Right-Angled Coxeter Polytopes, Hyperbolic 6-manifolds, and a Problem of Siegel

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- The n-dimensional manifold Siegel problem: Determined the minimum possible volume obtained by an orientable hyperbolic n-manifold.
- Our solution for $n = 6$ will be described in this talk.

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- A compact orientable M satisfies $\chi(M) \in 2\mathbb{Z}$, so the minimum volume is most likely achieved by a noncompact manifold.

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- For $n = 5$, the minimum known volume is $7\zeta(3)/4$ where ζ is the Riemann zeta function.

Hyperbolic n-Space

- We work in the hyperboloid model of hyperbolic n-space

$$H^n = \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\}.$$

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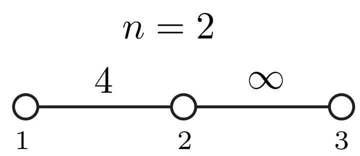
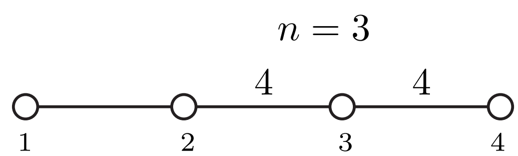
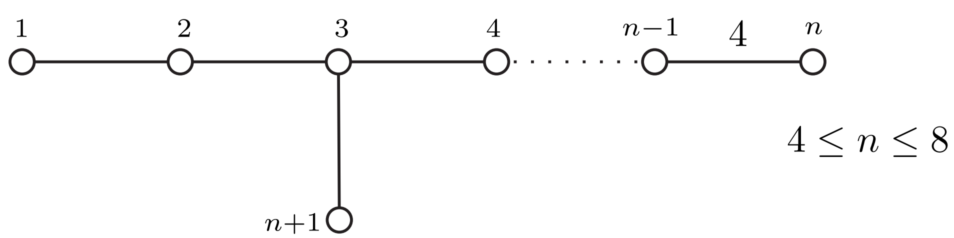
- A Coxeter diagram for Δ^n is given on the next slide.

Coxeter diagrams







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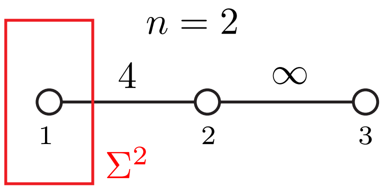

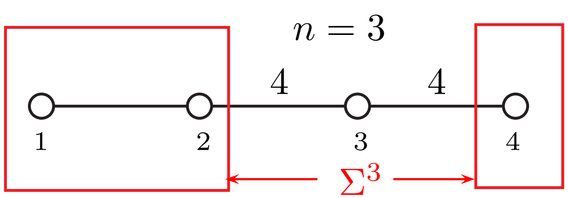

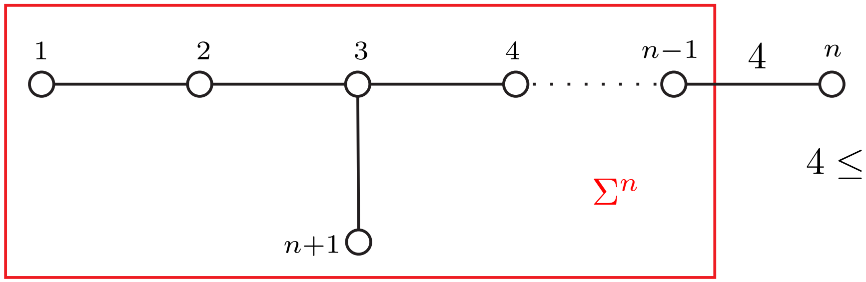

$4 \leq n \leq 8$

The right-angled polytope P^n

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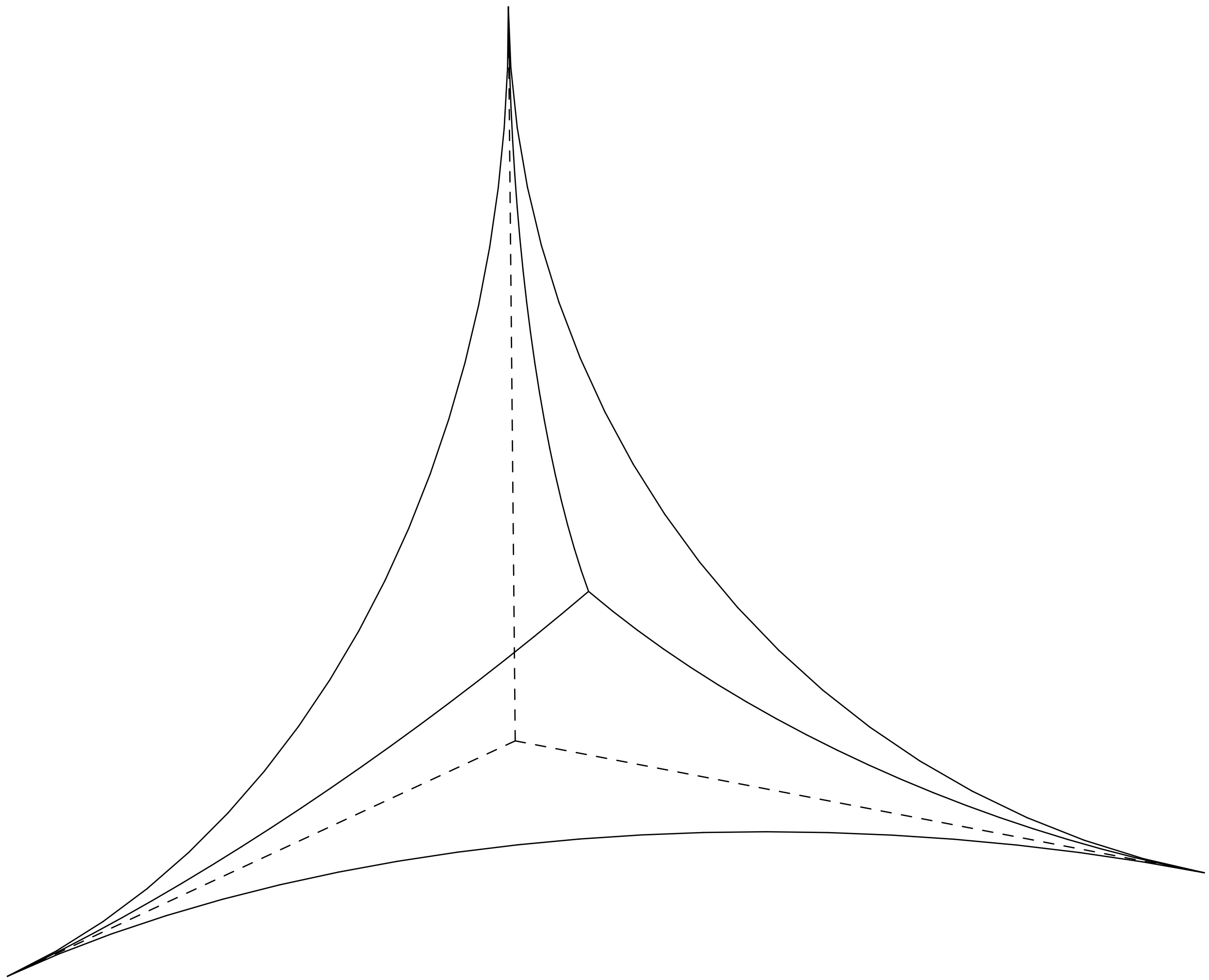
- The polytope P^n is right-angled for all n, and each side of P^n is congruent to P^{n-1} for all $n > 2$.

The right-angled polytope P^3



The Congruence Two Subgroup

- Let Γ_2^n be the congruence two subgroup of $\Gamma^n = PO_{n,1}\mathbb{Z}$, that is, the subgroup of all matrices in Γ^n that are congruent to the identity matrix modulo two.

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- $$\begin{pmatrix} -1 & -2 & 2 \\ -2 & -1 & 2 \\ -2 & -2 & 3 \end{pmatrix}$$
- The above matrix is in Γ_2^2 and represents the third Coxeter generator of Γ^2.

Γ_2^n is a right-angled Coxeter group

- Theorem: For $n = 2, \ldots, 7$, the congruence two subgroup Γ_2^n of Γ^n is a hyperbolic reflection group with Coxeter polytope the right-angled polytope P^n. Moreover, $\Gamma^n/\Gamma_2^n \cong \Sigma^n$.

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- Corollary: For $n = 2, \ldots, 7$, every maximal finite subgroup of Γ_2^n is either elementary of order 2^n and conjugate to the stabilizer of an actual vertex of P^n or is elementary of order 2^{n-1} and conjugate to the stabilizer of a line edge of P^n.

The right-angled polytope P^6

- The polytope P^6 has 72 actual vertices and 27 ideal vertices, 432 ray edges, 216 line edges, 1089 P^2-faces, 720 P^3-faces, 216 P^4-faces and 27 sides each congruent to P^5.

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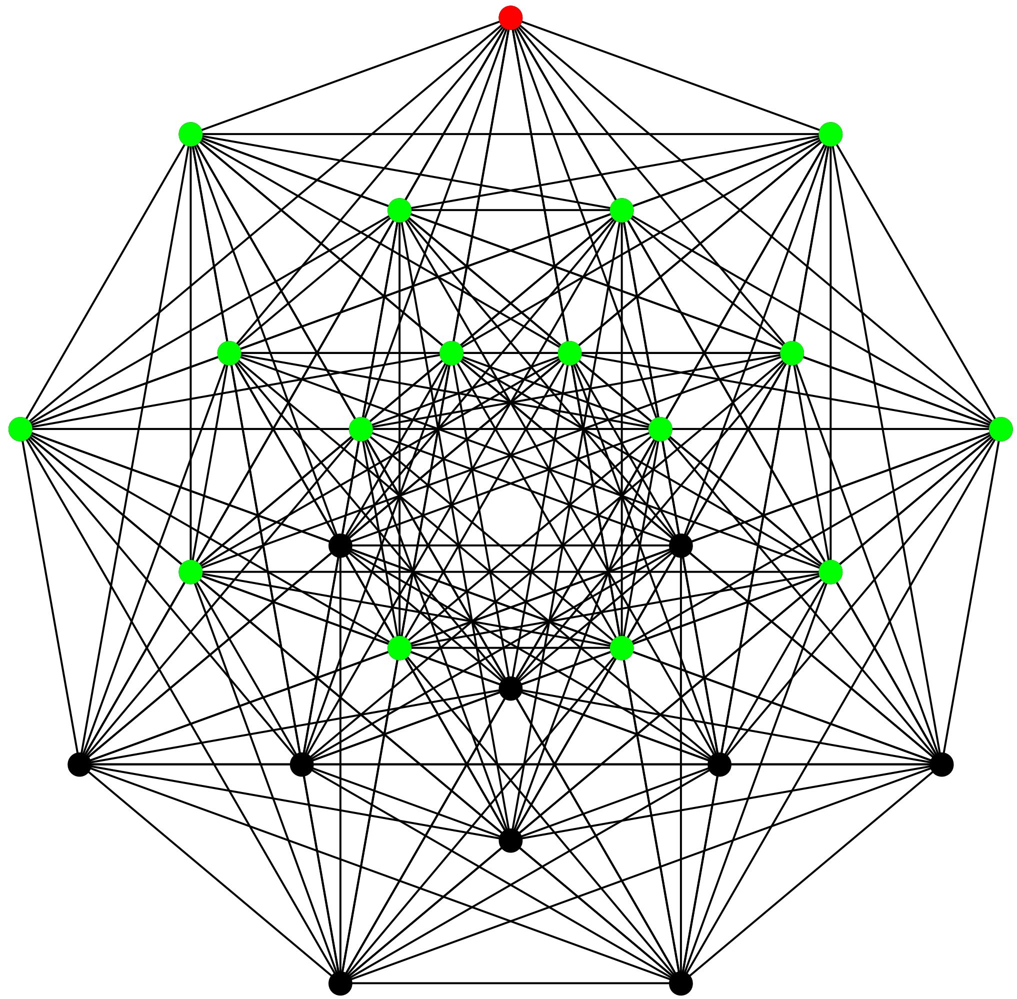
- The Euclidean dual of P^6 is a semi-regular 6-dimensional polytope discovered by Gosset in 1900.
- The Gosset 6-polytope combinatorially parametrizes the arrangement of the 27 straight lines in a general cubic surface.

1-Skeleton of the Gosset 6-Polytope



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- By the Gauss-Bonnet theorem, P^6 has volume $\pi^3/15$ which is one-eighth the smallest volume possible for a hyperbolic 6-manifold.
- We constructed an orientable hyperbolic 6-manifold M of the smallest possible volume $8\pi^3/15$ and $\chi(M) = -1$ by gluing together eight copies of P^6 along there sides.

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- The group Γ is generated by $\Gamma \cap \Gamma_2^6$ and the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & -1 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \quad \text{with } \det(A) = 1.$$

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- The polytope Q^6 is a fundamental polytope for $\Gamma \cap \Gamma_2^6$.
- The group $\Gamma \cap \Gamma_2^6$ is generated by 252 elements of the form $s_i k_i$ where s_i is the reflection in the ith side of Q^6 and $k_i \in \mathrm{K}^6$, with $\det(k_i) = -1$, and so $s_i k_i$ is orientation preserving for each i.

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- The group $\Gamma \cap \Gamma_2^6$ is torsion-free because each maximal finite subgroup of Γ_2^6 maps isomorphically into K^6 under the isomorphism $\Gamma_2^6/(\Gamma \cap \Gamma_2^6) \cong \mathrm{K}^6$.

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- The quotient $\Gamma/(\Gamma \cap \Gamma_2^6)$ is a cyclic group of order 8, since A projects to a matrix $\overline{A}$ in Σ^6 of order 8.

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- Now $A = r_2\overline{A}$ where $r_2 = \mathrm{diag}(1, -1, 1, 1, 1, 1, 1)$.
- Rewriting the equation $1 = (hA^4)(hA^4)$ in terms of $\overline{A}$ leads to a linear equation

$$(I + \overline{A}_*^4)(v) = v_9 + v_{11} + v_{12} + v_{14} + v_{20} + v_{21}.$$

in the $\mathbb{Z}/2$ -vector space $(\Gamma \cap \Gamma_2^6)/[\Gamma_2^6, \Gamma_2^6]$ with basis $v_7, \ldots, v_{27}$ which has no solution. Here v_i is the image of the Q^6 side-pairing map r_ik_i. Hence Γ is torsion-free.

H^6/Γ has minimum volume

- As $\chi(\Gamma_2^6) = -1/8$ and $\Gamma_2^6/(\Gamma \cap \Gamma_2^6)$ has order 64, we have that
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$$\chi(\Gamma) = -8/8 = -1.$$
- The orientable hyperbolic manifold $M = H^6/\Gamma$ has the smallest possible volume $8\pi^3/15$ by the Gauss-Bonnet Theorem.

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$$\chi(\Gamma) = -8/8 = -1.$$
- The orientable hyperbolic manifold $M = H^6/\Gamma$ has the smallest possible volume $8\pi^3/15$ by the Gauss-Bonnet Theorem.
- The manifold M is noncompact with five cusps.

H^6/Γ has minimum volume

- As $\chi(\Gamma^6_2) = -1/8$ and $\Gamma^6_2/(\Gamma \cap \Gamma^6_2)$ has order 64, we have that
$$\chi(\Gamma \cap \Gamma^6_2) = -64/8 = -8.$$
- As $\Gamma/\Gamma \cap \Gamma^6_2$ has order 8, we have that
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- $H_1(M) \cong (\mathbb{Z}/2)^4 \oplus \mathbb{Z}/8$.

References

- The details for this talk are available in our preprint that can be downloaded from lanl.arXiv.org.

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- The preprint version of this talk will appear in Mathematische Annalen.