 6.15	Professor H.Duistermaat I	"Eigenvalues and Closed Geodesics"

Thursday 18th

9.30 - 10.45	Professor H.Duistermaat II	"Clustering of Eigenvalues"
11.15 - 12.30	Professor I.M.Singer	"Reidemeister Torsion"
5.00 - 6.15	Mr. J. Dodziuk	"Combinatorial Laplacians"

Laplacian in Graph and Riemannian Settings

In Riemannian Geometry the Laplacian on functions is defined as

$$\Delta u = -\frac{1}{\sqrt{g}} \sum \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial u}{\partial x^j} \right) = -g^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + \dots$$

where g_{ij} are components of the metric tensor, $g = \det(g_{ij})$, and $(g^{ij}) = (g_{ij})^{-1}$. Thus the Laplace operator contains in it the complete information about the geometry.

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$$Lu(x) = \sum_{y \sim x} (u(x) - u(y)) = -m(x) \left(\frac{1}{m(x)} \left(\sum_{y \sim x} u(y) \right) - u(x) \right).$$

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Note that according to the first expression $z \sim x$ if and only if $L\delta_z(x) = -1$. Thus the combinatorial Laplacian contains the full information about the graph.

A continuous analog of the second expression for Lu.

We do it in $\mathbb{R}^2$ to simplify the notation. By Taylor's Theorem

$$u(x,y) = u(x_0,y_0)+(xu_x+yu_y)+(1/2)(x^2u_{xx}+2xyu_{xy}+y^2u_{yy})+O(r^3)$$

where the subscripts denote partial derivatives and partials are evaluated at (x_0, y_0).

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$$\frac{1}{2\pi r}\int_{C_r} u\,ds = u(x_0,y_0) - \frac{r^2}{4}\Delta u(x_0,y_0) + O(r^3).$$

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$$\Delta u(x_0,y_0) = -\lim_{r\to 0}\frac{4}{r^2}\left(\frac{1}{2\pi r}\int_{C_r} u\,ds - u(x_0,y_0)\right).$$

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This is analogous to our second expression for Lu

$$Lu(x) = -m(x)\left(\frac{1}{m(x)}\left(\sum_{y\sim x} u(y)\right) - u(x)\right).$$

The Maximum Principle

Proposition. Suppose $Lu \geq 0$ and for every $y \sim x$ $u(y) \geq u(x)$. Then $u(y) = u(x)$ for every neighbor y of x.

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Proof.

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The proposition ought to be called "the minimum principle". Applying it to $-u$ and reversing all inequalities one obtains "the maximum principle." In particular, a harmonic function ($Lu = 0$) cannot attain an "interior" extremum.

Harnack inequality

Proposition. Suppose $x \sim y$ are two neighboring vertices of K and $u \geq 0$ is a function on V with $Lu(x) \geq 0$ and $Lu(y) \geq 0$. Then

$$\frac{1}{m(y)} u(x) \leq u(y) \leq m(x)u(x).$$

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Proof.

$$0 \leq Lu(x) = m(x)u(x) - \sum_{x \sim z} u(z) \leq m(x)u(x) - u(y).$$

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The last two spaces become Hilbert spaces if equipped with the natural inner products

$$(u,v) = \sum_{x \in V} u(x)v(x) \quad \text{and} \quad (\phi,\psi) = \frac{1}{2} \sum_{[x,y] \in \tilde{E}} \phi([x,y])\psi([x,y])$$

respectively.

L^2, self-adjointness, and the spectrum

There are natural maps from functions to cochains and back.

$$du([x,y]) = u(y) - u(x) \quad \text{and} \quad d^*\phi(x) = \sum_{y\sim x} \phi([y,x])$$

which are adjoints with respect to the inner products defined above. d is the difference analog of the gradient while $-d^*$ is the analog of the divergence.

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Clearly

$$(Lu, v) = (d^*du, v) = (du, dv)$$

if at least one of u, v has finite support.

L^2, self-adjointness, and the spectrum - continued

Our graphs will be always connected and for the most part infinite. If the valence function $m(x)$ is bounded, the Laplacian is a bounded operator on on $\ell_2(K)$. It is also symmetric and hence self-adjoint.

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This is analogous to the fact that the Laplacian Δ on a complete Riemannian manifold M with the domain $C_0^\infty(M)$ is essentially self-adjoint.
In view of the theorem we can talk unambiguously about the spectrum of L and derive invariants of the graph from it. In particular,

$$\lambda_0(K) = \inf\{\lambda \in \mathrm{Spec}(L)\} = \inf\left\{\frac{(du, du)}{(u, u)} \mid u \in C_c^0(K) \setminus \{0\}\right\}$$

is a very important one.

Cheeger's constant and bounds on $\lambda_0(K)$

Define, for a finite subgraph N of K,

$$h(N) = \frac{\#\{x \in V \mid x \in N, \exists y \in V, y \notin N, y \sim x\}}{\#\{y \in V \mid y \in N\}} = \frac{L(\partial N)}{A(V)}$$

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Theorem. Suppose K satisfies $m(x) \leq m$ for all $x \in V$. Then

$$\frac{h^2}{2m} \leq \lambda_0(K) \leq h.$$

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- An analog for finite graphs is more important and gave rise to an explosion of research on expanding graphs.
- The appearance of m in the denominator of the lower bound is counterintuitive. Understanding the formulation that would not have this defect came in a recent work of Bauer, Keller and Wojciechowski using the new notion of intrinsic metric on a graph to modify the way that the size of the boundary is measured.

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The appropriate isoperimetric (Cheeger) constant is

$$h = h(K) = \inf_{N \subset V, |N| \leq (1/2)|V|} \frac{L(\partial N)}{A(N)}$$

and the estimates above hold for λ_1 and h i.e.

$$\frac{h^2}{2m} \leq \lambda_1(K) \leq h.$$

It is worth pointing out that the two results (about λ_0 for infinite graphs and λ_1 for finite ones, at least the lower bounds, are proved in practically the same way.

To see the connection note that the first eigenfunction ϕ_1 of L is perpendicular to constants, i.e. $\sum_{x \in V} \phi(x) = 0$. Replacing ϕ by its negative if necessary, we can assume that $\#\{x \in V \mid \phi(x) > 0\} \leq (1/2)\#V$.

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$$\lambda_1 = \frac{(L\phi, \phi_+)}{(\phi_+, \phi_+)} \geq \frac{(L\phi_+, \phi_+)}{(\phi_+, \phi_+)} = \frac{(d\phi_+, d\phi_+)}{(\phi_+, \phi_+)}$$

since for points x where $\phi(x) > 0$ $L\phi(x) \geq L\phi_+(x)$.

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since for points x where $\phi(x) > 0$ $L\phi(x) \geq L\phi_+(x)$.Thus to give a lower bound for λ_1 we estimate the Rayleigh-Ritz quotient of a function with finite support which is precisely what we need to do to estimate λ_0 in the case of an infinite graph.

Surjectivity of the Laplacian

T. Ceccherini-Silberstein, M. Coornaert, JD

Theorem. Suppose K is an infinite connected graph. Then $L : C^0(K) \longrightarrow C^0(K)$ is surjective.

Remarks.

- For finite graphs, the image of L is perpendicular to constants.
- The proof uses only the maximum principle. Therefore the theorem holds for a large class of operators.

Outline of proof. Consider the equation $Lu = f$ for a fixed, arbitrary f.

Step 1. Take an exhaustion of the graph by finite subgraphs and solve the difference equation on each subgraph.

Step 2. Pass to the limit to get the solution on the whole graph.

Step 1

Fix $x_0 \in V$ and consider $B_n = \{x \in V \mid d(x, x_0) \leq n\}$. Let K_n be the full subgraph with B_n as the set of vertices. Let F_n be the set of all real-valued functions on B_n. Define $L_n : F_n \longrightarrow F_n$ as follows.

$$L_n u = (L\tilde{u})\,|_{B_n}$$

where $\tilde{u}$ denotes the extension by zero of u to V.

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Lemma. L_n is surjective.

Proof.

We show that L_n is injective. Suppose $u \in F_n$ is in the kernel of L_n. Then $\tilde{u}$ is harmonic on B_n and vanishes on its boundary. By the maximum principle $\tilde{u}$ and hence u are identically zero. □

Fix $f \in C^0(K)$. The lemma implies that the set $S_n = \{u \in F_n \mid L_n u = f|_{B_n}\}$ is nonempty for every n.

Step 2

For $m \geq n$, let $r_{m,n} : F_m \longrightarrow F_n$ be the restriction $r_{m,n}u = u|_{B_n}$. Consider the sets

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These sets are affine subspaces of F_n, are nonempty by the Lemma, and form a decreasing sequence, i.e. $X_{m+1,n} \subset X_{m,n}$. It follows that they stabilize in the sense that there exists $m_0(n)$ such that $X_{m_0(n),n} = \bigcap_{m \geq n} X_{m,n} =: U_n$.

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Lemma. For every $n \geq 1$, $r_{n+1,n} : U_{n+1} \longrightarrow U_n$ is surjective.

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Lemma. For every $n \geq 1$, $r_{n+1,n} : U_{n+1} \longrightarrow U_n$ is surjective.

Proof.

Take $u \in U_n$ and choose $m \geq \max\{m_0(n), m_0(n+1)\}$. There exists $v \in S_m$ such that $r_{m,n}v = u$. Now $u' = r_{m,n+1}v \in U_{n+1}$ and $r_{n+1,n}u' = u$. □

Now take $u_1 \in U_1$ and choose inductively $u_{n+1} \in U_{n+1}$ so that $u_{n+1}|_{B_n} = u_n$. Then define u on V by $u(x) = u_n(x)$ if $x \in B_n$. Clearly, u is well defined and satisfies $Lu = f$.