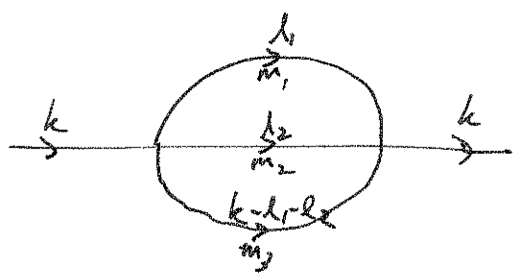
Higher normal functions as Feynman integrals

In this talk I'd like to describe an approach to the Bloch–Vanhove computation reported on yesterday by David, which seems promising for working a lot of similar integrals, and for going to higher dimensions and non-modular settings. The idea is to let the computations be guided by an algebraic cycle, and comes from work I did with C. Doran a few years ago. ([DK] = "Alg. K-theory of toric hypersurfaces", CNTP 5 (2011), pp. 397-600)

Recall from David's talk the 2-loop sunset integral in 2 dimensions

QD context



$$\int_{\mathbb{R}^4} \frac{d^2\ell_1\, d^2\ell_2}{(\ell_1^2-m_1^2+i\epsilon)(\ell_2^2-m_2^2+i\epsilon)((\ell_1+\ell_2-k)^2-m_3^2+i\epsilon)}$$

which becomes

upon the introduction of Schwinger parameters (and setting $k^2 =: 1/t$)

$$I(t;\underline{m}) := \int_0^\infty\int_0^\infty \frac{dx\,dy}{\underbrace{(m_1^2x+m_2^2y+m_3^2)(x+y+xy) - \tfrac{1}{t}xy}_{=\Phi}}$$

For most of the talk I will set masses equal, viz. $I(t)$.

Theorem 1 (Laporta, Remiddi; Broadhurst): $I(t)$ satisfies the inhomogeneous equation

$$\left\{t(9t-1)(t-1)\partial_t^2 + (3t-1)(3t+1)\partial_t + \frac{1-3t}{t}\right\} I = -\frac{6}{m^2}$$

Others have worked on this example from an AG perspective: Adams, Bogner, Weinzierl, Müller-Stach, Zayadeh

(I'll give a bit more context than "strictly necessary") ②

To state the next result, we rewrite $\frac{-t}{xy}$ times the denominator:

$$1 - t(x+y+1)(x^{-1}+y^{-1}+1) \; =: \; 1 - t\,\phi(x,y)$$

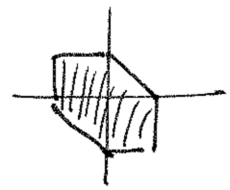
"Laurent polynomial"

and recognize

$$⑥ \quad E_t := \overline{\{(x,y) \in \mathbb{C}^* \times \mathbb{C}^* \mid 1 - t\,\phi(x,y) = 0\}} \subset \mathbb{P}_\Delta$$

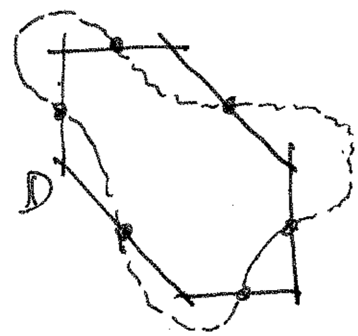
← compactify

as an elliptic curve. Here $\mathbb{P}_\Delta$ is the toric surface (compactifying $(\mathbb{C}^*)^2$) arising from the Newton polytope

$\Delta =$ 

of ϕ, and E is elliptic b/c Δ has a unique interior pt. & integral polar polytope

The picture is



and $E_t \cap \mathbb{D} = 6$ marked 6-torsion points, which suggests the congruence sgp.

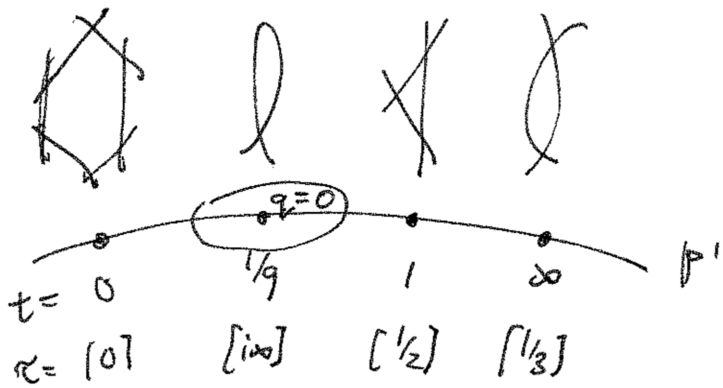
$$\Gamma_1(6) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; a \underset{(6)}{\equiv} 1 \underset{(6)}{\equiv} d, \;\; c \underset{(6)}{\equiv} 0 \right\}.$$

In fact, the family has a modular parametrization

$$\begin{array}{ccccc}
\text{“}\bigcup_{t \in \mathbb{P}^1 \setminus \mathcal{L}} E_t\text{”} & =: & \mathcal{E}_{\text{toric}} & \longrightarrow & \mathbb{P}^1 \setminus \mathcal{L} \\
 & & \cong \uparrow & & \cong \uparrow H = \text{Hauptmodul} = \left(q + 7 \cdots \frac{\eta(q^2)\,\eta(q^6)^5}{\eta(q^3)\,\eta(q)^5}\right)^{-1} \\
\text{“}\bigcup_{\tau \in \Gamma_1(6)\backslash\mathfrak{h}} \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau\text{”} & =: & \mathcal{E}_{\text{modular}} & \longrightarrow & \Gamma_1(6)\backslash\mathfrak{h}
\end{array}$$

where $q = e^{2\pi i \tau}$, $\tau \in \mathfrak{h}$

which admits a compactification with singular fibers



While the modular interpretation is not needed to obtain Thm. 1, it is helpful for stating

<u>Theorem 2 (Bloch, Vanhove):</u> We have

$$I(H(q)) = \frac{-1}{\sqrt{3}} \frac{\eta(q)^6 \eta(q^6)}{\eta(q^2)^3 \eta(q^3)^2} \Big\{ 5 D_2(\zeta_6) + 3i \overbrace{\sum_{k \geq 1} \Big(\mathrm{Li}_2(\zeta_6 q^k) + \mathrm{Li}_2(\zeta_6^2 q^k) - \mathrm{Li}_2(\zeta_6^4 q^k) - \mathrm{Li}_2(\zeta_6^5 q^k) \Big)}^{\text{also: } 2\sqrt{3}i \sum_{k \geq 1} \frac{\chi_6(k)}{k^2} \frac{q^k}{1-q^k}} \Big\}$$

with special value $I(1/9) = I(H(0)) = -\frac{5}{\sqrt{3}} D_2(\zeta_6)$.

think of as a complex-analytic extension of a sum of elliptic dilogarithms (evaluated at 6-torsion)

It turns out that both results can be derived with little effort by plugging into the machine of [DK] once one knows that

<u>$I(t)$ is the higher normal function associated to an algebraic cycle</u>.

What I'll explain in the remainder of the talk is what a HNF is and how the cycle "motivates" the Theorems.

§4 <u>Higher normal function</u>

Bloch's higher Chow groups give a geometric representation of the graded pieces of algebraic K-theory, and come with cycle-class maps to Jac. tori:

$$\underset{\text{alg. var.}}{CH^p(X,n)} \cong \underbrace{CH^p\left(X\times(\mathbb{P}^1\setminus\{1\},\{0,\infty\})^n\right)}_{\substack{\text{"algebraic cycles"}\\ \Sigma\text{ algebraic submanifolds w./constraints}\\ \equiv\text{-relation}}} \underset{\otimes\mathbb{Q}}{\cong} Gr^p_\gamma K_n^{alg}(X)$$

$$\xrightarrow{AJ} H^{2p-n-1}(X,\mathbb{C})/\{F^p(\cdot\cdot) + H^{2p-n-1}(X,(2\pi i)^p\mathbb{Z})\}.$$

(Compute in terms of coh. class of currents on X)

<u>Special case</u>: $X=E$, $p=n=2$. (<u>Note</u>: if $X=RS$, $p=1$, $n=0$, recover Abel's map.)

$$\xi \in \underset{\substack{\text{graph}\\ \text{in } E\times(\mathbb{P}^1)^2}}{CH^2(E,2)} \cong \ker\left\{\frac{\mathbb{C}(E)^\times \wedge \mathbb{C}(E)^\times}{\langle f\wedge(1-f)\rangle} \xrightarrow[\text{Tame}]{} \bigoplus_{p\in E}\mathbb{C}^\times\right\}$$

$$\text{Milnor symbols } \{f,g\} \longmapsto \lim_{x\to p}(-1)^{\mathrm{ord}_p(f)\mathrm{ord}_p(g)}\frac{f(x)^{\mathrm{ord}_p(g)}}{g(x)^{\mathrm{ord}_p(f)}}$$

$$AJ\downarrow$$

$$R_\xi \in \mathrm{Hom}\big(\underset{\mathbb{Z}\alpha+\mathbb{Z}\beta}{H_1(E,\mathbb{Z})},\ \mathbb{C}/(2\pi i)^2\mathbb{Z}\big)$$

AJ is computed by sending $\{f,g\} \longmapsto R_{\{f,g\}} := \log f\, d\log g - 2\pi i \log g\, \delta_{T_f}$, $T_f := f^{-1}(\mathbb{R}^-)$
and integrating the latter over (loops) 1-cycles. If $\xi = \sum_i\{f_i,g_i\}$, $T_\xi = \sum_i T_{f_i}\cap T_{g_i}$, and $d(R_\xi) = -(2\pi i)^2\delta_{T_\xi}$. Taking Γ 1-chain with $\partial\Gamma = T_\xi$, $\Rightarrow$
$\tilde{R}_\xi := R_\xi + (2\pi i)^2\delta_\Gamma$ is closed, hence gives a coh. class $[\tilde{R}_\xi]\in H^1(E,\mathbb{Z})$.

Now suppose $\underset{}{\overset{\tilde{R}_t}{\xi_t}} \in CH^2(E_t,2)$ is a family of classes (descended from $\Xi\in CH^2(\mathcal{E},2)$), and ω_t a holomorphic family of (1,0)-forms, i.e. $\omega_t \in \Gamma(\mathbb{P}^1\setminus\mathcal{L}, \Omega^1_{\mathcal{E}/\mathbb{P}^1})$. Then we may define the (holo/multivalued) higher normal fcn

$$\boxed{V(t) := \langle[\tilde{R}_t],[\omega_t]\rangle}.$$

over $\mathbb{P}^1$.

Poincaré pairing $H^1\times H^1\to\mathbb{C}$: wedge & integrate

§2 The inhomogeneous equation

Given a family of cohomology classes $\eta \in \Gamma(\mathbb{P}^1\setminus\Sigma, \mathcal{H}^1_{\mathcal{E}/\mathbb{P}})$, we have

the Gauss-Manin connection: $\nabla_{\partial_t}\eta \in \Gamma(\mathbb{P}\setminus\Sigma, \mathcal{H}^1_{\mathcal{E}/\mathbb{P}})$ has periods $= \frac{d}{dt}$ of the periods of η.

$\langle \tilde{\eta}, \partial/\partial t\rangle$

One way to compute it is to lift η to $\tilde{\eta} \in A^1(\mathcal{E})$ and contract $d\tilde{\eta}$ with a lift $\widetilde{\partial/\partial t}$.

For $\tilde{R}_\xi$, this lift is R_Ξ, and $dR_\Xi = \sum_i d\log f_i \wedge d\log g_i \; (+ \ldots) \Rightarrow$

$\nabla_{\delta_t}\tilde{R}_\xi =: \omega_t$ is a family of holo $(1,0)$-forms. Such a family
($t\partial_t$)

satisfies an "Picard Fuchs" equation $(\nabla_{\delta_t}^2 + g_1\nabla_{\delta_t} + g_2)[\omega_t] = 0$, meaning the

periods $\int_\alpha \omega_t$, $\int_\beta \omega_t$ satisfy $D_{PF}(\cdot) := (\delta_t^2 + \underset{tf_1}{\underset{\shortparallel}{g_1}}\delta_t + \underset{tf_2}{\underset{\shortparallel}{g_2}})(\cdot) = 0$.

Now compute

$$\delta_t\langle \tilde{R}, \omega\rangle = \langle \nabla_{\delta_t}\tilde{R}, \omega\rangle + \langle \tilde{R}, \nabla_{\delta_t}\omega\rangle$$
$$= \underset{0}{\cancel{\langle \omega, \omega\rangle}} + \langle \tilde{R}, \nabla_{\delta_t}\omega\rangle$$
$$\delta_t^2\langle \tilde{R}, \omega\rangle = \langle \underset{\omega}{\nabla_{\delta_t}\tilde{R}}, \nabla_{\delta_t}\omega\rangle + \langle \tilde{R}, \nabla_{\delta_t}^2\omega\rangle$$
$$=: y(t) + \langle \tilde{R}, \nabla_{\delta_t}^2\omega\rangle \qquad (\text{note } y \in \mathbb{C}(t)^*)$$
$$\Rightarrow D_{PF}\, V = y(t) + \underset{0}{\cancel{\langle \tilde{R}, \nabla_{PF}\,\omega\rangle}}, \qquad \longleftarrow \text{general fact about HNF.}$$

where (writing $g_i = tf_i$) $\delta_t y_t = \underset{0}{\cancel{\langle \nabla_{\delta_t}\omega, \nabla_{\delta_t}\omega\rangle}} + \langle \omega, \nabla_{\delta_t}^2\omega\rangle$

$$= -tf_1\langle \omega, \nabla_{\delta_t}\omega\rangle - tf_2\underset{0}{\cancel{\langle \omega, \omega\rangle}}$$

giving $\frac{dy}{dt} = -f_1 y \Rightarrow y \underset{(*)}{=} \kappa e^{-\int f_1 dt}$.

Specialize: $\xi_t := \{-x, -y\} \in CH^2(E_t, 2)\,(\otimes\mathbb{Q})$

$$\Rightarrow \omega_t = \operatorname{Res}_{E_t}\left\{\frac{dx/x \wedge dy/y}{1 - t\phi(x,y)}\right\} =: \operatorname{Res}\,\Omega_t$$

from the I_6 or 0

$$\Rightarrow \text{in } D_{PF},\; f_1 = \frac{18t-10}{(9t-1)(t-1)} = \frac{h'}{h} \underset{(*)}{\Rightarrow} y = \frac{\kappa}{h(t)} = \frac{+6(2\pi i)}{(9t-1)(t-1)}.$$

(to get PF eqn., use $\int_\beta \omega_t = \frac{1}{2\pi i}\int_{|x|=|y|=1}\Omega_t = 2\pi i \sum_n [\phi^n]_0 t^n$ where $[\phi^n]_0 = \sum \binom{n}{a,b,c}^2$)

§3 Recognizing the HNF

← trick that works on surprising variety of integrals of this form: BEK wheel w/3 spokes, Apéry–Beukers ζ, etc.

Write $I = \frac{-t}{2\pi i\, m^2} J$,

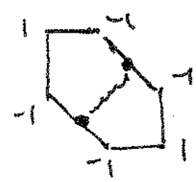
$$J(t) = 2\pi i \int_0^\infty\!\!\int_0^\infty \underbrace{\frac{dx/x \wedge dy/y}{1 - t\phi}}_{\hat{\Omega}_t} \,, \quad \text{and}$$

Consider the symbol $\Xi = \{-x, -y\}$ (on $\mathbb{C}^* \times \mathbb{C}^*$, or even on $\mathbb{P}_\Delta$), with $T_\Xi = \mathbb{R}_+ \times \mathbb{R}_+ \subset \mathbb{P}_\Delta$. We have on $\mathbb{P}_\Delta$

$$dR_\Xi = \overset{d\log x \wedge d\log y}{\Omega_\Xi} - (2\pi i)^2 \delta_{T_\Xi} + \left(\text{Res terms supp. on } \mathbb{D}\right).$$

So

$$\begin{aligned} J(t) &= 2\pi i \int_{\mathbb{P}_\Delta} \delta_{T_\Xi} \wedge \hat{\Omega}_t \\ &= -\frac{1}{2\pi i} \int_{\mathbb{P}_\Delta} dR_\Xi \wedge \hat{\Omega}_t \\ &= \frac{1}{2\pi i} \int_{\mathbb{P}_\Delta} R_\Xi \wedge \underbrace{d\hat{\Omega}_t}_{(2\pi i)\iota_* \mathrm{Res}\, \hat{\Omega}_t} \\ &= \int_{E_t} R_{\xi_t} \wedge \omega_t \end{aligned}$$

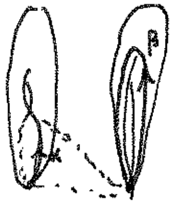


← Done? Not yet: need to add the δ_Γ. For generic t (actually for $t \notin \mathbb{R}$), there is no T_{ξ_t}, but we aren't done b/c ξ_t needs to be "moved" before computing. Upshot: need to add chain connecting pts shown.

$$\underset{\left(\text{modulo } \frac{(2\pi i)^2}{6}\cdot \text{pds. of } \omega_t\right)}{\equiv} \int_{E_t} \tilde{R}_{\xi_t} \wedge \omega_t = V.$$

$$\Longrightarrow \boxed{D_{PF}\, J(t) = \frac{\kappa}{h(t)}}.$$

§4 Holomorphic solution (about $t=1/4$)



Pulling back under Hauptmodul H,

$$\omega = \mathcal{A}\, dz = \mathcal{A}(\beta - \tau\alpha)$$ [where we regard 1-cycles as cohom. classes by duality]

and

$$\nabla_{\partial_\tau}\tilde{R} = \mathcal{E}\, dz$$ where $\begin{cases}\mathcal{A} \\ \mathcal{E}/(2\pi i)^2 = E\end{cases}$ is a $\begin{cases}\text{modular form} \\ \text{Eisenstein series}\end{cases}$ of weight $\begin{cases}1\\3\end{cases}$.

(Using the fact that the higher cycle obtained from Ξ only has residues on the $t=0/\tau=i0$ fiber, $\overset{[DK]}{\Rightarrow}$ $E(\tau) = \frac{-2}{(2\pi i)^3}\sum'_{(m,n)\in\mathbb{Z}^2}\frac{\hat{\varphi}(m,n)}{(m\tau+n)^3}$ with $\hat{\varphi} = 6\sqrt{3}\, i\, \chi_{-6}(m)$.)

So $\tilde{R} = \left(R_\beta(0) + \int_0 \tau\mathcal{E}\, d\tau\right)\alpha - \left(\int_0 \mathcal{E}\, d\tau\right)\beta \implies$

$$V = \langle \tilde{R}, \omega\rangle = \langle \; '' \;, \mathcal{A}\beta - \tau\mathcal{A}\alpha\rangle$$
$$= \mathcal{A}\left\{R_\beta(0) + \int_0 \tau\mathcal{E}\, d\tau - \tau\int_0 \mathcal{E}\, d\tau\right\}$$
$$= \mathcal{A}\left\{R_\beta(0) + \iint_0 \mathcal{E}\, d\tau\right\}.$$

The miracle that $\iint_0$ works on $\mathcal{E}$ is best seen from

$$\mathcal{E} = 12\sqrt{3}\, i\, (2\pi i)^2 \sum_{m\geq 1} q^m \sum_{r|m} r^2 \chi_{-6}\left(\tfrac{m}{r}\right)$$

$$\overset{\iint}{\rightsquigarrow}\; 12\sqrt{3}\, i \underbrace{\sum_{m\geq 1} q^m \sum_{r|m}\left(\tfrac{r}{m}\right)^2 \chi\left(\tfrac{m}{r}\right)}_{\underset{(k=\frac{m}{r})}{=} \sum_{k\geq 1}\frac{\chi(k)}{k^2}\underbrace{\sum_{r\geq 1}(q^k)^r}_{\frac{q^k}{1-q^k}}}$$

which recovers the formula in Thm. 2 up to the special value, which is read off from a formula in [DK] to be $R_\beta(0) = -6\sqrt{3}\, i\, L(\chi_{-6}, 2)$ (from the form of $\hat{\varphi}$).

Remark: in the absence of modular behavior, one can still compute the special value by $\mathbb{P}^1$ normalizing the singular curve and integrating R from 0 to ∞.

You may ask what a special value of an L-fcn. is doing there, and in fact that is one of the points of the theory: the limit of a "$K_2(E)$" class at a singular fiber is a $K_3^{ind}(\mathbb{F})$ element, and the limit of AJ is the Borel regulator (or [illegible]

35 __Bloch – Vanhove motive__ (dual of) Write $T = (\mathbb{C}^*)^2$, $E^* = E \cap T$.

They make use of the relative motive

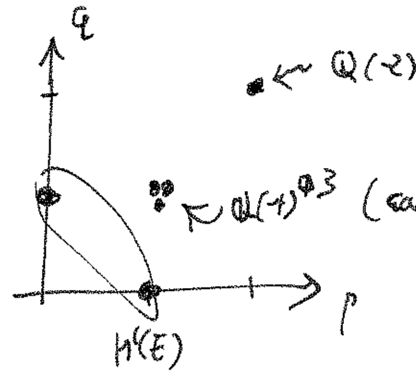
$$\mathbb{M} = (T, E^*)$$

which sits in an exact sequence of MHS

$$\begin{array}{ccccccc} & 0 & & & & & \\ & \downarrow & & & & & \\ & H^1(E) & & & & & \\ & \downarrow & & & & & \\ 0 \to & \dfrac{H^1(E^*)}{H^1(T)} & \to & H^2(\mathbb{M}) & \to & H^2(T) & \to 0 \\ & \downarrow & & & & \wr\wr & \\ & \mathbb{Q}(-1)^{\oplus 3} & & & & \mathbb{Q}(-2) & \\ & \downarrow & & & & & \\ & 0 & & & & & \end{array}$$

Note: this is not split, but we can lift a generator of $\mathbb{Q}(-2)$ under maps preserving the weight resp. Hodge filtration & take the difference. This gives the AJ class.

which gives a picture



(comes from punctures on ell. curve)

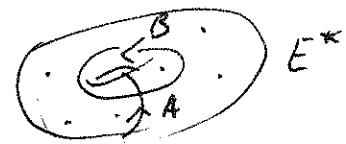
really 3 extensions

and an extension class $\mathcal{E}_{\underline{m},t} \in \mathrm{Ext}^1_{MHS}(\mathbb{Q}(-2), H^1(E^*)/H^1(T))$ which is what the AJ class measures. When the masses are equal:

- the 6 points of $E \cap \mathbb{D}$ are (relatively) torsion $\Rightarrow$ wt. 2 $\leftrightarrow$ wt. 1 ext. splits
- the Tame symbol of $\{-x, -y\}$ vanishes $\Rightarrow$ wt 4 $\leftrightarrow$ wt. 2 ext. splits

$\Longrightarrow$ $\mathcal{E}_t$ descends to $\tilde{\mathcal{E}}_t \in \mathrm{Ext}^1_{MHS}(\mathbb{Q}(-2), H^1(E))$, which is what $\tilde{R}_t$ computed above. In the non-equal-masses case, $AJ/\mathcal{E}$ is still computed by (the non-closed current R_t, and $J(t)$ by $\int R \wedge \omega$; there are formulas for R in [DK].

Somewhat surprisingly, while one can directly compute the periods of R around all 1-cycles on E^* mapping to A on E, to get the B-period one makes use of



local mirror symmetry; it is written as a generating function of Gromov-Witten invariants (instanton #'s) of $K_{\mathbb{P}_\Delta}$. I don't know if this will be the best way to go for producing formulas of use to physics, but it should be fun to see what one gets from this perspective in the unequal mass case.