Why does predictability vary from forecast to forecast?

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Talk Outline

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Perfect predictability

- Uncertainty in predictions occurs for two major reasons.
- An inaccurate specification of the initial or boundary conditions.
- An inaccurate dynamical model.
- We consider only the first case here as this is amenable to a comprehensive theoretical exploration.

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Predictability Definitions

- Simplest idea involves uncertainty. The smaller this is made the greater the predictability achieved.
- A more useful idea takes into fundamental account that there is usually prior knowledge of a prediction random variable. This prior usually derives from historical or climatological observations of the system.
- Prediction can be viewed then as a modification of a prior to a posterior random variable. This perspective underlies learning theory as well as Bayesian statistics.
- This modification often involves more than just uncertainty reduction. Consider the following examples.....

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Simple Examples

- Suppose that climatologically the maximum temperature today is 14°C with a standard deviation of 4°C.
- Imagine two different (hypothetical) statistical predictions for today. First is 14°C with standard deviation 0.5°C. Second is 8°C with a standard deviation of 4°C.
- The usefulness of the first prediction derives from the reduction in uncertainty in the posterior from that of the prior. The usefulness of the second prediction derives not from any reduction in uncertainty but in the large difference of the prediction from "normal" or technically because the posterior mean differs markedly from the prior mean.

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- These different notions of predictability have traditionally been measured using a large variety of metrics (RMS error, anomaly correlation etc). They both derive however from a shift between a prior and posterior probability distribution.
- In the first case the variances of the distributions differ while the means are identical. In the second case the means differ but the variances are identical. Is there a universal way of measuring the shift?

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Talk Outline

Information theoretic measures

- In learning theory the shift between prior and posterior is a measure of how much knowledge has been acquired. It is traditionally measured using the relative entropy of the two distributions. Let the posterior distribution be p and the prior be q.

Relative entropy

$$D(p,q) \equiv \int p \log\left(\frac{p}{q}\right) dx$$

- This is non-negative and measures effectively the "distance" between p and q. It is invariant under general non-linear variable changes and is a non increasing function of time for Markov processes. It is also not symmetric between prior and posterior as one might expect given that learning is directed.

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Information theoretic measures

- Note that uncertainty changes alone due to prediction can be measured using an entropy difference.

Entropy Difference

$$H(q) - H(p) \equiv \int q \log(q)\, dx - \int p \log(p)\, dx$$

- Note that this measure does not satisfy the list of nice mathematical properties noted above for the relative entropy. It is, for example, only invariant under linear transformations of dynamical variables. It is however useful in our discussion as it measures the simplest kind of predictability discussed earlier.

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Gaussian distributions

- The relative entropy can be computed analytically in this instance. Let the prior random variable be denoted by y and the posterior by x and variances by σ

Gaussian relative entropy

$$\begin{aligned} D(p\|q) &= \tfrac{1}{2}\left[\log\left(\det\left(\sigma_y\right)/\det\left(\sigma_x\right)\right)+tr\left(\sigma_x\left(\sigma_y\right)^{-1}\right)-n\right] \\ &+ \tfrac{1}{2}(\overline{\mathrm{x}}-\overline{\mathrm{y}})^t\sigma_y^{-1}(\overline{\mathrm{x}}-\overline{\mathrm{y}}) \end{aligned}$$

- The first term on the RHS here is the entropy difference and measures reduction in uncertainty. The third term is the shift in means. The second term is usually not important. We call the first line the dispersion since it depends only on variances while we call the second line the signal. The dispersion amounts to a multivariate ensemble spread measure. It is easily shown that the signal is the sum of the squares of EOF anomalies divided by their climatological variances.

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Basic Motivation

- Anyone who has done practical forecasting knows some predictions are more useful than others. We study here reasons for this variation from a theoretical perspective.
- At any prediction time lag most dynamical systems exhibit large variations in the measure proposed above. We seek explanatory mechanisms for this variation since it appears to underly much utility variation.
- Since mechanisms for variation in predictability are likely system dependent we study three widely known but very different dynamical systems.

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1. Predictability Definition
   - Basic Concepts
   - Universal information theoretic measures
2. Predictability variation
   - Introduction
   - Chaos
   - Stochastic Models
   - Turbulence
   - Mechanisms for predictability variation
3. References

Lorenz 1963 Model

- Characterized by three non-linearly interacting variables which exhibit similar times scales.
- The prior distribution is a strange attractor so is highly non-Gaussian. Posterior distributions become rapidly non-Gaussian as they relax toward the prior.
- Gaussian indicators such as signal and dispersion (which recall amounts to an ensemble spread measure) do not work in the non-Gaussian regime.
- Entropy difference i.e. uncertainty reduction, on the other hand, is a highly reliable determinant of predictability.

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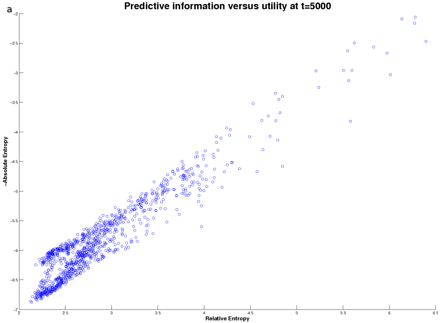

Shown is the relationship between entropy difference and relative entropy approximately mid way through relaxation to the prior. Each dot is a different prediction whose initial condition means were chosen at random from the prior.

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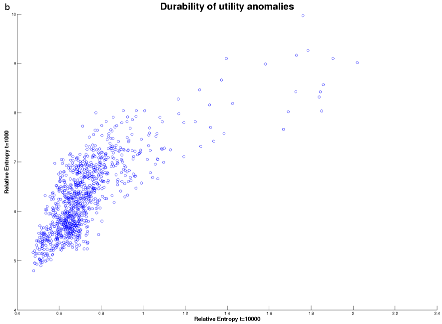

Predictions from this dynamical system exhibit considerable “durability” in that the level of predictability tends to persist for a considerable fraction of the relaxation (predictability limit) time. Shown is the predictability at early times (vertical) versus the predictability at later times (horizontal).

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Stochastic Models

- Intended to represent dynamical systems with fast and slow modes. The simplest and most widely used models of this type are linear systems with additive stochastic forcing.
- Models of this type have been proposed to explain general SST climate variability; ENSO and aspects of mid-latitude atmospheric turbulence.
- With Gaussian initial conditions all posterior and prior distributions are Gaussian. In addition one may prove that the variances of posterior distributions depend only on time and not on the choice of initial condition.
- It follows that variations in the signal are the sole determinant of predictability variations. The mechanism for this variation is that some initial conditions have large slow mode anomalies and other initial conditions do not. These modal anomalies persist during a prediction until eroded by the fast modes (i.e. the noise forcing).

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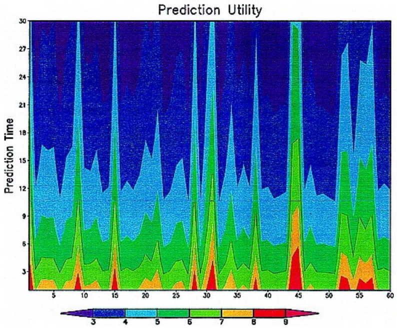
Stochastic Models

Like the chaotic model, stochastic model predictions also exhibit considerable durability. Shown is predictability for a random set of predictions (horizontal axis) for increasing prediction time (vertical axis). This result can be demonstrated analytically as well.



Talk Outline

Mid-latitude atmospheric turbulence

- This is characterised both by modes of similar times scales interacting non-linearly as well as by the presence of fast and slow modes. One might expect predictability characteristics intermediate between the first two examples. This was studied in a T42 and 5 vertical level dry global primitive equation model with orography and a good simulation of storm tracks.
- In general distributions are quasi-Gaussian so we can use the analytical expression above as a good approximation. Numerical results show that signal actually dominates dispersion at all prediction lags. Shown is the correlation between signal and dispersion with relative entropy. This is calculated using 48 statistical predictions whose initial condition means were drawn from the model climatology at random.

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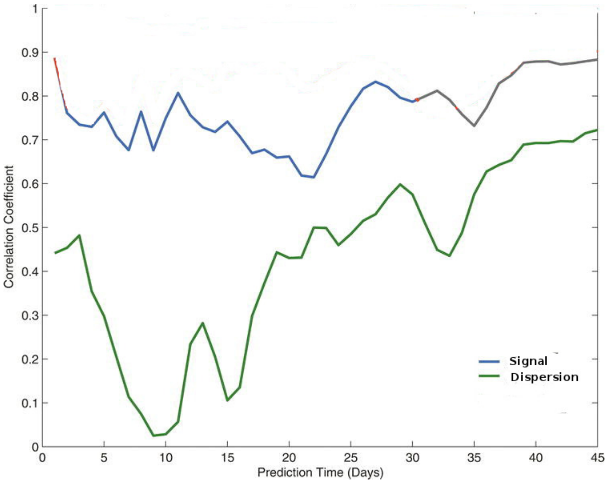
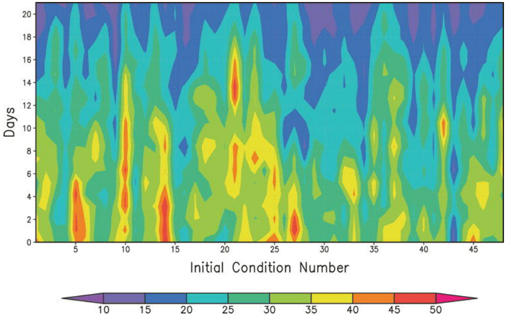


Mid-latitude atmospheric turbulence

Like the two earlier simpler models this more realistic model exhibits predictability durability although not quite as strongly. Shown is the signal for a random set of initial conditions (horizontal axis) and prediction times (vertical axis):



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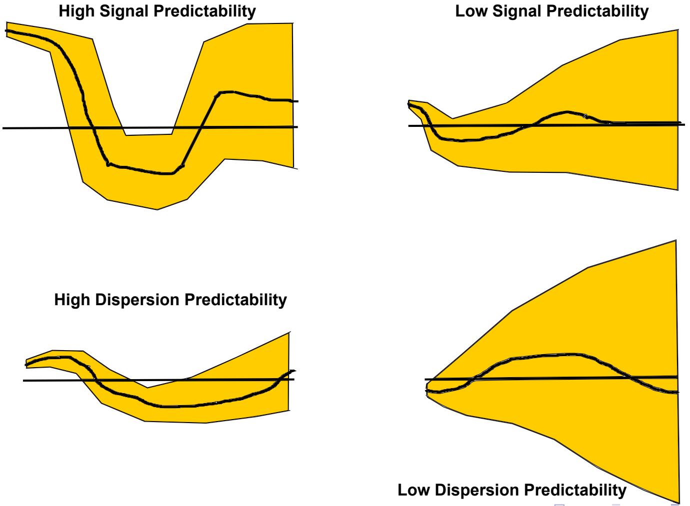
Different mechanisms for predictability variation



Different mechanisms for predictability variation

1. **Slow mode anomalies in the initial conditions.** Sometimes for purely random reasons, modes exhibiting slow temporal decorrelation are anomalously present in the initial conditions. This feature persists for a considerable time into a prediction thus increasing predictability. The reverse situation i.e. little anomalous slow mode presence in the initial condition results in reduced predictability since relaxation tends to be more rapid.
2. **Initial condition instability.** Sometimes again for purely random reasons the initial conditions of a prediction may be "more than normally" unstable to uncertainty growth. This could be linear or non-linear instability. Such a scenario results in decreased predictability. The reverse situation i.e. greater stability results in enhanced predictability.
3. **Which effect is more important?** Evidence gathered to date from realistic models of turbulence tends to favour the first mechanism but this conclusion needs deeper investigation.

For Further Reading I

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