Superstrings in $AdS_5 \times S^5$: some perturbative results

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- Scaling function from light-cone $AdS_5 \times S^5$ superstring
S.Giombi, R.Ricci, R.Roiban, A.T., C.Vergu, 1002.0018

- Exact 1-loop energy of spinning string in $AdS_5 \times S^5$
M.Beccaria, G.Dunne, V.Forini, M.Pawellek, A.T., 1001.4018

General aims:

- understand quantum gauge theories at any coupling

[applications to both perturbative and non-perturbative issues]

- understand string theories in non-trivial backgrounds

[e.g. RR ones for flux compactifications]

AdS/CFT duality:

- relates the two questions suggesting solving them together rather than separately is best strategy
- relates simplest most symmetric theories

use of symmetries on both sides to make progress

Integrability:

Existence of powerful hidden symmetries allowing to solve problem “in principle”

Strategy:

solve simplest most symmetric ("harmonic oscillator") case
then hope to treat other cases "in perturbation theory"

"Harmonic oscillator" (or "Ising", or "WZW"):

planar $\mathcal{N} = 4$ SYM theory = free superstring in $AdS_5 \times S^5$
most symmetric 4-d gauge th. = most symmetric 10-d string th.

$\mathcal{N} = 4$ SYM:

• maximal supersymmetry; conformal invariance;
• integrability? its precise meaning? in which observables?
could be expected in anomalous dimensions
[1-loop gluonic sector – known emergence of XXX spin chain:
Lipatov; Faddeev-Korchemsky, ...]
• in fact, ∞ of hidden symmetries should play broader role:
"inherited" via AdS/CFT from 2-d integrable QFT –
string σ-model: use 2-d int. QFT to solve 4-d CFT

Superstring in $AdS_5 \times S^5$:

• integrable in "canonical" sense:

sigma-model on symmetric space

classical equations admit infinite number of conserved charges

closely related (via Pohlmeyer reduction) to

(super) sine-Gordon and non-abelian Toda eqs

e.g. special motions of strings are described by

the integrable 1-d mechanical systems (Neumann, etc.)

• integrability extends to quantum level:

evidence directly on string-theory side to 2 loops

and also indirectly via AdS/CFT "bootstrap" reasoning

Quantum integrability: should control

• spectrum of closed string energies: $R \times S^1$

[anom. dim's of 2-d primary operators = vertex ops on $R^{1,1}$]

• correlation functions of vertex operators (to which extent?)*

[closed-string scattering amplitudes]

*not clear even in flat space; string field theory is not "integrable"

Integrability = hidden infinite dimensional symmetry
– if valid in quantum string theory –
i.e. at any value of string tension $\frac{\sqrt{\lambda}}{2\pi}$ – any $\lambda = g^2_{\rm YM} N_c$
should be "visible" then – via AdS/CFT – in
perturbative SYM theory

Integrability should then control:
• spectrum of dimensions of gauge-inv. single $\rm tr$ primary operators
[or spectrum of gauge-theory energies on $R \times S^3$]
• correlation functions of these operators ? (to which extent ?!)

What about scattering amplitudes and Wilson loops?
Amplitudes – IR divergent; Cusped Wilson loops – UV divergent
Hidden (Yangian) symmetries broken at loop level in a “useful” way?

Are there “better” observables? (from integrability point of view)
Cross-sections? Effective actions?
Relation to correlation functions of gauge-inv. ops.?
[today’s papers by Alday, Maldacena, Eden, Korchemsky, Sokatchev]

Hints from string theory ?
were crucial in the past (amplitudes $\leftrightarrow$ WL’s, ...)

Recent remarkable progress:

Spectrum of states

I. Spectrum of "long" operators = "semiclassical" string states
determined by Asymptotic Bethe Ansatz (2002-2007)
• its final (BES) form found after intricate superposition of information from perturbative gauge theory (spin chain, BA,...) and perturbative string theory (classical and 1-loop phase,...), use of symmetries (S-matrix), and assumption of exact integrability
• consequences checked against all available gauge and string data

Key example I:
cusp anomalous dimension $\text{Tr}(\Phi D^S \Phi)$

$$f(\lambda \ll 1) = \frac{\lambda}{2\pi^2}\Big[1 - \frac{\lambda}{48} + \frac{11\lambda^2}{2^8 \cdot 45} - (\frac{73}{630} + \frac{4\zeta^2(3)}{\pi^6})\frac{\lambda^3}{2^7} + ...\Big]$$

$$f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi}[1 - \frac{3\ln 2}{\sqrt{\lambda}} - \frac{K}{(\sqrt{\lambda})^2} - ...]$$

Extensions to subleading terms in large S expansion (see below)

II. Spectrum of “short” operators = all quantum string states

Thermodynamic Bethe Ansatz (2005-2009)

• reconstructed from ABA using solely
methods/intuition of 2-d integrable QFT, i.e. string-theory side
(how to incorporate wrapping terms directly on gauge-theory side?)

• highly non-trivial construction – lack of 2-d Lorentz invariance
in the standard “BMN-vacuum-adapted” l.c. gauge

• in few cases ABA “improved” by Luscher corrections is enough:
[Janik et al]
5-loop Konishi dimension and 5-loop minimal twist op. dimension

• crucial to check predictions against perturbative gauge and string data

Key example II:
anomalous dimension of Konishi operator $\mathrm{Tr}(\bar{\Phi}_i \Phi_i)$

$$\gamma(\lambda \ll 1) = \frac{12\lambda}{(4\pi)^2}\Big[1 - \frac{4\lambda}{(4\pi)^2} + \frac{28\lambda^2}{(4\pi)^4} - [208 - 48\zeta(3) + 120\zeta(5)]\frac{\lambda^3}{(4\pi)^6} + 8[158 + 72\zeta(3) - 54\zeta^2(3) - 90\zeta(5) + 315\zeta(7)]\frac{\lambda^4}{(4\pi)^8} + ...\Big]$$

$$\gamma(\lambda \gg 1) = 2\sqrt[4]{\lambda} + \mathrm{b}_0 + \frac{b_1}{\sqrt[4]{\lambda}} + \frac{b_2}{(\sqrt[4]{\lambda})^2} + \frac{b_3}{(\sqrt[4]{\lambda})^3} + ...$$

Suppose can sum up small λ expansion and re-expand at large λ
(finite radius of convergence at $N_c = \infty$)
values of $\mathrm{b}_0, b_1, b_2, ...$?
directly from string theory ?
from TBA/Y-system that should be describing string spectrum ?
[talk by Gromov]

Many open questions:

Analytic form of strong-coupling expansion from TBA/Y-system?
Matching onto string spectrum in near-flat-space expansion?
No level crossing?
Strong-coupling expansion is Borel (non)summable...
exponential corrections $e^{-a\sqrt{\lambda}}$ like in cusp anomaly case?
...

Deeper issues:
Solve string theory from first principles –
• fundamental variables? preserve 2-d Lorentz invariance?
• prove quantum integrability?
lattice version of "supercoset" sigma model?
[cf. talk by Volin]

Planar N=4 SYM – $AdS_5 \times S^5$ string duality:
4d CFT vs 2d CFT
planar correlators of single-tr conformal primary ops in SYM
= correlators of closed-string vertex ops on 2-sphere
equality of the generating functionals

$$\langle e^{\Phi \cdot O} \rangle_{4d} = \langle e^{\Phi \cdot V} \rangle_{2d}$$

O = primary SYM operator of dimension Δ
V = corresponding marginal string vertex operator

$$\Phi \cdot O = \int d^4x' \ \Phi(x') O(x')$$

$$\Phi \cdot V = \int d^4x' \ \Phi(x') V(x', z, ...)$$

$$V = \int d^2\xi \ \mathrm{V}(\xi; x', z, ...)$$

Poincare patch: $ds^2 = z^{-2}(dz^2 + dx^m dx_m)$

$$V = K(\partial X \partial X + ...) \ ,$$

$$K(x - x'; z) = c\big[z + z^{-1}(x - x')^2\big]^{-\Delta}$$

$$K(x - x'; z)_{z \to 0} = \delta^{(4)}(x - x')$$

2-point and 3-point correlators: 4d conf. invariance

$$\langle O_1(x) O_2(x') \rangle_{4d} = \frac{\delta_{\Delta_1,\Delta_2}}{|x-x'|^{2\Delta_1}}$$

$$\langle O_1(x) O_2(x') O_3(x'') \rangle_{4d}$$

$$= \frac{C_{123}}{|x-x'|^{\Delta_1+\Delta_2-\Delta_3} |x-x''|^{\Delta_1+\Delta_3-\Delta_2} |x'-x''|^{\Delta_2+\Delta_3-\Delta_1}}$$

Similar relations for correlators of corresponding V's (2d dim=2).

Problems :

- compute the spectrum, i.e. functions $\Delta(\lambda, Q)$

$\lambda = g^2_{\rm YM} N$, string tension $T = \frac{\sqrt{\lambda}}{2\pi}$

$Q = (S_1, S_2, J_1, J_2, J_3; ..., ...)$ – charges characterizing O_Δ

- compute $C_{123}(\lambda, Q_1, Q_2, Q_3)$

higher-point correlators – via OPE

Progress of 7 years: spectrum is described by integrable system
Δ's of "long" operators (no wrapping): Asymptotic Bethe Ansatz
Δ's of all operators: Thermodynamic Bethe Ansatz
structure fixed by highly non-trivial combination of arguments
from both gauge (small λ) and string (large λ) sides

- gauge theory: dilatation operator, spin chain interpretation, BA
- string theory = 2d sigma model: non-trivial phase in BA

ABA $\rightarrow$ TBA for closed string of finite length ($R \times S^1$)
conjectured to describe wrapping contributions at weak coupling
TBA (or Y-system + additional conditions): complete proposal
[Arutyunov, Frolov; Gromov, Kazakov, Vieira;
Bombardelli, Fioravanti, Tateo, 2009]

TBA: highly non-trivial construction
justified using string (2d sigma model) logic
but so far explored and checked mostly at weak coupling
[Gromov et al; Arutyunov et al; Balog, Hegedus 10]
structure and correspondence with string theory
still to be understood

As for other integrable sigma models formal solution
is to be checked against direct perturbative expansion:
importance of perturbative computations in
quantum $AdS_5 \times S^5$ GS superstring theory

Anomalous dimensions or energies of states on $R \times S^3$
= string energies:
$\Delta(\lambda; S_1, S_2, J_1, J_2, J_3; ...)$

complicated functions of many variables
should be studied using various expansions in different limits
need better understanding of patters of behaviour

Gauge states/operators vs string states:

1. compare states with same global $SO(2,4) \times SO(6)$ charges
e.g., (S, J) folded spinning string
dual to "sl(2) sector" operator $\mathrm{Tr}(D_+^S \Phi^J)$
2. assume no "level crosing" while changing λ:
min/max energy (S, J) states should be in correspondence

• Perturbative gauge theory: $\lambda \ll 1$
$\Delta \equiv E = S + J + \gamma(S, J, \lambda)$
$\gamma = \lambda\gamma_1 + \lambda^2\gamma_2 + \ldots$
fix S, J, ... and expand in λ; then may expand in large S, J

• Semiclassical string theory: $\sqrt{\lambda} \gg 1$
$E = S + J + \gamma(\mathcal{S}, \mathcal{J}, \sqrt{\lambda})$
$\gamma = \sqrt{\lambda} q_0 + q_1 + \frac{1}{\sqrt{\lambda}} q_2 + \ldots$
fix semiclassical parameters $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$, $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$
and expand in $\frac{1}{\sqrt{\lambda}}$; then may expand in large/small $\mathcal{S}$, $\mathcal{J}$

different limits: to match may need to resum expansions

Special limits:

(i) "Fast strings" – "locally-BPS" long operators

GT: $J \gg 1, \quad \frac{S}{J}$=fixed

ST: $\mathcal{J} \gg 1, \quad \frac{S}{\mathcal{J}}$=fixed

$$E = S + J + \frac{\lambda}{J}\left[h_{10} + \frac{1}{J}h_{11} + \frac{1}{J^2}h_{12} + ...\right]$$
$$+\frac{\lambda^2}{J^3}\left[h_{20} + \frac{1}{J}h_{21} + \frac{1}{J^2}h_{22} + ...\right]$$
$$+\frac{\lambda^3}{J^5}\left[h_{30}(\lambda) + \frac{1}{J}h_{31}(\lambda) + ...\right] + ...$$

$h_{nm} = h_{nm}(\frac{S}{J})$ – m-loop string contributions ($J = \sqrt{\lambda}\mathcal{J}$)

$h_{10}, h_{11}, h_{20}, h_{21}$ – same in ST and GT: direct agreement

[Frolov, AT 03; Beisert, Minahan, Staudacher, Zarembo 03; ...]

captured by effective Landau-Lifshitz model

on both string and gauge (spin chain) side

"non-renormalization":

low-derivative terms in Landau-Lifshitz action are protected

$h_{30}(\lambda \ll 1) = a_0 + \lambda a_1 + ...,$
$h_{30}(\lambda \gg 1) = b_0 + \frac{b_1}{\sqrt{\lambda}} + ...$
$a_0 \neq b_0$ implies non-trivial interpolation functions
in dressing phase in ABA [Beisert, AT, 05]

$h_{14}, h_{15}, ...$ and $h_{22}, h_{23}, ...$ also not protected:
$\frac{1}{J^5}$ and higher terms (can be re-arranged)

(ii) "Fast long strings"

GT: $S \gg J \gg 1, \quad j \equiv \frac{J}{\ln S}$=fixed
ST: $\mathcal{S} \gg \mathcal{J} \gg 1, \quad \ell \equiv \frac{\mathcal{J}}{\ln \mathcal{S}}$=fixed = $\frac{j}{\sqrt{\lambda}}$
[Belitsky, Gorsky, Korchemsky 06; Frolov, Tirziu, AT 06;
Alday, Maldacena 07, Freyhult, Rej, Staudacher 07;...]

Subcases: small or large ℓ, j

(iia) $\ell \ll 1, \quad \ln \mathcal{S} \gg \mathcal{J}$

$$E = S + \mathrm{f}(\ell, \sqrt{\lambda}) \ln \mathcal{S} + \ldots$$

$$\mathrm{f}(\ell, \sqrt{\lambda}) = f_0(\ell) + \frac{1}{\sqrt{\lambda}} f_1(\ell) + \frac{1}{(\sqrt{\lambda})^2} f_2(\ell) + \ldots$$

$$\mathrm{f}_{\ell \to 0} = f(\lambda) + \ell^2 \sum_{n=0}^{\infty} \frac{c_n (\ln \ell)^n + d_n (\ln \ell)^{n-1} + \ldots}{(\sqrt{\lambda})^{n-1}} + \mathcal{O}(\ell^4)$$

c_n, d_n fixed by $O(6)$ model truncation [Alday, Maldacena 07]

2-loop string computation [Roiban, AT 07]

$f_2(\ell) = -K + \ell^2 \Big[8(\ln \ell)^2 - 6 \ln \ell + q_{02} \Big] + O(\ell^4)$

$q_{02\,\mathrm{string}} =?= -\frac{3}{2} \ln 2 + \frac{7}{4} - 2K$

(K=Catalan's constant)

comparison to ABA at strong coupling

$q_{02\,\mathrm{ABA}} = -\frac{3}{2} \ln 2 + \frac{11}{4}$

[Gromov 08; Basso, Korchemsky 08; Volin 08]

resolution requires redoing 2-loop string computation on $R^{1,1}$

(iib) $\ell \gg 1$, $\mathcal{J} \gg \ln \mathcal{S}$, i.e. $j = \frac{J}{\ln S} = \sqrt{\lambda}\ell \gg 1$

$$E = S + \mathrm{f}(\lambda, \ell) \ln S + \ldots ,$$

$$\mathrm{f}(\lambda, \ell)_{\ell \gg 1} = j + \frac{\lambda}{j}\Big[\mathrm{c}_{10} + \frac{1}{j}\mathrm{c}_{11} + \frac{1}{j^2}\mathrm{c}_{12} + \ldots\Big]$$

$$+ \frac{\lambda^2}{j^3}\Big[\mathrm{c}_{20} + \frac{1}{j}\mathrm{c}_{21} + \ldots\Big] + \frac{\lambda^3}{j^5}\Big[\mathrm{c}_{30}(\lambda) + \frac{1}{j}\mathrm{c}_{31}(\lambda) + \ldots\Big] + \ldots$$

c_{nm} – m-loop string contributions
$\mathrm{c}_{10} = \frac{1}{2\pi^2}$, $\mathrm{c}_{11} = -\frac{4}{3\pi^2}$, $\mathrm{c}_{20} = -\frac{1}{8\pi^4}$, $\mathrm{c}_{21} = \frac{4}{5\pi^5}$,
protected [BGK, FTT 06; Beccaria 08]

$\mathrm{c}_{12}\lambda \frac{\ln^4 S}{J^3} = \mathrm{c}_{12} \frac{1}{\sqrt{\lambda}} \frac{\ln^4 S}{\mathcal{J}^3}$

2-loop string coeff. = 1-loop SYM coeff. ?
ABA prediction (finite size term): $\mathrm{c}_{12} = \frac{1}{3\pi^2}$
both at weak and strong coupling [Volin 08,09]

direct check of non-renormalization requires
2-loop string computation on $R^{1,1}$

(iii) "Slow long strings" – "long" far-from-BPS operators $\mathrm{Tr}(D_+^S \Phi^J)$
GT: $\ln S \gg J, \ J$=twist=fixed
ST: $\ln \mathcal{S} \gg \mathcal{J}, \ \mathcal{J}$=fixed (e.g. =0)

$$E = S + f(\lambda) \ln S + h(\lambda, J) + \frac{k(\lambda, J)}{\ln S} + ... + O(\frac{1}{S})$$
$$f_{\lambda \gg 1} = c_1 \sqrt{\lambda} + c_2 + ... , \qquad f_{\lambda \ll 1} = b_1 \lambda + b_2 \lambda^2 + ...$$

scaling functions f and h not sensitive to wrappings:
described by ABA
[Beisert,Eden,Staudacher 06; Freyhult,Zieme 09]

$\ln S \gg J$: wrapping contributions suppressed for leading terms
[at 5 loops wrapping corrections start at $\frac{\ln^2 S}{S^2}$
Banjok, Janik, Lukowski 08; Lukowski, Rej, Velizhanin, 09]
$1/S$ term fixed by reciprocity

$k = 0$: prediction (?) of linear integral equation from ABA
[Fioravanti, Grinza, Rossi 09]

no $\frac{1}{\ln S}$ correction at weak coupling (for fixed J)
but for $\lambda \gg 1$ k receives string 1-loop contribution:
finite size correction ($\int dp \to 2\pi L \sum_n, \quad L \sim \ln S$)

$\mathcal{J} = 0$: $\quad k = k_1 + \frac{k_2}{\sqrt{\lambda}} + ...,\qquad k_1 = -\frac{5}{12}\pi$
[Schafer-Nameki,Zamaklar; Beccaria,Dunne,Forini,Pawellek,AT]
Casimir effect of S^5 massless modes ($\ln S$=length)

matching weak coupling expansion would require resummation

$$\mathcal{J} \neq 0 : \qquad E_1 = -\frac{1}{12}\frac{\lambda}{J^2 + \frac{\lambda}{\pi}\ln^2 S}\ln S$$

+ exponential ("Luscher") corrections
of 4 massive ($\sim \mathcal{J}$) modes

first term is protected: 1-loop ST = 1-loop GT
5=1+4 = ABA + wrapping contribution [Gromov]

$k_2 = 0$? requires 2-loop string computation on $R \times S^1$

String background: folded spinning string in $AdS_3 \times S^1$

Folded spinning string in flat space:

$X_1 = \epsilon\ \sin\sigma\ \cos\tau, \quad X_2 = \epsilon\ \sin\sigma\ \sin\tau$

$$ds^2 = -dt^2 + dX_i dX_i = -dt^2 + d\rho^2 + \rho^2 d\phi^2$$

$$t = \epsilon\tau\ , \quad \rho = \epsilon\sin\sigma\ , \quad \phi = \tau$$

tension $T = \frac{1}{2\pi\alpha'} \equiv \frac{\sqrt{\lambda}}{2\pi}$

energy $E = \epsilon\sqrt{\lambda}$ and spin $S = \frac{\epsilon^2}{2}\sqrt{\lambda}$ – Regge relation:

$$E = \sqrt{2\sqrt{\lambda}S}$$

Folded spinning string in AdS_3: [Gubser,Klebanov,Polyakov 02]

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\phi^2$$

$$t = \kappa\tau, \quad \phi = w\tau, \quad \rho = \rho(\sigma)$$

$$\sinh\rho = \epsilon\,\mathrm{sn}(\kappa\epsilon^{-1}\sigma,\ -\epsilon^2)\ , \quad 0 < \rho < \rho_{\max}$$

$$\coth\rho_{\max} = \frac{w}{\kappa} \equiv \sqrt{1 + \frac{1}{\epsilon^2}}$$

ϵ measures length of the string

$$\kappa = \epsilon \,{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; -\epsilon^2)$$

classical energy $E_0 = \sqrt{\lambda}\mathcal{E}_0$ and spin $S = \sqrt{\lambda}\mathcal{S}$

$$\mathcal{E}_0 = \epsilon \,{}_2F_1(-\frac{1}{2}, \frac{1}{2}; 1; -\epsilon^2), \qquad \mathcal{S} = \frac{\epsilon^2\sqrt{1+\epsilon^2}}{2} \,{}_2F_1(\frac{1}{2}, \frac{3}{2}; 2; -\epsilon^2)$$

solve for ϵ – analog of Regge relation

$$\mathcal{E}_0 = \mathcal{E}_0(\mathcal{S}) \, , \qquad E_0 = \sqrt{\lambda}\, \mathcal{E}_0(\frac{S}{\sqrt{\lambda}})$$

short/long string – flat space/AdS interpolation:

$$\mathcal{E}_0(\mathcal{S} \ll 1) = \sqrt{2\mathcal{S}} + ...$$
$$\mathcal{E}_0(\mathcal{S} \gg 1) = \mathcal{S} + \tfrac{1}{\pi} \ln \mathcal{S} + ...$$

$\mathcal{S} \to \infty$: folds reach the boundary ($\rho = \infty$)
solution drastically simplifies: length $\kappa \sim \ln \mathcal{S} \to \infty$

$$t = \kappa\tau, \quad \phi = \kappa\tau, \quad \rho = \kappa\sigma\ , \quad \kappa \sim \epsilon \sim \ln \mathcal{S} \to \infty$$

$E = S$ from massless end points at AdS boundary (null geodesic)
$E - S \approx \frac{\sqrt{\lambda}}{\pi} \ln S$ from tension/stretching of the string
quantum superstring corrections to E respect $S + \ln S$ form

Generalization to $J \neq 0$: $AdS_3 \times S^1$ [Frolov, AT 03]

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\phi^2 + d\varphi^2$$

homogeneous large spin limit: $\ln S \to \infty$, $\frac{\mathcal{J}}{\ln \mathcal{S}}$=fixed

$$t = \kappa\tau, \quad \phi = \kappa\tau, \quad \rho = \mu\sigma\ , \quad \varphi = \nu\tau$$

$$\kappa^2 = \mu^2 + \nu^2\ , \qquad \mu = \frac{1}{\pi}\ln\mathcal{S}\ , \quad \nu = \mathcal{J}$$

$$E = S + \sqrt{J^2 + \frac{\lambda}{\pi}\ln^2 S}$$

$$E - S = \frac{\sqrt{\lambda}}{\pi}\mathrm{f}(\ell)\ \ln S\ , \qquad \ell = \pi\frac{\mathcal{J}}{\ln\mathcal{S}}$$

$$\mathrm{f}(\ell) = \sqrt{\ell^2 + 1} + \frac{\mathrm{f}_1(\ell)}{\sqrt{\lambda}} + \frac{\mathrm{f}_2(\ell)}{(\sqrt{\lambda})^2} + ...$$

How to compute quantum string corrections to energy?

Compute $E = \langle E \rangle,\ S = \langle S \rangle, J = \langle J \rangle$
find $E = E(S, J)$
starting point: string sigma model path integral at fixed charges
free energy with chemical potentials
(cf. quantization of non-topological solitons)
[Roiban, AT 07; Giombi, Ricci, Roiban, AT, Vergu 10]
conformal gauge (impose Virasoro) or l.c. gauge

$$Z = e^{-\beta \Sigma(\kappa,\nu)} = \mathrm{Tr}[e^{-\beta \widetilde{H}_{2d}}]$$

$$\widetilde{H}_{2d} = H_{2d} + \kappa(E - S) - \nu J$$

$$\Sigma = \mathcal{F}(\widehat{\nu}) L\ , \qquad L = \frac{\sqrt{\lambda}}{\pi} \ln S\ , \qquad \widehat{\nu} = \frac{\sqrt{\lambda}\nu}{L}$$

$$\mathrm{f}(\ell) \equiv \frac{\pi}{\sqrt{\lambda}} \frac{E - S}{\ln S} = \sqrt{1 + \widehat{\nu}^2}\ \Big[\mathcal{F}(\widehat{\nu}) - \widehat{\nu}\frac{d\mathcal{F}(\widehat{\nu})}{d\widehat{\nu}}\Big]$$

$$\ell \equiv \frac{\pi}{\sqrt{\lambda}} \frac{J}{\ln S} = \widehat{\nu}\mathcal{F}(\widehat{\nu}) - (1 + \widehat{\nu}^2)\frac{d\mathcal{F}(\widehat{\nu})}{d\widehat{\nu}}$$

non-trivial relation between generalized scaling function
and string partition function starting from 2 loops:

$$\mathrm{f}_0 = \sqrt{1+\ell^2}\,, \qquad \mathrm{f}_1 = \frac{\mathcal{F}_1(\ell)}{\sqrt{1+\ell^2}}$$

$$\mathrm{f}_2 = \frac{1}{\sqrt{1+\ell^2}}\Big[\mathcal{F}_2(\ell) + \frac{1}{2}\Big(\frac{\ell\mathcal{F}_1(\ell)}{\sqrt{1+\ell^2}} - \sqrt{1+\ell^2}\,\frac{d\mathcal{F}_1(\ell)}{d\ell}\Big)^2\Big]$$

$$= \frac{\mathcal{F}_2(\ell)}{\sqrt{1+\ell^2}} + \frac{1}{2}(1+\ell^2)^{3/2}\Big(\frac{d\mathrm{f}_1}{d\ell}\Big)^2$$

Compute partition function expanding near classical solution

Start with GS $\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}$ action
in AdS l.c. gauge: important technical simplification

Superstring theory in $AdS_5 \times S^5$

bosonic coset $\frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

generalized to supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ [Metsaev, AT 98]

$$S = T \int d^2\sigma \Big[G_{mn}(x)\partial x^m \partial x^n + \bar{\theta}(D + F_5)\theta\partial x + \bar{\theta}\theta\bar{\theta}\theta\partial x \partial x + ... \Big]$$

tension $T = \frac{\mathrm{R}^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$

Conformal invariance: $\beta_{mn} = R_{mn} - (F_5)^2_{mn} = 0$

Classical (Luscher-Pohlmeyer 76) integrability of coset σ-model true also for $AdS_5 \times S^5$ superstring [Bena, Polchinski, Roiban 02]

Much progress in understanding of implications of (semi)classical and quantum integrability

Poincare coordinates ($m = 0, 1, 2, 3;\ M = 1, ..., 6$):

$$ds^2 \frac{1}{z^2}(dx^+ dx^- + dx^* dx + dz^M dz^M)$$
$$= \frac{1}{z^2}(dx^m dx_m + dz^2) + du^M du^M \ , \quad u^M u^M = 1$$
$$x^{\pm} = x^3 \pm x^0 \ , \qquad x, x^* = x^1 \pm \mathrm{i} x^2 \ , \quad z^M = z u^M$$

AdS l.c. gauge: [Metsaev, Thorn, AT '00]

$$\sqrt{-g} g^{\alpha\beta} = \mathrm{diag}(-z^2, z^{-2}) \ , \quad x^+ = \tau \ , \qquad \Gamma^+ \vartheta^I = 0$$
$$I = \frac{1}{2} T \int d\tau \int d\sigma \ \mathcal{L} \ , \qquad T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$$
$$\mathcal{L} = \dot{x}^* \dot{x} + (\dot{z}^M + \mathrm{i} z^{-2} z_N \ \eta_i \rho^{MN\,i}{}_j \eta^j)^2$$
$$+\mathrm{i}(\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i - h.c.) - z^{-2}(\eta^2)^2 + z^{-4}(x'^* x' + z'^M z'^M)$$
$$+2\mathrm{i}\Big[\ z^{-3} \eta^i \rho^M_{ij} z^M (\theta'^j - \mathrm{i} z^{-1} \eta^j x') + h.c.\Big]$$

$\mathcal{L}$: only quartic in fermions with "standard" kinetic terms
non-trivial issue of regularization preserving symmetries
UV divergences should cancel

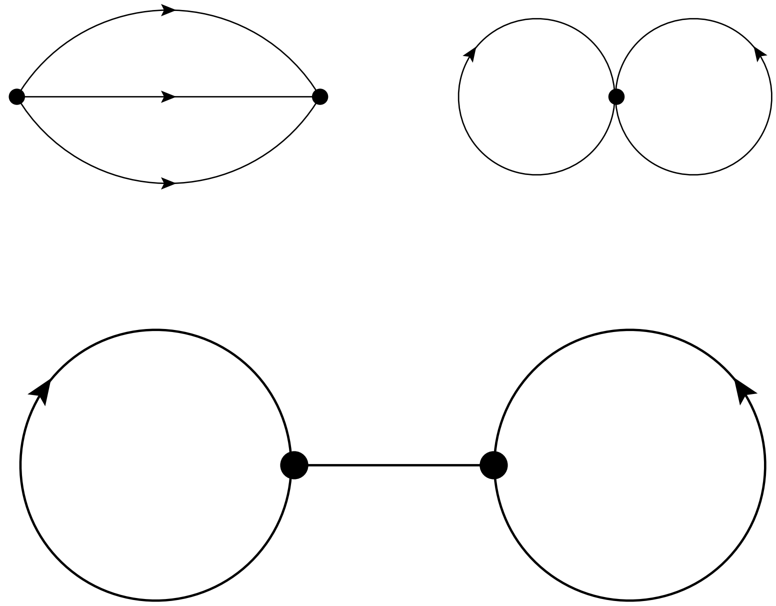
Background: infinite (S,J) folded string
$\to$ "null cusp" + rotation in $z_5 + i z_6 = z e^{i\varphi}$

$$z = \sqrt{\frac{\kappa}{\mu}}\sqrt{\frac{\tau}{\sigma}}\ , \quad x^+ = \tau\ , \quad x^- = -\frac{\kappa}{2\mu}\ \frac{1}{\sigma}\ , \quad \varphi = \frac{\widehat{\nu}}{2\kappa}\ln\tau$$

$L \sim \ln S \to \infty$: "decompactification" – string on $R^{1,1}$
one loop: reproduce conf. gauge result [Frolov,Tirziu,AT 06]

$$E = S + \frac{\sqrt{\lambda}}{\pi}\mathrm{f}(\ell)\ \ln S$$

$$\mathrm{f}_1(\ell) = \frac{1}{\sqrt{1+\ell^2}}\Big[\sqrt{1+\ell^2} - 1 + 2(1+\ell^2)\ln(1+\ell^2)$$
$$- \ \ell^2\ln\ell^2 - 2(1+\tfrac{1}{2}\ell^2)\ln[\sqrt{2+\ell^2}(1+\sqrt{1+\ell^2})]\Big]$$
$$= -3\ln 2 - 2\ell^2(\ln\ell - \frac{3}{4}) + \ell^4(\ln\ell - \frac{3}{8}\ln 2 - \frac{1}{16}) + O(\ell^6)$$



2-loop computation of $\ln Z$ and thus of f_2:
straightforward but very involved:
non-diagonal propagator for 8+8 fields;
lack of 2d Lorentz covariance

Small ℓ expansion tractable and gives:

$$\mathrm{f}_2 = -K \ + \ell^2 (8 \ln^2 \ell - 6 \ln \ell - \frac{3}{2} \ln 2 + \frac{11}{4}) + \ell^4 (-6 \ln^2 \ell - \frac{7}{6} \ln \ell + 3 \ln 2 \ln \ell - \frac{9}{8} \ln^2 2 + \frac{11}{8} \ln 2 + \frac{3}{32} K - \frac{233}{576}) + \mathcal{O}(\ell^6)$$

full agreement with ABA [Gromov 08]

Large ℓ expansion:

$$\mathrm{f}_2 = \frac{c}{\ell^3} + O(\frac{1}{\ell^4})$$

c appears to match $\frac{1}{\pi^2} \mathrm{c}_{12} = \frac{1}{3\pi^4}$ from ABA [Volin 08]
[Giombi, Ricci, Roiban, AT, in progress]

Small ℓ expansion at higher loops:
leading log ℓ terms generated by non-1PI graphs
can be resummed to all orders
[Giombi et al; Roiban, talk at IGST '10]

$$\mathcal{F} = \sqrt{1 + \frac{2}{\sqrt{\lambda}}\mathcal{F}_1}\ , \qquad \mathcal{F}_1 = -2\widehat{\nu}^2 \ln \widehat{\nu}$$

$$\mathrm{f}(\lambda, \ell)_{\text{ leading} \ln \ell} = \sqrt{1 + \frac{\ell^2}{1 + \frac{4}{\sqrt{\lambda}} \ln \ell}}$$

agreement with ABA [Gromov 08]

Highly non-trivial checks of quantum integrability
of $AdS_5 \times S^5$ superstring and consistency of ABA

Finite spin / finite length corrections?
comparison with TBA at strong coupling with $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$, etc fixed ?

1-loop order:
full agreement (including exp corr's) guaranteed [Gromov 09]

2-loop order: still need a non-trivial check

Finite $\mathcal{S} = \frac{S}{\sqrt{\lambda}}, \ \ \mathcal{J} = 0$:
use (i) exact elliptic folded string solution and (ii) $R \times S^1$
tractable at 1 loop – Lame operators
[Beccaria, Dunne, Forini, Pawellek, AT]
but seems hard to extend to 2 loops

Important simplification if want only $\frac{1}{\ln S}$ term:
use asymptotic (rational) solution, but on $R \times S^1$
was shown to be enough to reproduce 1 - loop coeff [BDFPT]

2-loop computation of $\frac{1}{\ln S}$ term ($\mathcal{J} = 0$):

only diagrams with massless propagators may contribute
detailed analysis of such diagrams with $p = \frac{2\pi n}{L}, \quad L = 2 \ln S$:
sum of such diagrams is UV and IR finite
and does not contain $\frac{1}{L}$ term (no “Casimir” term at 2 loops)
[Giombi, Ricci, Roiban, AT, to appear]

remains to be reproduced from TBA
(comparison may depend on how $\mathcal{J} \to 0$ limit is taken)

3-point functions: semiclassics at strong coupling?

$C_{123} = C^{(0)}_{123}(1 + \lambda \sum_{n=1}^{3} c_n \gamma_n^{(1)} + ...)$
[Okuyuama, Tseng04; Grosardt, Plefka 10]

If exponentiation ($\exp \sum_{n=1}^{3} c_n \gamma_n$) then
$e^{a\sqrt{\lambda}}$ behaviour at strong coupling for operators with large spins
can be captured by semiclassical approximation
as is known to be true for 2-point function ?
[Janik, Surowka, Wereszczynski 10; Buchbinder, AT 10]

would be first step to see if (some?) 3-point correlators
are also described by an integrable system

Conclusions

• understanding of perturbative quantum $AdS_5 \times S^5$ superstring theory consistent with quantum integrability;
technical advantages of AdS l.c. gauge

• 2-loop string computation with a free spin parameter
– confirmation of ABA at strong coupling (beyond doubt)

• interpolation to weak coupling and order of limits issues
for non-trivial spins still to be understood

• TBA at strong coupling at 2 loops: still remains to be checked
$\frac{1}{\ln S}$ term as a testing ground?

• "short" operators vs quantum string states:
check of TBA for Konishi operator remains an open issue
requires
(i) analysis of TBA at strong coupling beyond
semiclassical (large spin) limit
(ii) understanding of quantum superstring spectrum
in near flat space expansion

Strong-coupling test of TBA against string theory for Konishi state?

Still open question about subleading terms
in strong-coupling expansion of Konishi dimension:

$$\gamma(\lambda \gg 1) = 2\sqrt[4]{\lambda} + b_0 + \frac{b_1}{\sqrt[4]{\lambda}} + \frac{b_2}{(\sqrt[4]{\lambda})^2} + \frac{b_3}{(\sqrt[4]{\lambda})^3} + ...$$

TBA: $b_1 \approx 2$ [Gromov, Kazakov, Vieira, 2009; Frolov, 2010]

Semiclassical string theory argument: $b_1 = 1$ [Roiban, AT 2009]
based on several assumptions (order of limits, etc.)
Need to push further perturbative string theory computations
(near flat space expansion, AdS l.c. gauge, ...)
as well as develop analytic methods on TBA side

Semiclassical string theory: universality of b_1?

integer for rational solutions but not for elliptic ones?

Folded spinning string and pulsating string cases

[Tirziu, AT 2008; Beccaria, Dunne, Forini, Pawellek, AT 2010; Beccaria, Dunne, Macorini, Tirziu, AT, in progress]

Folded spinning string in AdS_3

$$E = \sqrt{2S\sqrt{\lambda}}\Big(1 + \frac{\frac{3}{8}S + \frac{3}{2} - 4\log 2}{\sqrt{\lambda}} + \dots\Big) + 1 + \dots$$

Folded spinning string in $\mathbb{R} \times S^2$

$$E = \sqrt{2J\sqrt{\lambda}}\Big(1 + \frac{\frac{1}{8}J + 2 - 4\log 2}{\sqrt{\lambda}} + \dots\Big) + 2 + \dots$$

Pulsating string in AdS_3

$$E = \sqrt{2N\sqrt{\lambda}}\Big(1 + \frac{\frac{5}{8}N + \frac{5}{2} - 4\log 2}{\sqrt{\lambda}} + \dots\Big) + 1 + \dots$$

Pulsating string in $\mathbb{R} \times S^2$

$$E = \sqrt{2N\sqrt{\lambda}}\Big(1 + \frac{-\frac{1}{8}N + 1 - 4\log 2}{\sqrt{\lambda}} + \dots\Big) + 2 + \dots$$

Relation to Konishi states: $J = 2,\ S = 2$, ... ?