Multiple polylogarithms and Feynman integrals

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joint work with

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Outline:

Part 1: Multiple polylogarithms and Feynman parameters

- Multiple polylogarithms in several variables (with F. Brown)
- The criterion of linear reducibility
- Integration over Feynman parameters
- Minor-closedness (with M. Lüders)

Part 2: A case "beyond multiple polylogs" (with L. Adams and S. Weinzierl)

- The two-loop sunrise graph with arbitrary masses
- A second order differential equation
- A solution in terms of elliptic integrals

Part 1:
Multiple polylogarithms in several variables

Let

- k be a field (either $\mathbb{R}$ or $\mathbb{C}$),
- M a smooth manifold over k,
- $\gamma : [0, 1] \to M$ a smooth path on M,
- $\omega_1, \ldots, \omega_n$ smooth differential 1-forms on M,
- $\gamma^\star(\omega_i) = f_i(t)dt$ the pull-back of ω_i to $[0, 1]$

Def.: The *iterated integral* of $\omega_1, \ldots, \omega_n$ along γ is

$$\int_\gamma \omega_n \ldots \omega_1 = \int_{0 \leq t_1 \leq \ldots \leq t_n \leq 1} f_n(t_n)dt_n \ldots f_1(t_1)dt_1.$$

We use the term *iterated integral* for k-linear combinations of such integrals.

We obtain different classes of functions by choosing different finite sets of 1-forms Ω.

- $\Omega_1 = \left\{ \frac{dt}{t}, \frac{dt}{t-1} \right\}$, $\omega_0 \equiv \frac{dt}{t}$, $\omega_1 \equiv \frac{dt}{t-1}$
  - **classical polylogarithms**: $\mathrm{Li}_n(z) = \int_\gamma \underbrace{\omega_0 \ldots \omega_0}_{n-1 \text{ times}} \omega_1 =$

    $\int_{0 \le t_1 \le \ldots \le t_n \le 1} \frac{dt_n}{t_n} \cdots \frac{dt_2}{t_2} \frac{z dt_1}{1 - z t_1}$
  - **multiple polylogarithms in one variable**: $\mathrm{Li}_{n_1, \ldots, n_r}(z) = (-1)^r \int_\gamma \underbrace{\omega_0 \ldots \omega_0}_{n_r - 1} \omega_1 \ldots \underbrace{\omega_0 \ldots \omega_0}_{n_1 - 1} \omega_1$, where γ a smooth path in $\mathbb{C} \backslash \{0, 1\}$ with end-point z

- $\Omega_n^{\mathrm{Hyp}} = \left\{ \frac{dt_1}{t_1}, \frac{dt_1}{t_1 - 1}, \frac{t_2 dt_1}{t_1 t_2 - 1}, \ldots, \frac{(\prod_{i=2}^n t_i) dt_1}{\prod_{i=1}^n t_i - 1} \right\}$: **hyperlogarithms**
  (Poincare, Kummer 1840, Lappo-Danilevsky 1911)

  including **harmonic polylogarithms** (Remiddi, Vermaseren 1999), **two-dimensional harmonic polylogarithms** (Gehrmann, Remiddi '01)

Let Ω_n be the set of differential 1-forms $\frac{df}{f}$
with $f \in \left\{t_1,\ \dots,\ t_n,\ \prod_{a \le i \le b} t_i - 1\right\}$, for $1 \le a \le b \le n$:

$$\Omega_n = \left\{ \frac{dt_1}{t_1}, \dots, \frac{dt_n}{t_n}, \frac{d\left(\prod_{a \le i \le b} t_i\right)}{\prod_{a \le i \le b} t_i - 1} \text{ where } 1 \le a \le b \le n \right\}$$

Examples:

$$\Omega_1 = \left\{ \frac{dt_1}{t_1}, \frac{dt_1}{t_1 - 1} \right\} \ (\rightarrow \text{multiple polylogs in one variable})$$

$$\Omega_2 = \left\{ \frac{dt_1}{t_1}, \frac{dt_2}{t_2}, \frac{dt_1}{t_1 - 1}, \frac{dt_2}{t_2 - 1}, \frac{t_1\, dt_2 + t_2\, dt_1}{t_1 t_2 - 1} \right\}$$

From this Ω_n we want to construct iterated integrals which are *homotopy invariant*, i.e.

$$\int_{\gamma_1} \omega_n ... \omega_1 = \int_{\gamma_2} \omega_n ... \omega_1 \text{ for homotopic paths } \gamma_1, \gamma_2 .$$

Consider tensor products $\omega_1 \otimes ... \otimes \omega_m \equiv [\omega_1|...|\omega_m]$ over $\mathbb{Q}$.

Define an operator D by

$$D([\omega_1|...|\omega_m]) = \sum_{i=1}^{m} [\omega_1|...|\omega_{i-1}|d\omega_i|\omega_{i+1}|...\omega_m] + \sum_{i=1}^{m-1} [\omega_1|...|\omega_{i-1}|\omega_i \wedge \omega_{i+1}|...|\omega_m].$$

Def.: A $\mathbb{Q}-$linear combination of tensor products

$$\xi = \sum_{l=0}^{m} \sum_{i_1, \dots, i_l} c_{i_1, \dots, i_l} [\omega_{i_1}|...|\omega_{i_l}], \ c_{i_1, \dots, i_l} \in \mathbb{Q}$$

is called *integrable word* if

$$D(\xi) = 0.$$

Consider the integration map

$$\sum_{l=0}^{m} \sum_{i_1, \ldots, i_l} c_{i_1, \ldots, i_l} [\omega_{i_1} | \ldots | \omega_{i_l}] \mapsto \sum_{l=0}^{m} \sum_{i_1, \ldots, i_l} c_{i_1, \ldots, i_l} \int_\gamma \omega_{i_1} \ldots \omega_{i_l}$$

Theorem (Chen '77): Under certain conditions on Ω this map is an isomorphism from *integrable words* to *homotopy invariant iterated integrals*.

(also see Lemma 1.1.3 of Zhao's lecture)

Our class of homotopy invariant functions:

- Construct the integrable words of 1-forms in Ω_n.

  (for an explicit construction see CB, Brown '12
  and cf. Duhr, Gangl, Rhodes '11, Goncharov et al '10)

- By the integration map obtain the set of **multiple polylogarithms in several variables** $\mathcal{B}(\Omega_n)$.

Properties of $\mathcal{B}(\Omega_n)$ (Brown '05):

- They are well-defined functions of n variables, corresponding to end-points of paths.
- On these functions, functional relations are algebraic identities.
- They can be decomposed to an explicit basis.
- $\mathcal{B}(\Omega_n)$ is closed under taking primitives.
- Let $\mathcal{Z}$ be the $\mathbb{Q}$-vector space of multiple zeta values. The limits at 0 and 1 of functions in $\mathcal{B}(\Omega_n)$ are $\mathcal{Z}$-linear combinations of elements in $\mathcal{B}(\Omega_{n-1})$.

Consequence:

Let F_n be the vector space of rational functions with denominators in $\left\{t_1, \ldots, t_n, \prod_{a \leq i \leq b} t_i - 1\right\}, 1 \leq a \leq b \leq n.$

Consider integrals of the type MPL

$$\int_0^1 dt_n \sum_j f_j \beta_j \text{ with } f_j \in F_n,\ \beta_j \in \mathcal{B}(\Omega_n).$$

We can compute such integrals. The results are $\mathcal{Z}$-linear combinations of elements in $\mathcal{B}(\Omega_{n-1})$, multiplied by elements in F_{n-1}.

Concept: Map *Feynman integrals* to integrals of this type and evaluate them.

When is this possible?

Scalar Feynman integrals

For a generic Feynman graph G with N edges and loop-number (first Betti number) L we consider the scalar Feynman integral

$$I(\Lambda) = \int \prod_{i=1}^{L} \frac{d^D k_i}{i\pi^{D/2}} \prod_{j=1}^{N} \frac{1}{\left(-q_j^2 + m_j^2\right)^{\nu_j}}, \quad N,\, L,\, \nu_j \in \mathbb{Z},\, D \in \mathbb{C},$$

Λ : external parameters, i.e. kinematical invariants and masses m_j; q_j : momenta

Using the "Feynman trick" we can re-write this as

$$I(\Lambda) = \frac{\Gamma(\nu - LD/2)}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_0^\infty \dots \int_0^\infty \left(\prod_{i=1}^{N} dx_i x_i^{\nu_i - 1} \right) \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{\mathcal{U}^{\nu-(L+1)D/2}}{(\mathcal{F}(\Lambda))^{\nu - LD/2}},$$

where $\nu = \sum_{j=1}^{N} \nu_j$, $\epsilon = (4-D)/2$.

$\mathcal{U}$ and $\mathcal{F}$ are the first and the second Symanzik polynomial.

Labelling the edges of G with Feynman parameters x_1, ..., x_N, we obtain the Symanzik polynomials as:

$$
\begin{aligned}
\mathcal{U} &= \sum_{\text{spanning trees } T \text{ of } G} \prod_{\text{edges } \notin T} x_i \\
\mathcal{F}_0 &= - \sum_{\text{spanning 2-forests } (T_1, T_2)} \left(\prod_{\text{edges } \notin (T_1, T_2)} x_i \right) \left(\sum_{\text{edges } \notin (T_1, T_2)} q_i \right)^2, \\
\mathcal{F} &= \mathcal{F}_0 + \mathcal{U} \sum_{i=1}^{N} x_i m_i^2 .
\end{aligned}
$$

Assumption: We are interested in integrands with $\mathcal{U}$ and/or $\mathcal{F}$ in the denominator and arguments of polylogs:

$$\frac{\text{(multiple) polylogs of } \{\mathcal{U}, \mathcal{F}\}}{\{\mathcal{U}, \mathcal{F}\}}$$

i.e. a Feynman integral which is finite from the beginning or appropriately renormalized (see Kreimer's talk)

Approach: Try to integrate out all Feynman parameters:

- After integration over x_i, consider the set of **polynomials in the denominator and in arguments of (possible) multiple polylogs in the integrand**.
  Condition: If there is a next Feynman parameter x_j in which all of these polynomials are **linear**, we can continue.
- Map the integral over x_j to an integral over t_n of the type MPL and integrate over t_n.

  Alternatively: Integrate over x_j directly, using an appropriate class of iterated integrals.
  (see recent work by E. Panzer, C. Duhr, F. Wissbrock, ...)

Question: For which polynomials (i.e. which graph) does this approach succeed?

Linear reduction algorithm (Brown '08)

- If the polynomials $S = \{f_1, \dots, f_n\}$ are linear in a Feynman parameter x_{r_1}, consider:
$$f_i = g_i x_{r_1} + h_i, \; g_i = \frac{\partial f_i}{x_{r_1}}, \; h_i = f_i|_{x_{r_1}=0}$$
- $S_{(r_1)}$ = irreducible factors of $\{g_i\}_{1\le i\le n}, \{h_i\}_{1\le i\le n}, \{h_i g_j - g_i h_j\}_{1\le i<j\le n}$
- iterate for a sequence $(x_{r_1}, x_{r_2}, \dots, x_{r_n}) \Rightarrow S_{(r_1)}, S_{(r_1, r_2)}, \dots, S_{(r_1, \dots, r_n)}$
- take intersections:

$$\begin{aligned} S_{[r_1, r_2]} &= S_{(r_1, r_2)} \cap S_{(r_2, r_1)} \\ S_{[r_1, r_2, \dots, r_k]} &= \cap_{1\le i\le k} S_{[r_1, \dots, \hat{r}_i, \dots, r_k](r_i)}, \; k > 3 \end{aligned}$$

$$x_{r_1}, x_{r_2}, \dots, x_{r_n} \Rightarrow S_{(r_1)}, S_{[r_1, r_2]}, \dots, S_{[r_1, \dots, r_n]}$$

Def.: A Feynman graph G is called *linearly reducible*, if the set $\{\mathcal{U}_G, \mathcal{F}_G\}$ is linearly reducible, i.e. there is a $(x_{r_1}, x_{r_2}, \dots, x_{r_n})$ such that for all $1 \le k \le n$ every polynomial in $S_{[r_1, \dots, r_k]}$ is linear in $x_{r_{k+1}}$.

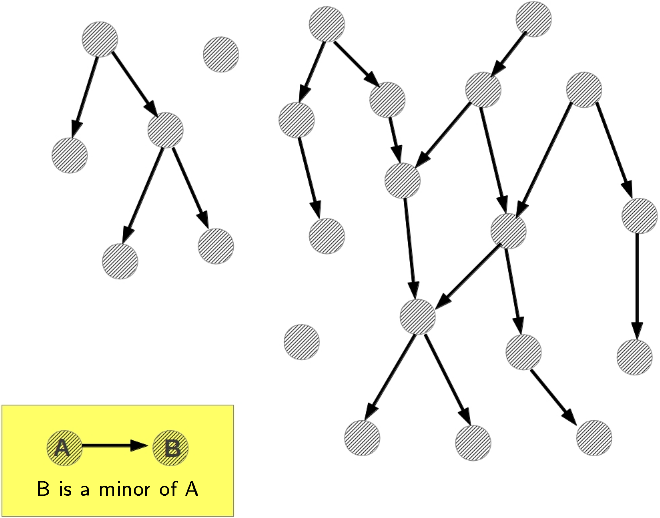
For e an edge of G consider the deletion $(G\backslash e)$ and contraction $(G//e)$ of e

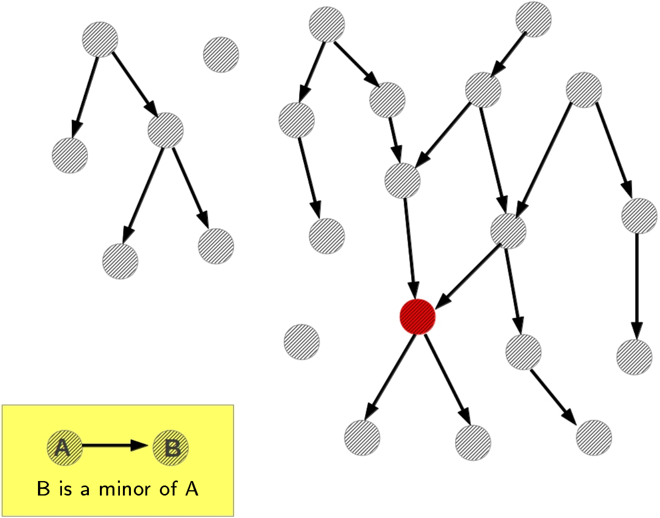
The deletion and contraction of different edges is commutative.

$\Rightarrow$If C, D are disjoint sets of edges of G then $G\backslash D//C$ is a unique graph.
Any such graph is called *minor* of G.

Def.: A set $\mathcal{G}$ of graphs is called *minor-closed* if for each $G \in \mathcal{G}$ all minors belong to $\mathcal{G}$ as well.

Example: The set of all planar graphs is minor-closed.


A
B
B is a minor of A


A
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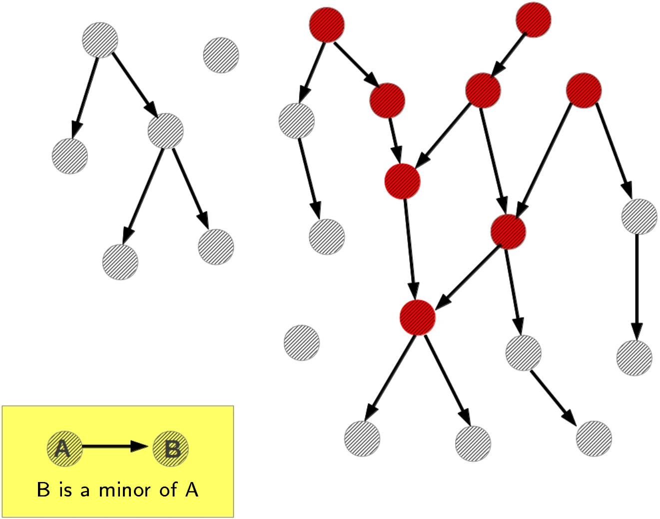
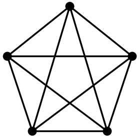
B is a minor of A

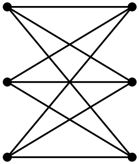

A
B
B is a minor of A

Let $\mathcal{H}$ be a finite set of graphs.
Define $\mathcal{G}_{\mathcal{H}}$ to be the set of graphs **whose minors do not belong** to $\mathcal{H}$.
Then the graphs in $\mathcal{H}$ are called *forbidden minors* of $\mathcal{G}_{\mathcal{H}}$. The set $\mathcal{G}_{\mathcal{H}}$ is minor-closed.

Theorem (Robertson and Seymour): Any minor-closed set of graphs can be defined by a finite set of forbidden minors.

Example:
The set of planar graphs is the set of all graphs which have neither K_5 nor $K_{3,3}$ as a minor. (Wagner's theorem)





Theorem (Brown '09, CB and Lüders '13):
The set of linearly reducible Feynman graphs is minor-closed.

We should search for the forbidden minors!

A first case study (with M. Lüders):

- Let Λ be the set of massless Feynman graphs with four on-shell legs. (On-shell condition: $p_i^2 = 0$, $i = 1, \ldots, 4$)
- At two loops we find all graphs to be linearly reducible.
- At three loops we find first forbidden minors.
- Four loops are running on our computers and confirm the forbidden three-loop minors so far.

Part 2:

The sunrise graph - a case beyond multiple polylogarithms

Consider the sunrise graph with arbitrary masses:

In $D = 2$ dimensions we obtain the finite Feynman integral

$$S_{D=2}(t) = \int_\sigma \frac{\omega}{\mathcal{F}_G},$$

with

$$\omega = x_1\, dx_2 \wedge dx_3 + x_2\, dx_3 \wedge dx_1 + x_3\, dx_1 \wedge dx_2$$

$$\mathcal{F}_G\left(t,\, m_1^2,\, m_2^2,\, m_3^2\right) = -x_1 x_2 x_3 t + (x_1 x_2 + x_2 x_3 + x_1 x_3)(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2), \;\; t = p^2,$$

$$\sigma = \left\{[x_1 : x_2 : x_3] \in \mathbb{P}^2 | x_i \geq 0,\, i = 1,\, 2,\, 3\right\}$$

Simple observation: As $\mathcal{F}_G$ is not linear in any x_i, the graph is not linearly reducible.

(Incomplete) history of sunrises:
Equal mass case:

- Broadhurst, Fleischer, Tarasov (1993): result with hypergeometric functions
- Groote, Pivovarov (2000): Cutkosky rules $\Rightarrow$ imaginary part expressed by elliptic integrals
- Laporta, Remiddi (2004): solving a second-order differential equation $\Rightarrow$ result by integrals over elliptic integrals
- Bloch, Vanhove (in progress): a new result involving the elliptic dilogarithm (see talks by Broadhurst and Kerr)

Arbitrary mass case:

- Berends, Buza, Böhm, Scharf (1994): result with Lauricella functions
- Caffo, Czyz, Laporta, Remiddi (1998): system of four first-order differential equations (and numerical solutions)
- Groote, Körner, Pivovarov (2005): integral representations involving Bessel functions
- Müller-Stach, Weinzierl, Zayadeh (2012): one second-order differential equation

Our goal: Solve the new differential equation (as Laporta and Remiddi did for equal masses) and obtain a result involving elliptic integrals

A result for D dimensions is known from Berends, Buza, Böhm and Scharf (1994):

$$S_D(t) =$$

$$(-t)^{D-3}\left(\frac{\Gamma(3-D)\Gamma(\frac{D}{2}-1)^3}{\Gamma(\frac{3}{2}D-3)} F_C\left(3-D,\, 4-\tfrac{3}{2}D;\, 2-\tfrac{D}{2},\, 2-\tfrac{D}{2},\, 2-\tfrac{D}{2};\, \tfrac{m_1^2}{t},\, \tfrac{m_2^2}{t},\, \tfrac{m_3^2}{t}\right)\right.$$

$$\frac{\Gamma(2-\frac{D}{2})\Gamma(1-\frac{D}{2})\Gamma(\frac{D}{2}-1)^2}{\Gamma(D-2)}\left(F_C\left(3-D,\, 2-\tfrac{D}{2};\, \tfrac{D}{2},\, 2-\tfrac{D}{2},\, 2-\tfrac{D}{2};\, \tfrac{m_1^2}{t},\, \tfrac{m_2^2}{t},\, \tfrac{m_3^2}{t}\right)\left(-\frac{m_1^2}{t}\right)^{\frac{D}{2}-1}\right.$$

$$+F_C\left(3-D,\, 2-\tfrac{D}{2};\, 2-\tfrac{D}{2},\, \tfrac{D}{2},\, 2-\tfrac{D}{2};\, \tfrac{m_1^2}{t},\, \tfrac{m_2^2}{t},\, \tfrac{m_3^2}{t}\right)\left(-\frac{m_2^2}{t}\right)^{\frac{D}{2}-1}$$

$$\left.+\, F_C\left(3-D,\, 2-\tfrac{D}{2};\, 2-\tfrac{D}{2},\, 2-\tfrac{D}{2},\, \tfrac{D}{2};\, \tfrac{m_1^2}{t},\, \tfrac{m_2^2}{t},\, \tfrac{m_3^2}{t}\right)\left(-\frac{m_3^2}{t}\right)^{\frac{D}{2}-1}\right)$$

$$+\Gamma(1-\tfrac{D}{2})^2\left(F_C\left(1,\, 2-\tfrac{D}{2};\, \tfrac{D}{2},\, \tfrac{D}{2},\, 2-\tfrac{D}{2};\, \tfrac{m_1^2}{t},\, \tfrac{m_2^2}{t},\, \tfrac{m_3^2}{t}\right)\left(\frac{m_1^2 m_2^2}{t^2}\right)^{\frac{D}{2}-1}\right.$$

$$+F_C\left(1,\, 2-\tfrac{D}{2};\, \tfrac{D}{2},\, 2-\tfrac{D}{2},\, \tfrac{D}{2};\, \tfrac{m_1^2}{t},\, \tfrac{m_2^2}{t},\, \tfrac{m_3^2}{t}\right)\left(\frac{m_1^2 m_3^2}{t^2}\right)^{\frac{D}{2}-1}$$

$$\left.\left.+\, F_C\left(1,\, 2-\tfrac{D}{2};\, 2-\tfrac{D}{2},\, \tfrac{D}{2},\, \tfrac{D}{2};\, \tfrac{m_1^2}{t},\, \tfrac{m_2^2}{t},\, \tfrac{m_3^2}{t}\right)\left(\frac{m_2^2 m_3^2}{t^2}\right)^{\frac{D}{2}-1}\right)\right)$$

with the Lauricella function

$$F_C(a_1, a_2; b_1, b_2, b_3; x_1, x_2, x_3) = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} \frac{(a_1)_{j_1+j_2+j_3}(a_2)_{j_1+j_2+j_3}}{(b_1)_{j_1}(b_2)_{j_2}(b_3)_{j_3}} \frac{x_1^{j_1} x_2^{j_2} x_3^{j_3}}{j_1! j_2! j_3!}$$

and the Pochhammer symbol $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$

Using Euler-Zagier sums $Z_1(n) = \sum_{j=1}^{n} \frac{1}{j}$, $Z_{11}(n) = \sum_{j=1}^{n} \frac{1}{j} Z_1(j-1)$ we can expand this result in $D = 2$ and obtain:

$$S_{D=2}(t) = -\frac{1}{t} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \left(\frac{j_{123}!}{j_1! j_2! j_3!}\right)^2 \left(\frac{m_1^2}{t}\right)^{j_1} \left(\frac{m_2^2}{t}\right)^{j_2} \left(\frac{m_3^2}{t}\right)^{j_3}$$

$$(12 Z_{11}(j_{123}) + 6 Z_1(j_{123}) Z_1(j_{123}) - 8 Z_1(j_{123}) (Z_1(j_1) + Z_1(j_2) + Z_1(j_3))$$

$$4 (Z_1(j_1) Z_1(j_2) + Z_1(j_2) Z_1(j_3) + Z_1(j_3) Z_1(j_1)) +$$
$$2 (2 Z_1(j_{123}) - Z_1(j_2) - Z_1(j_3)) \ln\left(-\frac{m_1^2}{t}\right) + 2 (2 Z_1(j_{123}) - Z_1(j_3) - Z_1(j_1)) \ln\left(-\frac{m_2^2}{t}\right)$$

$$+2 (2 Z_1(j_{123}) - Z_1(j_1) - Z_1(j_2)) \ln\left(-\frac{m_3^2}{t}\right)$$

$$+ \ln\left(-\frac{m_1^2}{t}\right) \ln\left(-\frac{m_2^2}{t}\right) + \ln\left(-\frac{m_2^2}{t}\right) \ln\left(-\frac{m_3^2}{t}\right) + \ln\left(-\frac{m_1^2}{t}\right) \ln\left(-\frac{m_3^2}{t}\right))$$

We obtain a five-fold nested sum.
Can we obtain a result avoiding multiple nested sums?

Start from the second order differential equation (Müller-Stach, Weinzierl, Zayadeh '12):

$$\left(p_0(t) \frac{d^2}{dt^2} + p_1(t) \frac{d}{dt} + p_2(t) \right) S(t) = p_3(t)$$

p_0, p_1, p_2, p_3 are polynomials in t (of degrees 7, 6, 5, 4) and in m_1^2, m_2^2, m_3^2 and p_3 involves $\ln\left(\frac{m_i^2}{\mu^2}\right)$

Ansatz for the solution:

$$S(t) = C_1 \psi_1(t) + C_2 \psi_2(t) + \int_0^t dt_1 \frac{p_3(t_1)}{p_0(t_1) W(t_1)} \left(-\psi_1(t)\psi_2(t_1) + \psi_2(t)\psi_1(t_1) \right)$$

with the solutions of the homogeneous equation ψ_1, ψ_2, constants C_1, C_2,

Wronski determinant $W(t) = \psi_1(t) \frac{d}{dt} \psi_2(t) - \psi_2(t) \frac{d}{dt} \psi_1(t)$

We will use

- complete elliptic integral of the first kind:

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

- complete elliptic integral of the second kind:

$$E(k) = \int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}} dx$$

- the moduli $k(t)$, $k'(t)$ satisfy $k(t)^2 + k'(t)^2 = 1$

Introduce the notation

$$x_1 = (m_1 - m_2)^2,\ x_2 = (m_3 - \sqrt{t})^2,\ x_3 = (m_3 + \sqrt{t})^2,\ x_4 = (m_1 + m_2)^2$$

Consider the **auxiliary elliptic curve** given by the equation

$$y^2 = (x - x_1)(x - x_2)(x - x_3)(x - x_4).$$

By the associated holomorphic 1-form dx/y one obtains the period integrals

$$\psi_1(t) = 2\int_{x_2}^{x_3} \frac{dx}{y} = \frac{4}{\xi(t)} K\left(k(t)\right),$$

$$\psi_2(t) = 2\int_{x_4}^{x_3} \frac{dx}{y} = \frac{4i}{\xi(t)} K\left(k'(t)\right)$$

$$\text{with } \xi(t) = \sqrt{(x_3 - x_1)(x_4 - x_2)},$$
$$k(t) = \sqrt{\tfrac{(x_3-x_2)(x_4-x_1)}{(x_3-x_1)(x_4-x_2)}},\ k'(t) = \sqrt{\tfrac{(x_2-x_1)(x_4-x_3)}{(x_3-x_1)(x_4-x_2)}},\ k(t)^2 + k'(t)^2 = 1$$

$\psi_1(t)$ and $\psi_2(t)$ **solve the homogeneous differential equation** for $S(t)$.

Furthermore, from integrating over $\frac{xdx}{y}$ we obtain

$$\phi_1(t) = \frac{4}{\xi(t)} \left(K(k(t)) - E\left(k(t)\right) \right)$$

$$\phi_2(t) = \frac{4i}{\xi(t)} E\left(k'(t)\right)$$

The period matrix of the elliptic curve is

$$\begin{pmatrix} \psi_1(t) & \psi_2(t) \\ \phi_1(t) & \phi_2(t) \end{pmatrix}$$

and we have the Legendre relation

$$\psi_1(t)\phi_2(t) - \psi_2(t)\phi_1(t) = \frac{8\pi i}{\xi(t)}.$$

These are appropriate functions to express the full solution in a compact way.

Full solution (Adams, CB, Weinzierl '13):

$$S(t) = \frac{1}{\pi}\left(\sum_{i=1}^{3} \mathrm{Cl}_2(\alpha_i)\right)\psi_1(t) + \frac{1}{i\pi}\int_0^t dt_1 \left(\eta_1(t_1) - \frac{b_1 t_1 - b_0}{3(x_2 - x_1)(x_4 - x_3)}\left(\eta_2(t_1) - \eta_1(t_1)\right)\right)$$

where

$$\eta_1(t_1) = \psi_2(t)\psi_1(t_1) - \psi_1(t)\psi_2(t_1)$$

$$\eta_2(t_1) = \psi_2(t)\phi_1(t_1) - \psi_1(t)\phi_2(t_1)$$

Clausen function: $\mathrm{Cl}_2(x) = \frac{1}{2i}\left(\mathrm{Li}_2(e^{ix}) - \mathrm{Li}_2(e^{-ix})\right)$

$\alpha_i = 2\arctan\left(\frac{\sqrt{\Delta}}{\delta_i}\right)$, Δ, δ_i : polynomials in m_1, m_2, m_3 of degrees 4 and 2 resp.

$b_i = d_i(m_1, m_2, m_3)\ln(m_1^2) + d_i(m_2, m_3, m_1)\ln(m_2^2) + d_i(m_3, m_1, m_2)\ln(m_3^2)$,

$d_1(m_1, m_2, m_3) = 2m_1^2 - m_2^2 - m_3^2$,

$d_0(m_1, m_2, m_3) = 2m_1^4 - m_2^4 - m_3^4 - m_1^2 m_2^2 - m_1^2 m_3^2 + 2m_2^2 m_3^2$

Conclusions:

- Multiple polylogarithms in several variables are homotopy invariant iterated integrals with particularly good properties. We want to use them to iteratively integrate out Feynman parameters.

- To decide whether the approach can succeed there is a criterion of linear reducibility on the graphs. The class of linearly reducible graphs is minor-closed. This allows for a convenient classification by forbidden minors.

- The sunrise integral with arbitrary masses is a case where we can express the result by integrals over elliptic integrals. This result can be built up from the period integrals of an (auxiliary) elliptic curve.