Implications of hyperbolic geometry to operator K-theory of arithmetic groups

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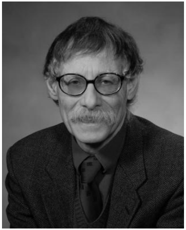
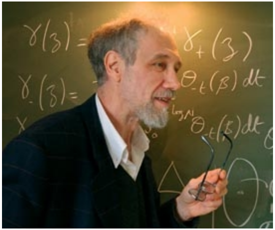
This talk will be about

- The Mathematical topics connected by the Baum/Connes assembly map
- An interesting example: the Bianchi groups
- Motivations for studying Bianchi groups
- The system of representation rings of their finite subgroups
- The equivariant K-homology of the Bianchi groups
- Torsion subcomplexes of the Bianchi groups

The (analytical) assembly map





Paul Frank Baum

Alain Connes

$$\mu_i : K_i^G(\underline{E}G) \longrightarrow K_i(C_r^*(G)), \qquad i \in \mathbb{N} \cup \{0\}$$

The Bianchi groups

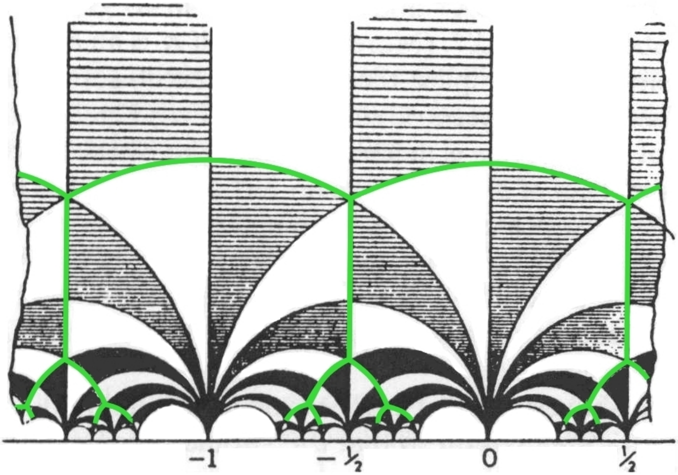
Definition

For m a positive square-free integer, let $\mathcal{O}_{-m}$ denote the ring of algebraic integers in the imaginary quadratic field extension $\mathbb{Q}[\sqrt{-m}]$ of the rational numbers.
The Bianchi groups *are the projective special linear groups* $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$.

Motivations

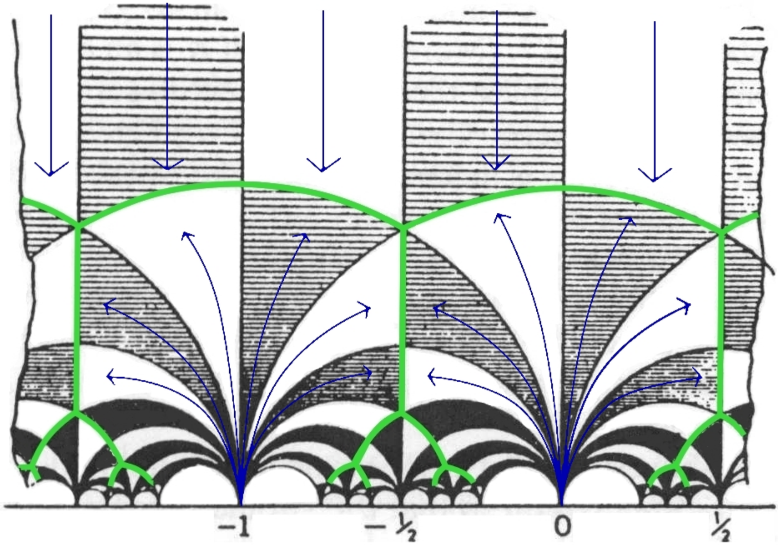
- Group theory
- Hyperbolic geometry
- Knot theory
- Automorphic forms
- Baum/Connes conjecture
- Algebraic K-theory
- Heat kernels
- Quantized orbifold cohomology

The modular tree for $PSL_2(\mathbb{Z})$

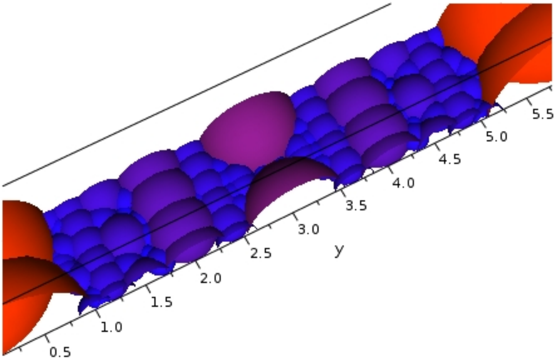


Underlying picture by Robert Fricke for Felix Klein's lecture notes, 1892

The $PSL_2(\mathbb{Z})$-equivariant retraction



A fundamental domain for $\Gamma = \mathrm{PSL}_2\left(\mathbb{Z}[\sqrt{-37}\,]\right)$



$$\begin{array}{ccccc} \mathrm{PSL}_2(\mathbb{Z}) & \hookrightarrow & \mathrm{PSL}_2(\mathbb{R}) & \circlearrowright & \mathcal{H}^2_{\mathbb{R}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{PSL}_2(\mathcal{O}_{-m}) & \hookrightarrow & \mathrm{PSL}_2(\mathbb{C}) & \circlearrowright & \mathcal{H}^3_{\mathbb{R}} \end{array}$$

Complex representation rings of the cell stabilisers

Character tables.

$\mathbb{Z}/2$	1	g
ρ_1	1	1
ρ_2	1	-1

Let $j = e^{\frac{2\pi i}{3}}$.

$\mathcal{A}_4$	1	(12)(34)	(123)	(132)
χ_1	1	1	1	1
χ_2	1	1	j	j^2
χ_3	1	1	j^2	j
χ_4	3	-1	0	0

Frobenius reciprocity: $(\phi|\tau\uparrow)_G = (\phi\downarrow|\tau)_H$

The Bredon chain complex

$$
\begin{array}{c}
0 \\
\downarrow \\
\bigoplus_{\sigma \in {}_{\Gamma}\backslash X^{(2)}} R_{\mathbb{C}}(\Gamma_\sigma) \\
\downarrow \Psi_2 \\
\bigoplus_{\sigma \in {}_{\Gamma}\backslash X^{(1)}} R_{\mathbb{C}}(\Gamma_\sigma) \\
\downarrow \Psi_1 \\
\bigoplus_{\sigma \in {}_{\Gamma}\backslash X^{(0)}} R_{\mathbb{C}}(\Gamma_\sigma) \\
\downarrow \\
0
\end{array}
$$

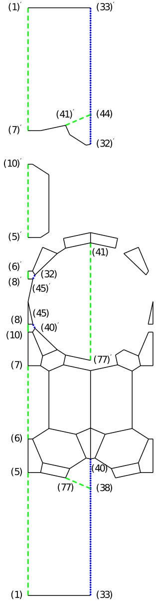
Theorem (R.)
Let $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$. Then, for $\mathcal{O}_{-m}$ principal, the equivariant K-homology of Γ has isomorphy types

	$m = 1$	$m = 2$	$m = 3$	$m = 7$	$m = 11$	$m \in \{19, 43, 67, 163\}$
$K_0^\Gamma(\underline{E}\Gamma)$	$\mathbb{Z}^6$	$\mathbb{Z}^5 \oplus \mathbb{Z}/2$	$\mathbb{Z}^5 \oplus \mathbb{Z}/2$	$\mathbb{Z}^3$	$\mathbb{Z}^4 \oplus \mathbb{Z}/2$	$\mathbb{Z}^{\beta_2} \oplus \mathbb{Z}^3 \oplus \mathbb{Z}/2$
$K_1^\Gamma(\underline{E}\Gamma)$	$\mathbb{Z}$	$\mathbb{Z}^3$	0	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z} \oplus \mathbb{Z}^{\beta_1}$,

where the Betti numbers are

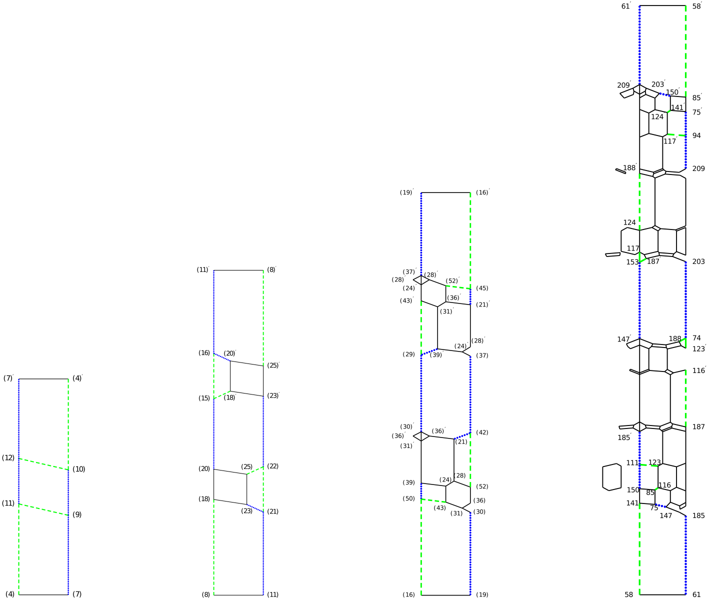
m	19	43	67	163
β_1	1	2	3	7
β_2	0	1	2	6.

Extracting the torsion subcomplexes



For a prime ℓ, consider the subcomplex of the orbit space consisting of the cells with elements of order ℓ in their stabiliser. We call it the ℓ*–torsion subcomplex*.

The non-Euclidean principal ideal domain cases



Geometry of the torsion elements

Theorem (R.)

For any vertex $v \in \mathcal{H}$, there is a natural bijection between the Γ–rotation axes passing through it and the non-trivial cyclic subgroups of its stabiliser.

Corollary (R.)

For any vertex $v \in \mathcal{H}$, the action of its stabiliser on the set of Γ–rotation axes passing through it, restricted from the action of Γ on $\mathcal{H}$, is given by conjugation of its non-trivial cyclic subgroups.

Geometry of the torsion elements

Theorem (R.)

Let v be vertex in hyperbolic 3-space. Then the number n of orbits of subdivided edges adjacent to v, with stabiliser in Γ *isomorphic to* $\mathbb{Z}/\ell\mathbb{Z}$*, is given as follows for* $\ell = 2$ *and* $\ell = 3$*.*

Isomorphy type of Γ_v	$\{1\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$
n for $\ell = 2$	0	2	0	3	2	1
n for $\ell = 3$	0	0	2	0	1	2.

Fritz Grunewald (1949-2010)

$$P^{\ell}(t) := \sum_{q=\mathrm{vcd}(\Gamma)+1}^{\infty} \dim_{\mathbb{F}_\ell} \mathrm{H}_q(\Gamma; \mathbb{Z}/\ell)\ t^q.$$

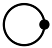
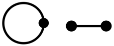
Group homology

Theorem (R.)

The ℓ–primary part of the integral homology of $\mathrm{PSL}_2(\mathcal{O}_{-m})$ depends in degrees greater than 2 (the virtual cohomological dimension) only on the homeomorphism type of the ℓ–torsion subcomplex.

The results in homological 3–torsion

Let $P_m^3(t) := \sum_{q=3}^{\infty} \dim_{\mathbb{F}_3} \mathrm{H}_q(\mathrm{PSL}_2(\mathcal{O}_{\mathbb{Q}[\sqrt{-m}]}); \mathbb{Z}/3)t^q$.

m specifying the Bianchi group	3–torsion subcomplex, homeomorphism type	$P_m^3(t)$
2, 5, 6, 10, 11, 15, 22, 29, 34, 35, 46, 51, 58, 87, 95, 115, 123, 155, 159, 187, 191, 235, 267		$\frac{-2t^3}{t-1}$
7, 19, 37, 43, 67, 139, 151, 163		$\frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}$
13, 91, 403, 427		$2\left(\frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}\right)$
39		$\frac{-2t^3}{t-1} + \frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}$

Thanks a lot
for your attention!