On the Integral Homology of Kleinian Groups

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The Basic Setup

Let $\Gamma \subset \mathrm{PSL}_2(\mathcal{O}_K)$ be a cofinite Kleinian group and $\mathcal{O}_K$ integers of a number field K (possibly some primes inverted).

Let E_k denote the space of homog. polynomials over $\mathcal{O}_K$ of degree k in two variables. Γ acts on E_k:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(X, Y) := P(aX + cY, bX + dY).$$

Put

$$E_{k,\ell} := E_k \otimes_{\mathcal{O}_K} \overline{E_\ell}.$$

We are interested in the amount of torsion in

$$H_1(\Gamma, E_{k,\ell}).$$

Terminology: $E_{k,\ell}$ is "weight (k,ℓ)".

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Why Torsion ?

When $\Gamma \subset \mathrm{PSL}(\mathcal{O}_K)$ is arithmetic, there is an infinite collection $\mathbb{T}$ of commuting endomorphisms (Hecke operators) on H_1.

$$\begin{array}{ccc} H_1(\Gamma, E_{k,\ell}(\mathcal{O}_K)) & \xrightarrow{\otimes \mathbb{C}} & H_1(\Gamma, E_{k,\ell}(\mathbb{C})) \\ \downarrow{\scriptstyle \otimes \mathbb{F}_p} & & \\ H_1(\Gamma, E_{k,\ell}(\mathbb{F}_p)) & & \end{array}$$

Because of the possible "large" p-torsion, mod p homology may carry more "arithmetic information" than the complex homology. The latter can be identified with certain automorphic forms.

How big is the torsion ?

A Bianchi group is $\mathrm{PSL}_2(\mathcal{O}_K)$ with K imaginary quadratic field.

$H^2(\mathrm{PGL}_2(\mathcal{O}_K), E_{n,n}(\mathcal{O}_K))$ with $K = \mathbb{Q}(\sqrt{-2})$

n	prime divisors of the size of torsion
12	[2, 3, 5, 7, 11, **37**]
13	[2, 3, 5, 13]
14	[2, 3, 5, 7, 11, 13, **110281**]
15	[2, 3, 5, 7]
16	[2, 3, 5, 7, 11, 13, **1671337**]
17	[2, 3, 5, 7, **103**]
18	[2, 3, 5, 7, 11, 13, 17, **3812807473**]
19	[2, 3, 5, 7, **907**]
20	[2, 3, 5, 7, 11, 13, 17, 19, **3511**, **879556698451244053**]

How big is the torsion ?

For an ideal $\mathfrak{a}$ of $\mathcal{O}_K$, define

$$\Gamma_0(\mathfrak{a}) := \{ g \in \mathrm{PSL}_2(\mathcal{O}_K) \mid g \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod \mathfrak{a} \}.$$

Here is a sample of torsion in

$$H_1(\Gamma_0(\mathfrak{p}), \mathbb{Z}) \simeq \Gamma_0(\mathfrak{p})^{ab}$$

for prime ideals $\mathfrak{p}$ with $K = \mathbb{Q}(\sqrt{-11})$.

$\mathbf{N}\mathfrak{p}$	**some** of the prime divisors of the size of the torsion
4931	59242366654994144915737, 397153057377536493107457514082773
4933	471591580131222099301009, 753357254439534230416253
4937	774606120056702384410790118960699805738139
4943	7533150099701393721041, 1806172579157695730540919793
4951	575858582707156517384453334853901
4973	2223356120717452698676440064717
4987	121708009502005164710374726093
4999	35270997998154652004835942597708494620078410433635847

Asymptotic behaviour for cocompact arithmetic cases

Theorem (Bergeron-Venkatesh)

Let Γ_1 be a cocompact arithmetic Kleinian group. Let $\{\Gamma_n\}$ be a decreasing tower of congruence subgroups of Γ_1 such that $\bigcap_n \Gamma_n = \{1\}$. Then for $k \neq \ell$

$$\lim_{n\to\infty} \frac{\log |H_1(\Gamma_n, E_{k,\ell})_{tor}|}{vol(\Gamma_n \backslash \mathcal{H}^3)} = \frac{1}{6\pi} \cdot c_{k,\ell} > 0$$

where $c_{k,\ell}$ is a rational number depending only on k, ℓ.

Marshall-Müller have a similar result on the asymptotics in the weight aspect (weights $(k, 0)$ as $k \to \infty$).

Asymptotics for non-cocompact arithmetic cases ?

Every arithmetic Kleinian group that is not co-compact is commensurable with a Bianchi group.

So let's fix a Bianchi group and take a look at the ratios

$$\frac{\log |H_1(\Gamma_0(\mathfrak{p}), \mathbb{Z})_{tor}|}{\mathrm{vol}(\Gamma_0(\mathfrak{p})\backslash\mathcal{H}^3)}$$

as $\mathfrak{p}$ ranges over prime ideals of $\mathcal{O}_K$ of residue degree one.

Note that $\{\Gamma_0(\mathfrak{p})\}_{\mathfrak{p}}$ is not a tower and $\bigcap_{\mathfrak{p}} \Gamma_0(\mathfrak{p})$ is not the identity anymore.

Here is a sample for $\mathrm{PSL}_2(\mathbb{Z}[i])$.

$N\mathfrak{p}$	the ratio		$N\mathfrak{p}$	the ratio
44453	0.05201		44729	0.05351
44497	0.05342		44741	0.05355
44501	0.05442		44753	0.05533
44533	0.05342		44773	0.05604
44537	0.05391		44777	0.05573
44549	0.05250		44789	0.05172
44617	0.05467		44797	0.05480
44621	0.05509		44809	0.05220
44633	0.05390		44893	0.05476
44641	0.05317		44909	0.05227
44657	0.05203		44917	0.05281
44701	0.05520		44953	0.05441

Note that

$$\frac{1}{6\pi} \simeq 0.0530516476972984452562945877908$$

A Conjecture

I inspected the behavior of the above sequence of ratios in the case of Euclidean Bianchi groups.

Conjecture

Let G be any Bianchi group. Then

$$\lim_{N\mathfrak{p}\to\infty} \frac{\log |H_1(\Gamma_0(\mathfrak{p}), \mathbb{Z})_{tor}|}{\mathrm{vol}(\Gamma_0(\mathfrak{p})\backslash\mathcal{H}^3)} = \frac{1}{6\pi}$$

where the limit is taken over prime ideals of residue degree one.

Note that $E_{0,0}(\mathbb{Z}) \simeq \mathbb{Z}$ and $c_{0,0} = 1$.

A Conjecture, cont'd

The above data suggest that the result of Bergeron-Venkatesh probably holds in bigger generality:

- non-cocompact arithmetic lattices (see upcoming thesis of Jean Raimbault),
- $E_{k,\ell}$'s with $k = \ell$,
- more relaxed conditions on $\{\Gamma_n\}_n$.

Work in progress of Abert-Bergeron-Biringer-Gelander-Nikolov-Raimbault-Samet suggests that one might only need to have the collection $\{\Gamma_n\}_n$ "locally converge" to the identity.

Asymptotics for non-arithmetic lattices ?

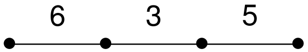
Consider the above sequence for congruence subgroups of non-arithmetic *hyperbolic tetrahedral groups*.

These are the index 2 subgroups consisting of orientation-preserving isometries in the discrete groups generated by reflections in the faces of hyperbolic tetrahedra whose dihedral angles are submultiples of π.

Lannér proved in 1950 that there are 32 of these groups. Only 7 of them are non-arithmetic. Among the non-arithmetic ones only one is cocompact.

A specific non-arithmetic case

Let $H(2)$ be the tetrahedral group attached to the Coxeter symbol



The tetrahedron associated to this Coxeter symbol has one ideal vertex. The volume of the tetrahedron is $\simeq 0.1715016613$. A presentation is:

$$H(2) = \langle a, b, c, \mid a^6 = b^2 = c^2 = (ca)^2 = (cb^{-1})^5 = (ab)^3 = 1 \rangle,$$

where

$$a := \begin{pmatrix} e^{i\pi/6} & 0 \\ 0 & e^{-i\pi/6} \end{pmatrix}, \quad b := \begin{pmatrix} i & -(\frac{1+\sqrt{5}}{2})i \\ 0 & -i \end{pmatrix}, \quad c := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Sample Data for $\Gamma_0(\mathfrak{p})$ subgroups of H(2)

$N(\mathfrak{p})$	rank	log(torsion)/volume
80089	47	0.00046147256599 8
94249	51	0.00039717749821 4
96721	72	0.00161519518746 0
96721	72	0.00165698128960 2
97969	52	0.00036262142827 4
100489	0	0.05302454966891 1
113569	56	0.00031660479790 1
120409	0	0.05336169831922 6
124609	0	0.05331047217697 9
128881	84	0.00127125170976 4
128881	84	0.00130261090167 1
134689	61	0.00025564809254 6
139129	62	0.00026269400067 7
146689	0	0.05318174735691 5

Asymptotics for non-arithmetic lattices

Data for the other non-arithmetic hyperbolic tetrahedral groups look the same:

• When the rank is 0, the "log(torsion)/volume" sequence seems to converge to $1/(6\pi)$.

• When the rank is positive, the sequence diverges from $1/(6\pi)$.

N.Bergeron and A.Venkatesh have a general philosophy that explains the above picture.

Contribution of the Regulator

Let Γ be a Kleinian group of finite covolume. Let $\{\Gamma_n\}_n$ be a sequence of subgroups of Γ which is locally converging to the identity. Recent work of Bergeron-Venkatesh suggests that

$$\frac{\log\left(|H_1(\Gamma_n, \mathbb{Z})_{tor}| \cdot R(\Gamma_n \backslash \mathcal{H}_3)\right)}{vol\left(\Gamma_n \backslash \mathcal{H}_3\right)} \longrightarrow \frac{1}{6\pi}.$$

Here $R(M)$ denotes the regulator of an hyperbolic 3-manifold M of finite volume.

Contribution of the Regulator, cont'd

Bergeron and Venkatesh believe that when Γ is arithmetic, the Hecke operators should force

$$\log(R(\Gamma_n\backslash\mathcal{H}_3)) \;/\; vol\Big(\Gamma_n\backslash\mathcal{H}_3\Big) \longrightarrow 0$$

and they conjecture this to be the case. It follows that

$$\log(|H_1(\Gamma_n,\mathbb{Z})_{tor}|) \;/\; vol\Big(\Gamma_n\backslash\mathcal{H}_3\Big) \longrightarrow 1/(6\pi).$$

The data coming from the Bianchi groups agree with this.

Contribution of the Regulator, cont'd

In the non-arithmetic case, we do not have Hecke operators.

When H_1 has rank 0, we have $R\left(\Gamma_n\backslash\mathcal{H}_3\right) = 1$ and

$$\log(|H_1(\Gamma_n, \mathbb{Z})_{tor}|) \ / \ vol\left(\Gamma_n\backslash\mathcal{H}_3\right) \longrightarrow 1/(6\pi).$$

But when H_1 has positive rank, the regulator contributes to the sum and

$$\log(|H_1(\Gamma_n, \mathbb{Z})_{tor}|) \ / \ vol\left(\Gamma_n\backslash\mathcal{H}_3\right) \ll 1/(6\pi).$$

The data coming from the non-arithmetic tetrahedral groups seem to agree with this speculation.

THANK YOU FOR YOUR ATTENTION!

Regulator

When M is compact, the regulator $R(M)$ can be given explicitly as

$$\left| \left(\int_{\gamma_i} \omega_j \right)_{i,j} \right|^{-2}$$

where $\omega_1, \ldots, \omega_n$ is an L^2-basis of harmonic 1-forms and $\gamma_1, \ldots, \gamma_n$ is a basis of $H_1(M, \mathbb{Z})_{free}$.

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