Helmholtz problems at large wavenumber k

J.M. Melenk

joint work with

S. Sauter (Zürich) M. Löhndorf (Vienna) S. Langdon (Reading)

TU Wien
Institut für Analysis und Scientific Computing

outline

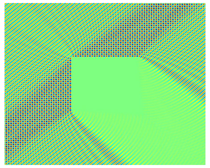
1. domain-based methods: convergence analysis for hp-FEM
2. discussion of some non-standard FEMs
3. BEM for Helmholtz problems

references used in the talks

- F. Ihlenburg: FE Analysis of acoustic scattering, Springer Verlag, 1998
- M. Ainsworth: Discrete dispersion relation for hp-version finite element approximation at high wave number, SINUM 42 (2004), 553-575
- J.M. Melenk, S. Sauter: Convergence Analysis for Finite Element Discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions, Math. Comp., 79 (2010), 1871-1914
- J.M. Melenk, S. Sauter: Wave-Number Explicit Convergence Analysis for Galerkin Discretizations of the Helmholtz Equation, ASC Rep. 31/2009
- I. Babuška, S. Sauter: Is the pollution effect of the FEM avoidable for the Helmholtz equation?, SIREV, 42 (2000), 451-484
- E. Perrey-Debain, O. Laghrouche, P. Bettess, J. Trevelyan: Plane wave basis finite elements and boundary elements for three-dimensional wave scattering,, Phil. Trans. R. Soc. Lond. A, 262 (2004), 561-577
- R. Tezaur and C. Farhat, three-dimensional discontinous Galerkin elements with plane waves and Lagrange multipliers for the solution of mid-frequency Helmholtz problems, IJNME, 66 (2006), 796-815
- C. Gittelson, R. Hiptmair, I. Perugia: Plane wave Discontinuous Galerkin methods, M^2AN, 43 (2009), 297-331
- A. Buffa, P. Monk, error estimates for the UWVF of the Helmholtz equation, M^2AN, 42 (2008), 925-940
- R. Hiptmair, A. Moiola, I. Perugia, Approximation by plane waves, SAM Report 2009-27, ETH Zürich.
- R. Hiptmair, A. Moiola, I. Perugia, Plane wave discontinuous Galerkin methods for the 2D Helmholtz equation: analysis of the p-version, SAM Report 2009-20, ETH Zürich
- A. Moiola, Approximation properties of plane wave spaces and application to the analysis of the plane wave discontinuous Galerkin method, SAM Report 2009-06, ETH Zürich
- Melenk: mapping properties of combined field Helmholtz boundary integral operators, ASC Rep. 01/2010
- M. Löhndorf, J.M. Melenk: wavenumber-explicit hp-BEM for high frequency scattering, ASC Rep. 02/2010
- S. Chandler-Wilde, P. Monk: wave-number-explicit bounds in time-harmonic scattering, SIMA 39 (2008), 1428-1455
- S. Chandler-Wilde, I. Graham: Boundary integral methods in high frequency scattering in: highly oscillatory problems, Cambridge University Press, B. Engquist, A. Fokas, E. Hairer, A. Iserles eds., 2009
- T. Betcke, S. Chandler-Wilde, I. Graham, S. Langdon, M. Lindner: condition number estimates for combined potential integral operators in acoustics and their boundary element discretization, to appear in Numer. Meths. PDEs

Helmholtz model problem

$$\begin{aligned} -\Delta u - k^2 u &= f && \text{in } \Omega,\\ b.c. && && \text{on } \partial\Omega,\\ (\text{radiation condition} && && \text{at } \infty) \end{aligned}$$



Examples:

- acoustic scattering problems
- electromagnetic scattering problems

goals:

- efficient and reliable numerical methods also for large $k > 0$.

Discretization

- domain-based discretizations $\rightsquigarrow$ FEM
- integral equation based discretizations $\rightsquigarrow$ BEM

the discretizations have the abstract form:

$$\text{find } u_N \in V_N \text{ s.t.} \quad a_k(u_N, v) = l(v) \qquad \forall v \in W_N.$$

how to choose V_N, W_N, a_k for large k?

fundamental issues

- approximability: approximate u from V_N well:
  - standard (polynomial based) approximation
  - nonstandard approximation (e.g., info from asymptotics)
- stability: ideally: (asymptotic) quasi-optimality, i.e.,

$$\|u - u_N\| \le C \inf_{v \in V_N} \|u - v\|$$

with some C independent of k in a norm $\|\cdot\|$ of interest

model problem

$$\begin{aligned} Lu := -\Delta u - k^2 u &= f && \text{in } \Omega, \quad \Omega \subset \mathbb{R}^d \text{ bounded},\\ Bu := \partial_n u - \mathbf{i} k u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

weak formulation

find $u \in H^1(\Omega)$ s.t. $a(u, v) = l(v) \qquad \forall v \in H^1(\Omega)$

where

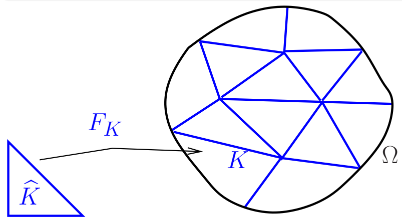
$$\begin{aligned} a(u, v) &:= \int_\Omega \nabla u \cdot \nabla \overline{v} - k^2 \int_\Omega u \overline{v} - \mathbf{i} k \int_{\partial\Omega} u \overline{v},\\ l(v) &:= \int_\Omega f \overline{v}. \end{aligned}$$

hp-FEM spaces V_N

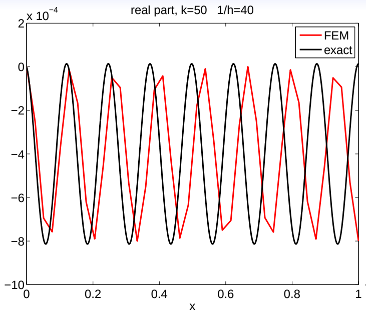
abstract FEM discretization

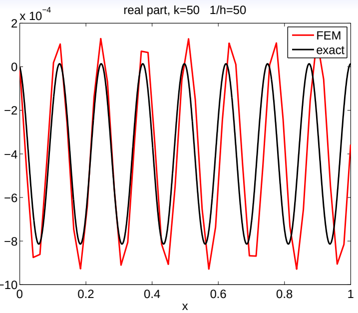
given $V_N \subset H^1(\Omega)$ find $u_N \in V_N$ s.t.

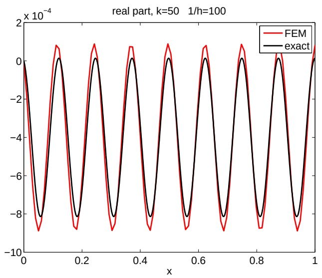
$$a(u_N, v) = l(v) \qquad \forall v \in V_N.$$



- $\mathcal{T}$ = triangulation of $\Omega \subset \mathbb{R}^d$ with element maps F_K
- $\operatorname{diam} K \sim h$ for all $K \in \mathcal{T}$
- $V_N := S^p(\mathcal{T}) := \{u \in H^1(\Omega) \,|\, u \circ F_K \in \mathcal{P}_p\}$
- $N := \dim V_N \sim h^{-d} p^d$

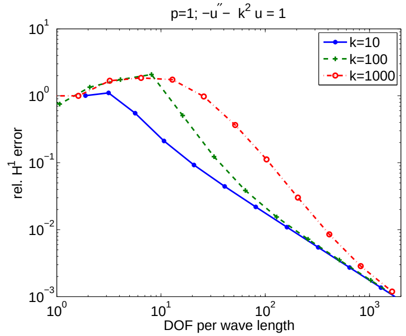






$$-u'' - k^2 u = 1 \quad \text{in } (0,1)$$
$$u(0) = 0, \qquad u'(1) - \mathbf{i}ku(1) = 0$$

stability/onset of asymptotic quasioptimality



- kh = constant is not sufficient to obtain given (relative) accuracy ("pollution")

mathematical analysis:

- Babuška & Ihlenburg: 1D, uniform meshes, arbitrary, fixed p
- Ainsworth: $d > 1$, infinite tensor product mesh, p-explicit

dispersion analysis: p.w. linears, uniform mesh

$$-u'' - k^2 u = f, \qquad u(0) = 0, \quad u'(1) - \mathbf{i}ku(1) = 0$$

cont. Green's fct: $$G(x,y) = k^{-1} \begin{cases} \sin kx e^{\mathbf{i}ky} & 0 < x < y \\ \sin ky e^{\mathbf{i}kx} & y < x < 1 \end{cases}$$

disc. Green's fct: $$G_h(x,y) = \frac{1}{h \sin k'h} \begin{cases} \sin k'x\,(A \sin k'y + \cos k'y) & 0 < x < y \\ \sin k'y\,(A \sin k'x + \cos k'x) & y < x < 1 \end{cases}$$

where

- $A = A(k,k',h) \in \mathbb{C}$ is a constant
- k' = discrete wave number
- dispersion relation $\cos k'h = \cos \dfrac{6 - 2k^2h^2}{6 + k^2h^2}$

For kh small, we get $k'h = kh - \frac{1}{24}(kh)^3 + \cdots$ and therefore

$$k' = k - \frac{1}{24} k^3 h^2 + \cdots$$

discrete dispersion analysis, I

Theorem (Babuska & Ihlenburg)

For the 1D model problem and piecewise polynomial approximation of degree $p \geq l$ on a uniform mesh, there holds for $kh < \pi$ and solution $u \in H^{l+1}(0,1)$:

$$\|u - u_N\|_{H^1(0,1)} \leq C_{p,l} \underbrace{\left[1 + k \left(\frac{kh}{2p}\right)^p\right]}_{\text{pollution}} \underbrace{\left(\frac{h}{2p}\right)^l |u|_{H^{l+1}(0,1)}}_{\text{best approximation error}}$$

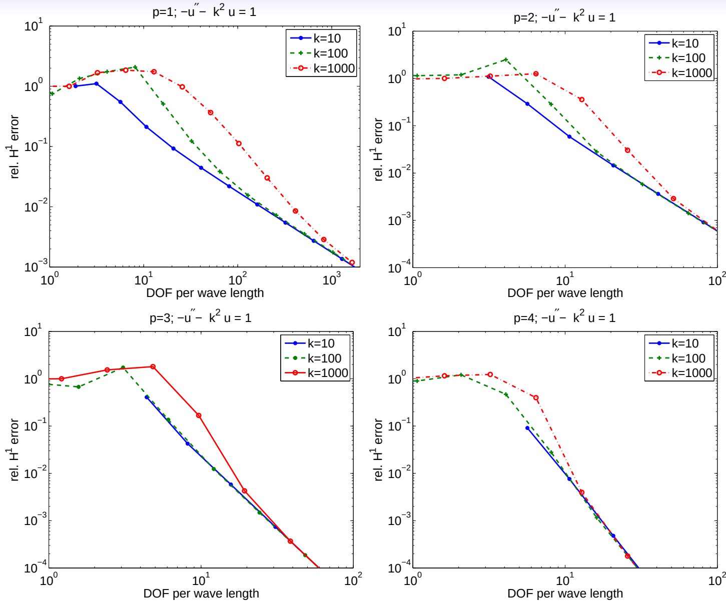
conclusion

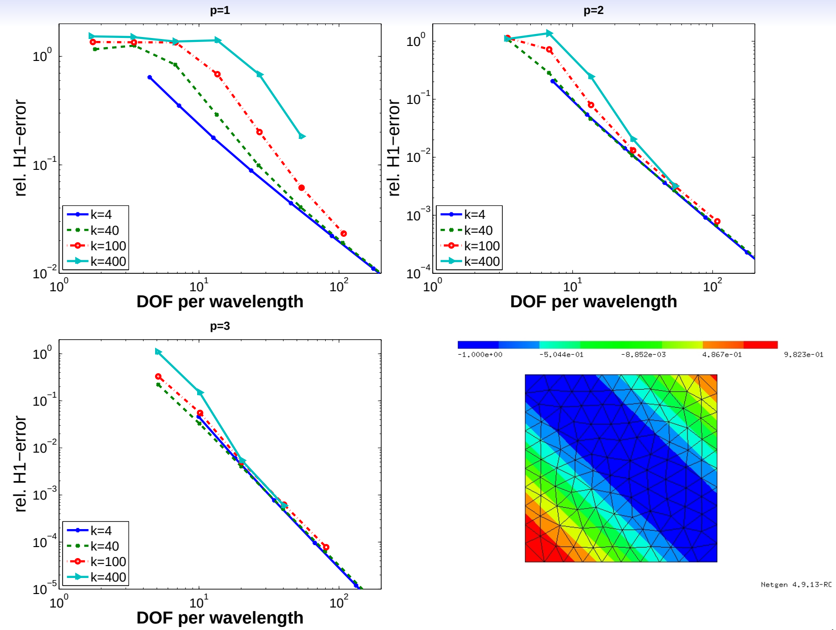
- the phase error ("pollution") is not as pronounced for higher order elements as for lower order elements.
- suggests that

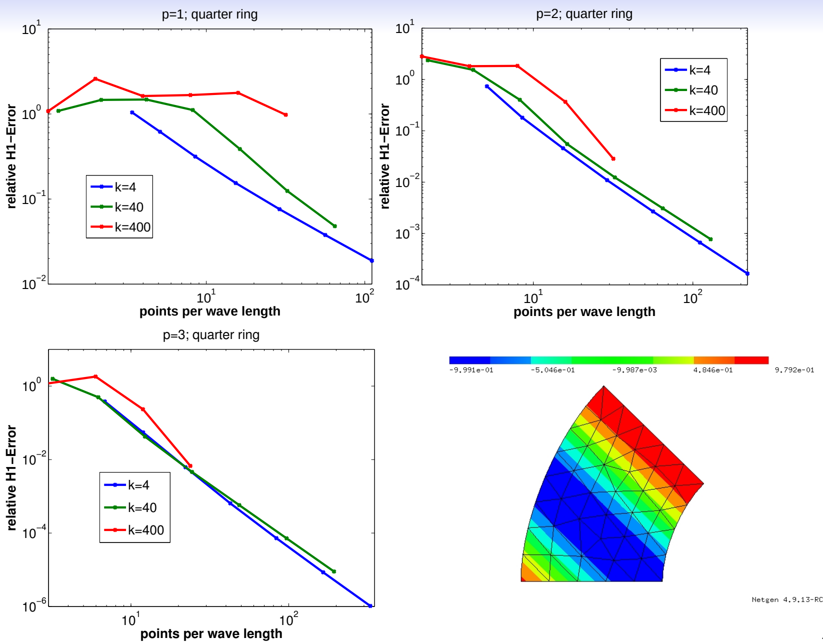
$$k \left(\frac{kh}{\sigma p}\right)^p \quad \text{small}$$

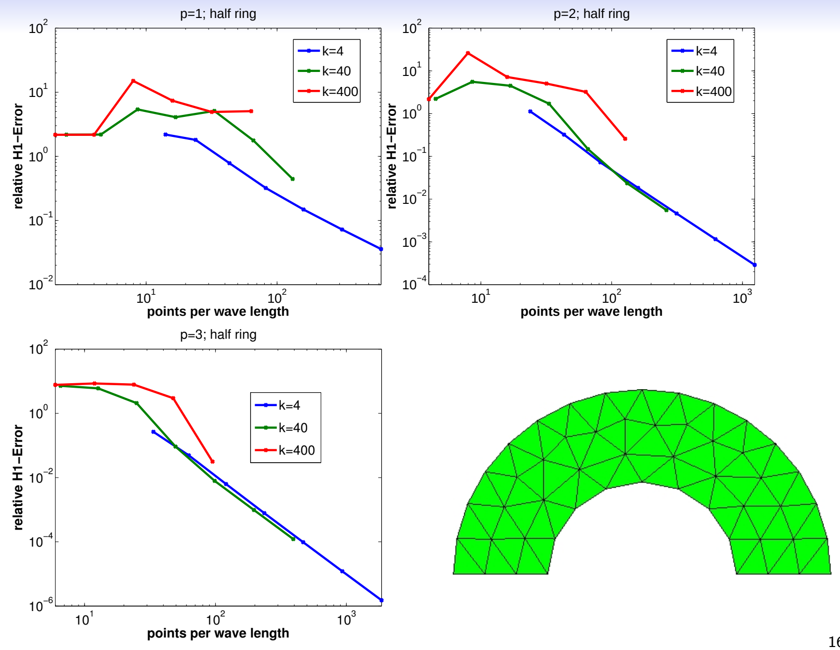
is an important condition

details


p=1; −u'' − k² u = 1
p=2; −u'' − k² u = 1
p=3; −u'' − k² u = 1
p=4; −u'' − k² u = 1
k=10
k=100
k=1000
rel. H¹ error
DOF per wave length


p=1
p=2
p=3
k=4
k=40
k=100
k=400
rel. H1−error
DOF per wavelength
Netgen 4.9.13-RC


p=1; quarter ring
p=2; quarter ring
p=3; quarter ring
k=4
k=40
k=400
relative H1−Error
points per wave length
Netgen 4.9.13-RC


p=1; half ring
p=2; half ring
p=3; half ring
k=4
k=40
k=400
relative H1−Error
points per wave length

stability analysis of hp-FEM

goals:

- show that the scale resolution conditions

$$\frac{kh}{p} \text{ small} \qquad \text{together with} \qquad p \geq C \log k$$

  is sufficient to guarantee quasi-optimality of the hp-FEM
- no uniform meshes ($\rightarrow$ no discrete Green's function)
- use only stability of the continuous problem

assumptions:

- geometry is (piecewise) analytic
- solution operator $f \mapsto u$ grows only polynomially in k (in a suitable norm)

techniques:

- view Helmholtz problems as "H^1-elliptic plus compact perturbation"
- study regularity of suitable adjoint problems

stability of the continuous problem

$$\begin{aligned} -\Delta u - k^2 u &= f && \text{in } \Omega \\ \partial_n u - \mathbf{i} k u &= g && \text{on } \partial\Omega \end{aligned}$$

Theorem

Let $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$ be a bounded Lipschitz domain. Then

$$\|u\|_{1,k} \leq Ck^{5/2}\left[\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right]$$

where

$$\|v\|_{1,k}^2 := |v|_{H^1(\Omega)}^2 + k^2\|v\|_{L^2(\Omega)}^2$$

remark

If Ω is star-shaped with respect to a ball, then

$$\|u\|_{1,k} \leq C\left[\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right]$$

remarks on stability of the continuous problem

- one option to get *a priori* estimates is the use of judicious test functions.
- For star shaped domains, an interesting test fct is $v = x \cdot \nabla u$ and then clever integration by parts (Rellich identities)
- here: use estimates for layer potentials

Theorem (quasioptimality of hp-FEM)

Let $\partial\Omega$ be analytic. Then there exist c_1, c_2, $C > 0$ independent of h, p, k s.t. for

$$\frac{kh}{p} \leq c_1 \qquad \text{and} \qquad p \geq c_2 \log k$$

there holds:

$$\|u - u_N\|_{1,k} \leq C \inf_{v \in V_N} \|u - v\|_{1,k}$$

where $\|v\|^2_{1,k} = |v|^2_{H^1(\Omega)} + k^2\|v\|^2_{L^2(\Omega)}$.

Remark

- if $p = O(\log k)$ then there is no "pollution"
- choice $p \sim \log k$ and $h \sim p/k$ leads to quasioptimality for a fixed number of points per wavelength
- generalization to polygonal Ω possible (see below)

the adjoint problem

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla \overline{v} - k^2 \int_\Omega u\overline{v} - \mathbf{i}k \int_{\partial\Omega} u\overline{v}.$$

adjoint solution operator S^*_k:

$$u^* = S^*_k(f) \text{ solves} \qquad a(v, u^*) = \int_\Omega v\overline{f} \qquad \forall v \in H^1(\Omega)$$

strong formulation:

$$\begin{aligned} -\Delta u^* - k^2 u^* &= f \quad \text{in } \Omega, \\ \partial_n u^* + \mathbf{i}ku^* &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

notation

adjoint solution operator S^*_k:

$$u^* = S^*_k(f) \text{ solves} \qquad a(v, u^*) = \int_\Omega v\overline{f} \qquad \forall v \in H^1(\Omega)$$

notation

- k-dependent norm: $\|v\|^2_{1,k} := \|\nabla v\|^2_{L^2(\Omega)} + k^2\|v\|^2_{L^2(\Omega)}$
- continuity: $|a(u, v)| \leq C_c\|u\|_{1,k}\|v\|_{1,k}$ (C_c indep. of k)
- adjoint approximation property:

$$\eta_N := \sup_{f \in L^2(\Omega)} \inf_{v \in V_N} \frac{\|S^*_k f - v\|_{1,k}}{\|f\|_{L^2(\Omega)}}$$

$$\eta_N = \sup_{f \in L^2(\Omega)} \inf_{v \in V_N} \frac{\|S^*_k f - v\|_{1,k}}{\|f\|_{L^2(\Omega)}}, \qquad \|v\|^2_{1,k} = \|\nabla v\|^2_{L^2(\Omega)} + k^2\|v\|^2_{L^2(\Omega)}$$

Theorem (quasioptimality)

If

$$2C_c k \eta_N \leq 1$$

then the Galerkin-FEM is quasi-optimal and $e := u - u_N$ satisfies

$$\begin{aligned} \|e\|_{1,k} &\leq 2C_c \inf_{v \in V_N} \|u - v\|_{1,k}, \\ \|e\|_{L^2(\Omega)} &\leq C_c\, \eta_N \|e\|_{1,k} \end{aligned}$$

- $\rightarrow$ study adjoint approximation property η_N
- $\rightarrow$ need regularity for S^*_k

quasioptimality: proof

- $a(u,v) = (\nabla u, \nabla v)_{L^2(\Omega)} - k^2(u,v)_{L^2(\Omega)} - \mathbf{i}k(u,v)_{L^2(\partial\Omega)}$
- $\|v\|_{1,k}^2 = \|\nabla v\|_{L^2}^2 + k^2\|v\|_{L^2}^2 = \operatorname{Re} a(v,v) + 2k^2\|v\|_{L^2}^2$ Gårding ineq.
- $\eta_N = \sup_{f\in L^2} \inf_{v\in V_N} \dfrac{\|S_k^* f - v\|_{1,k}}{\|f\|_{L^2}}$
- assumption: $C_c k \eta_N \leq 1/2$
- define ψ by $a(\cdot,\psi) = (\cdot, e)_{L^2}$, i.e. $\psi = S_k^* e$
- $\|e\|_{L^2}^2 = a(e,\psi) = a(e, \psi - \psi_N) \leq C_c \|e\|_{1,k}\|\psi - \psi_N\|_{1,k}$
- $\Longrightarrow \|e\|_{L^2}^2 \leq C_c\|e\|_{1,k}\eta_N\|e\|_{L^2} \Longrightarrow \|e\|_{L^2} \leq C_c\eta_N\|e\|_{1,k}$

$$\begin{aligned}\|e\|_{1,k}^2 &= \operatorname{Re} a(e,e) + 2k^2\|e\|_{L^2}^2 \\ &\leq \operatorname{Re} a(e, u - v_N) + 2k^2(C_c\eta_N)^2\|e\|_{1,k}^2 \\ &\leq C_c\|e\|_{1,k}\|u - v_N\|_{1,k} + \tfrac{1}{2}\|e\|_{1,k}^2\end{aligned}$$

$\Longrightarrow \|e\|_{1,k} \leq 2C_c \inf_{v\in V_N} \|u - v\|_{1,k}$.

Theorem (k-explicit regularity by decomposition)

Let $\partial\Omega$ be analytic. Then $u = S_k^\star(f)$ can be written as

$$u = u_{H^2} + u_{\mathcal{A}},$$

where for C, $\gamma > 0$ independent of k:

$$\begin{aligned}\|u_{H^2}\|_{H^2(\Omega)} &\leq C\|f\|_{L^2(\Omega)}, \\ \|\nabla^n u_{\mathcal{A}}\|_{L^2(\Omega)} &\leq Ck^{3/2}\gamma^n \max\{n,k\}^n \|f\|_{L^2(\Omega)} \qquad \forall n \in \mathbb{N}_0.\end{aligned}$$

implication for adjoint approximation η_N:

$$\begin{aligned}\inf_{v\in S^p(\mathcal{T}_h)} k\|u_{H^2} - v\|_{1,k} &\lesssim \left(\frac{kh}{p} + \frac{k^2h^2}{p^2}\right)\|f\|_{L^2(\Omega)} \\ \inf_{v\in S^p(\mathcal{T}_h)} k\|u_{\mathcal{A}} - v\|_{1,k} &\lesssim \left[k^{7/2}\left(\frac{kh}{\sigma p}\right)^p + \cdots\right]\|f\|_{L^2(\Omega)} \\ &\Longrightarrow \quad k\eta_N \text{ small, if } \frac{kh}{p} + k^{7/2}\left(\frac{kh}{\sigma p}\right)^p \text{ small}\end{aligned}$$

proof the decomposition result

1. analyze the Newton potential, i.e., the full space problem
2. analyze the bounded domain case by a fixed point argument

properties of the Newton potential $\mathcal{N}_k$

$$\begin{aligned}u &= \mathcal{N}_k(f) := G_k \star f \\ & \quad u \text{ solves } -\Delta u - k^2 u = f \quad \text{in } \mathbb{R}^d \\ G_k(z) &:= \begin{cases} -\frac{e^{\mathbf{i}k|z|}}{2\mathbf{i}k} & d = 1, \\ \frac{\mathbf{i}}{4}H_0^{(1)}(k\|z\|) & d = 2, \\ \frac{e^{\mathbf{i}k\|z\|}}{4\pi\|z\|} & d = 3. \end{cases} \\ \widehat{G}_k(\xi) &= c_d\frac{1}{|\xi|^2 - k^2}, \qquad c_d \in \mathbb{R}\end{aligned}$$

key ingredient of analysis: study the symbol of $\mathcal{N}_k$, i.e., $\widehat{G}_k(\xi)$

properties of the Newton potential, I

Theorem

Let $u = \mathcal{N}_k(f)$ and $\operatorname{supp} f \subset B_R$. Then:

$$k^{-1}\|u\|_{H^2(B_R)} + \|u\|_{H^1(B_R)} + k\|u\|_{L^2(B_R)} \le C\|f\|_{L^2(B_R)}$$

proof: localize the represen. $u = G_k \star f$ and analyze the symbol

- for $\chi \in C_0^\infty(\mathbb{R}^d)$ with $\chi \equiv 1$ on B_{2R} set

$$u_R(x) := \int_{\mathbb{R}^d} G(x-y)\chi(x-y)f(y)\,dy = (G_k\chi)\star f.$$

- Then: $u_R = u$ on B_R
- analyze symbol $\widehat{G_k\chi}$.
- Parseval gives estimates for $\|u_R\|_{L^2(\mathbb{R}^d)}$, $|u_R|_{H^1(\mathbb{R}^d)}$, $|u_R|_{H^2}$.

properties of the Newton potential $\mathcal{N}_k$, II

Theorem (decomposition lemma)

Let $\operatorname{supp} f \subset B_R$ and $u = \mathcal{N}_k(f) = G_k \star f$. Then, for every $\eta > 1$ the function $u|_{B_R}$ can be written as $u = u_{H^2} + u_{\mathcal{A}}$ where

$$\|\nabla^s u_{H^2}\|_{L^2(B_R)} \le C\left(1 + \frac{1}{\eta^2 - 1}\right)(\eta k)^{s-2}\|f\|_{L^2}, \quad s \in \{0,1,2\},$$

$$\|\nabla^s u_{\mathcal{A}}\|_{L^2(B_R)} \le C\eta\left(\sqrt{d}\eta k\right)^{s-1}\|f\|_{L^2} \qquad \forall s \in \mathbb{N}_0.$$

Corollary

For every $q \in (0,1)$, one can decompose $u|_{B_R} = u_{H^2} + u_{\mathcal{A}}$ s.t.

$$k\|u_{H^2}\|_{L^2(B_R)} + |u_{H^2}|_{H^1(B_R)} \le qk^{-1}\|f\|_{L^2}$$

$$\|u_{H^2}\|_{H^2(B_R)} \le C\|f\|_{L^2}$$

$u_{\mathcal{A}}$ *analytic*

proof

- key idea: decomposition $u = u_{H^2} + u_{\mathcal{A}}$ follows from decomposition of f in Fourier space (recall: $u = \mathcal{N}_k(f)$)
- select $\eta > 1$.
- write $f = L_{\eta k} f + H_{\eta k} f$, where the low pass filter $L_{\eta k}$ and the high pass filter $H_{\eta k}$ are

$$\mathcal{F}(L_{\eta k} f) = \chi_{B_{\eta k}}\widehat{f}, \qquad \mathcal{F}(H_{\eta k} f) = \chi_{\mathbb{R}^d \setminus B_{\eta k}}\widehat{f},$$

- define $u_{H^2} := \mathcal{N}_k(H_{\eta k} f)$.
- define $u_{\mathcal{A}} := \mathcal{N}_k(L_{\eta k} f)$.

bounds for u_{H^2}

- $u_{H^2} = \mathcal{N}_k(H_{\eta k} f) \Longrightarrow$

$$\widehat{u}_{H^2} = \widehat{G}_k \cdot \left(\chi_{\mathbb{R}^d \setminus B_{\eta k}}\widehat{f}\right) = c_d \frac{\chi_{\mathbb{R}^d \setminus B_{\eta k}}}{|\xi|^2 - k^2}\widehat{f}$$

- observe that for $|\xi| \ge \eta k$ (recall: $\eta > 1$)

$$\frac{1}{|\xi|^2 - k^2} \le \frac{\eta^2}{\eta^2 - 1}\frac{1}{(\eta k)^2}, \qquad \frac{|\xi|}{|\xi|^2 - k^2} \le \frac{\eta^2}{\eta^2 - 1}\frac{1}{(\eta k)^1}$$

$$\frac{|\xi|^2}{|\xi|^2 - k^2} \le \frac{\eta^2}{\eta^2 - 1}\frac{1}{(\eta k)^0}$$

- hence, for $s \in \{0,1,2\}$:

$$|u_{H^2}|_{H^s(\mathbb{R}^d)} = \||\xi|^s \widehat{u}_{H^2}\|_{L^2(\mathbb{R}^d)} \le C(\eta k)^{s-2}\|\widehat{f}\|_{L^2(\mathbb{R}^d)}$$

estimating $u_{\mathcal{A}}$

- $u_{\mathcal{A}} := \mathcal{N}_k(L_{\eta k} f)$. Hence, $u_{\mathcal{A}}$ solves
$$-\Delta u_{\mathcal{A}} - k^2 u_{\mathcal{A}} = L_{\eta k} f \qquad \text{on } \mathbb{R}^d$$
- Paley-Wiener: $L_{\eta k} f$ is analytic and, since $\operatorname{supp} \chi_{B_{\eta k}} \widehat{f} \subset B_{\eta k}$,
$$\|\nabla^s L_{\eta k} f\|_{L^2(\mathbb{R}^d)} \le C(\sqrt{d}\eta k)^s \|f\|_{L^2(\mathbb{R}^d)} \qquad \forall s \in \mathbb{N}_0$$
- recall a priori estimate:
$$\|u_{\mathcal{A}}\|_{H^1(B_{2R})} + k\|u_{\mathcal{A}}\|_{L^2(B_{2R})} \le C\|L_{\eta k} f\|_{L^2} \le C\|f\|_{L^2}$$
- use elliptic regularity (induction argument) to get
$$\|\nabla^s u_{\mathcal{A}}\|_{L^2(B_R)} \le C(\gamma \eta k)^{s-1} \|f\|_{L^2}$$

(for suitable $\gamma > 0$)

key ingredient of the proof:

Lemma (frequency splitting)

The decomposition $f = H_{\eta k} f + L_{\eta k} f$ has the following properties:

(i) $L_{\eta k}$ *is analytic*

(ii) $H_{\eta k} f$ *satisfies*

$$\begin{aligned}
\|H_{\eta k} f\|_{L^2(\mathbb{R}^d)} &\le \|f\|_{L^2(\mathbb{R}^d)} \\
\|H_{\eta k} f\|_{H^{-1}(\mathbb{R}^d)} &\le C(\eta k)^{-1} \|f\|_{L^2(\mathbb{R}^d)} \\
\|H_{\eta k} f\|_{H^{s'}(\mathbb{R}^d)} &\le C_{s,s'}(\eta k)^{-(s-s')} \|f\|_{H^s(\mathbb{R}^d)}
\end{aligned}$$

for $s' \le s$.

note: analogous splittings possible on manifolds (e.g., $\partial\Omega$)

decomposition theorem for bounded domains

recall: aim is the proof of the following theorem:

Theorem (k-explicit regularity)

Let $\partial\Omega$ be analytic. Then the solution $u = S_k(f)$ of

$$\begin{aligned}
-\Delta u - k^2 u &= f \quad \text{in } \Omega \\
\partial_n u - \mathbf{i} k u &= 0 \quad \text{on } \partial\Omega
\end{aligned}$$

can be written as

$$u = u_{H^2} + u_{\mathcal{A}},$$

where for C, $\gamma > 0$ independent of k:

$$\begin{aligned}
\|u_{H^2}\|_{H^2(\Omega)} &\le C\|f\|_{L^2(\Omega)}, \\
\|\nabla^n u_{\mathcal{A}}\|_{L^2(\Omega)} &\le C k^{3/2} \gamma^n \max\{n, k\}^n \|f\|_{L^2(\Omega)} \qquad \forall n \in \mathbb{N}_0.
\end{aligned}$$

key steps of the decomposition (general case)

$$\begin{aligned}
-\Delta u - k^2 u = f \in L^2(\Omega) &\qquad \text{in } \Omega \\
\partial_n u - \mathbf{i} k u = 0 &\qquad \text{on } \partial\Omega
\end{aligned}$$

goal: write $u = u^I_{H^2} + u^I_{\mathcal{A}} + \delta$, where

- $\|u^I_{H^2}\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$
- $u^I_{\mathcal{A}}$ is analytic with
$\|\nabla^s u^I_{\mathcal{A}}\|_{L^2(\Omega)} \le C k^{3/2} \max\{s, k\}^s \|f\|_{L^2(\Omega)} \qquad \forall s \in \mathbb{N}_0$
- δ solves
$$\begin{aligned}
-\Delta \delta - k^2 \delta = f_\delta \in L^2(\Omega) &\qquad \text{in } \Omega \\
\partial_n \delta - \mathbf{i} k \delta = 0 &\qquad \text{on } \partial\Omega
\end{aligned}$$

where, for a $q \in (0, 1)$, $\|f_\delta\|_{L^2(\Omega)} \le q\|f\|_{L^2(\Omega)}$

then: obtain decomposition of u by a geometric series argument

key steps of the decomposition

$$-\Delta u - k^2 u = f \in L^2(\Omega) \quad \text{in } \Omega$$
$$\partial_n u - \mathbf{i}ku = 0 \quad \text{on } \partial\Omega$$

Let S_k denote the solution operator for this problem.

1. decompose $f = L_{\eta k} f + H_{\eta k} f$. Set
   - $u_{\mathcal{A},1} := S_k(L_{\eta k} f)$.
     Then, $\|u_{\mathcal{A},1}\|_{1,k} \le C k^{5/2} \|L_{\eta k} f\|_{L^2}$. Furthermore, $u_{\mathcal{A},1}$ is analytic and satisfies the desired bounds (elliptic regularity)
   - $u_{H^2,1} := \mathcal{N}_k(H_{\eta k} f)$.
     $\|u_{H^2,1}\|_{H^2(\Omega)} \le C\|H_{\eta k} f\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$.
     $\|u_{H^2,1}\|_{H^1(\Omega)} \le Ck^{-1}\|H_{\eta k} f\|_{L^2(\Omega)} \le Ck^{-1}\|f\|_{L^2(\Omega)}$.
2. the remainder $\delta' := u - (u_{H^2,1} + u_{\mathcal{A},1})$ solves

$$\begin{aligned} -\Delta\delta' - k^2\delta' &= 0 \\ \partial_n \delta' - \mathbf{i}k\delta' &= \partial_n u_{H^2,1} - \mathbf{i}k u_{H^2,1} =: g \end{aligned}$$

together with $\quad \|g\|_{H^{1/2}(\partial\Omega)} \le C\|u_{H^2,1}\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$

key steps of the decomposition

δ' solves

$$\begin{aligned} -\Delta\delta' - k^2\delta' &= 0 \\ \partial_n \delta' - \mathbf{i}k\delta' &= g, \qquad \|g\|_{H^{1/2}(\partial\Omega)} \le C\|f\|_{L^2(\Omega)} \end{aligned}$$

3. Decompose g as $g = L^{\partial\Omega}_{\eta k} g + H^{\partial\Omega}_{\eta k} g$ with

$L^{\partial\Omega}_{\eta k} g \quad$ analytic

$$\|H^{\partial\Omega}_{\eta k} g\|_{H^{-1/2}(\partial\Omega)} \le C\frac{1}{\eta k}\|g\|_{H^{1/2}(\partial\Omega)} \le C\frac{1}{\eta k}\|f\|_{L^2(\Omega)}$$

4. define $u_{\mathcal{A},2}$ and $u_{H^2,2}$ as solutions of

$$\begin{aligned} -\Delta u_{\mathcal{A},2} - k^2 u_{\mathcal{A},2} &= 0 \\ \partial_n u_{\mathcal{A},2} - \mathbf{i}k u_{\mathcal{A},2} &= L^{\partial\Omega}_{\eta k} g \end{aligned} \qquad \begin{aligned} -\Delta u_{H^2,2} + k^2 u_{H^2,2} &= 0 \\ \partial_n u_{H^2,2} - \mathbf{i}k u_{H^2,2} &= H^{\partial\Omega}_{\eta k} g \end{aligned}$$

key steps of the decomposition

$$\begin{aligned} -\Delta u_{\mathcal{A},2} - k^2 u_{\mathcal{A},2} &= 0 \\ \partial_n u_{\mathcal{A},2} - \mathbf{i}k u_{\mathcal{A},2} &= L^{\partial\Omega}_{\eta k} g \end{aligned} \qquad \begin{aligned} -\Delta u_{H^2,2} + k^2 u_{H^2,2} &= 0 \\ \partial_n u_{H^2,2} - \mathbf{i}k u_{H^2,2} &= H^{\partial\Omega}_{\eta k} g \end{aligned}$$

5. $L^{\partial\Omega}_{\eta k}$ is analytic $\Longrightarrow$ $u_{\mathcal{A},2}$ is analytic with appropriate bounds
6. standard *a priori* bounds for $u_{H^2,2}$ ("positive definite Helmholtz problem"):

$$\begin{aligned} \|u_{H^2,2}\|_{1,k} &\le C\|H^{\partial\Omega}_{\eta k} g\|_{H^{-1/2}(\partial\Omega)} \le C\frac{1}{\eta k}\|f\|_{L^2(\Omega)}, \\ \|u_{H^2,2}\|_{H^2} &\le C\|H^{\partial\Omega}_{\eta k} g\|_{H^{1/2}(\partial\Omega)} \le C\|f\|_{L^2(\Omega)}. \end{aligned}$$

7. $\delta := u - (u_{\mathcal{A},1} + u_{\mathcal{A},2} + u_{H^2,1} + u_{H^2,2}) = \delta' - (u_{\mathcal{A},2} + u_{H^2,2})$ solves

$$\begin{aligned} -\Delta\delta - k^2\delta &= -2k^2 u_{H^2,2} =: f_\delta, \\ \partial_n\delta - \mathbf{i}k\delta &= 0, \end{aligned}$$

and $\|f_\delta\|_{L^2} = 2k^2\|u_{H^2,2}\|_{L^2} \le C\frac{1}{\eta}\|f\|_{L^2}$.
$\Longrightarrow$ selecting η sufficiently large concludes the argument.

Theorem (decomposition for convex polygons)

Let $\Omega \subset \mathbb{R}^2$ be a convex polygon. Then the solution u of

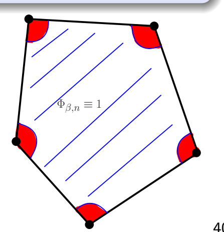
$$-\Delta u - k^2 u = f \quad \text{in } \Omega, \qquad \partial_n u - \mathbf{i}ku = 0 \quad \text{on } \partial\Omega$$

can be written as $u = u_{H^2} + u_{\mathcal{A}}$, where

$$\begin{aligned} \|u_{H^2}\|_{H^2(\Omega)} &\le C\|f\|_{L^2(\Omega)} \\ \|\Phi_{\beta,n}\nabla^{n+2}u_{\mathcal{A}}\|_{L^2(\Omega)} &\le C\gamma^n \max\{n,k\}^{n+1}\|f\|_{L^2(\Omega)} \quad \forall n \in \mathbb{N}_0 \end{aligned}$$

for some C, $\gamma > 0$ and $\beta \in [0,1)$.

$$\begin{aligned} r(x) &:= \text{min. distance to vertices} \\ r_c &:= \min\{1, \frac{n+1}{k}\} \\ \Phi_{\beta,n} &= \begin{cases} 1 & \text{if } r_c \ge 1 \\ \left(\frac{r}{\min\{1,\frac{n+1}{k}\}}\right)^{n+\beta} & \text{if } r_c < 1 \end{cases} \end{aligned}$$



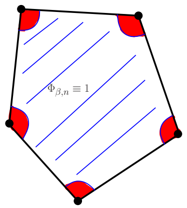
Theorem (quasi-optimality of hp-FEM, polygons)

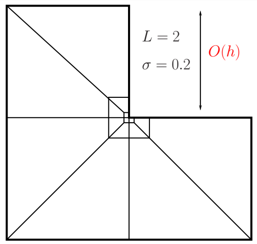
Let $\Omega \subset \mathbb{R}^2$ be a convex polygon. Let $\mathcal{T}_h^L$ be a mesh s.t.:

- *the restriction of $\mathcal{T}_h^L$ to $\Omega \setminus \cup_{j=1}^J B_{ch}(A_j)$ is quasi-uniform*
- *$\mathcal{T}_h^L$ restricted to $B_{ch}(A_j)$ is a geometric mesh with L layers.*

Then: the hp-FEM is quasi-optimal under the condition

$$\frac{kh}{p} \quad \text{sufficiently small} \qquad \text{and} \qquad p \sim L \geq c \log k.$$

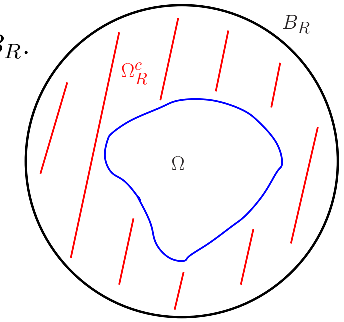




exterior Dirichlet problem

$$\begin{aligned} -\Delta u - k^2 u &= f \quad \text{in } \mathbb{R}^d \setminus \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \\ \partial_r u - \mathbf{i} k u &= o(r^{(1-d)/2}) \quad \to \infty. \end{aligned}$$

Assume additionally $\operatorname{supp} f \subset B_R$.



Theorem (exterior Dirichlet problem)

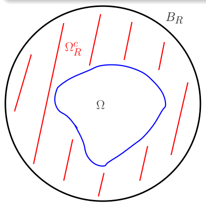
Assume:

- *$\partial\Omega$ is analytic*
- *the solution operator $f \mapsto u$ satisfies*

$$k\|u\|_{L^2(\Omega_R^c)} + \|u\|_{H^1(\Omega_R^c)} \leq C k^\alpha \|f\|_{L^2(\Omega_R^c)}$$

Then: the solution u can be written as $u = u_{H^2} + u_{\mathcal{A}}$ with

$$\begin{aligned} \|\nabla^n u_{\mathcal{A}}\|_{L^2(\Omega_R^c)} &\leq C\gamma^n \max\{n,k\}^n k^{\alpha-1} \|f\|_{L^2(\Omega_R^c)} \qquad \forall n \in \mathbb{N}_0 \\ \|u_{H^2}\|_{H^2(\Omega_R^c)} &\leq C\|f\|_{L^2(\Omega_R^c)}. \end{aligned}$$



quasi-optimality of hp-FEM

hp-FEM is quasi-optimal under the scale resolution condition, if the DtN-operator on ∂B_R is realized exactly.

summary

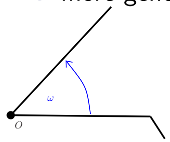
- basic mechanisms:
  1. operator has the form "elliptic + compact perturbation": $-\Delta u - k^2 u = (-\Delta u + k^2 u) - 2k^2 u$ (Gårding inequality)
  2. $\to$ asymptotic quasi-optimality
  3. k-explicit regularity for the (adjoint) problem in the form of an additive splitting permits to be explicit about k-dependence of onset of quasi-optimality
- pollution free methods by changing the discretization and/or ansatz functions?
  - 1D is possible: one can devise nodally exact methods ($\to$ pollution-free). This is due to the fact that the space of homogeneous solutions is finite dimensional (dim: 2)
  - pollution unavoidable for $d > 1$ for methods with fixed stencil (Babuska & Sauter)

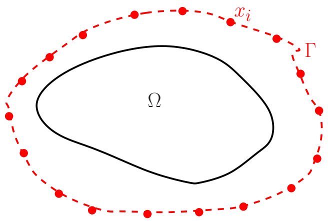
Trefftz type ansatz functions

idea: approximate solutions of $-\Delta u - k^2 u = 0$ with functions that solve the equation as well.

examples (2D):

- plane waves:
  $W(p) := \operatorname{span}\{e^{\mathbf{i}k\omega_n\cdot(x,y)}\,|\,n = 1,\ldots,p\}, \quad \omega_n = (\cos\frac{2\pi n}{p}, \sin\frac{2\pi n}{p})$
- cylindrical waves:
  $V(p) := \operatorname{span}\{J_n(kr)\sin(n\varphi),\ J_n(kr)\cos(n\varphi)\,|\,n = 0,\ldots,p\}$
- fractional Bessel functions (near corners of polygons):
  $\operatorname{span}\{J_{n\alpha}(kr)\sin(n\alpha\varphi)\,|\,n = 1,\ldots,N\}, \qquad \alpha = \frac{\pi}{\omega}$
- fundamental solutions:
  $\operatorname{span}\{G_k(|x - x_i|)\, i = 1,\ldots,N\}$
- more generally: "discretized" potentials





Trefftz type ansatz functions

reasons:

- improved approximation properties (error vs. DOF)
- greater potential for adaptivity (directionality)
- hope of reduction of pollution

stability analysis

- not clear that approach "coercive + compact perturbation" can be made to work for interesting cases
- → often, different, stable numerical formulations used such as
  - least squares
  - DG

Approximation properties of systems of plane waves for the approximation of u satisfying $-\Delta u - k^2 u = 0$ on $\Omega \subset \mathbb{R}^2$

$$W(p) \quad := \quad \operatorname{span}\{e^{\mathbf{i}k\omega_n\cdot(x,y)}\,|\,n = 1,\ldots,p\}, \quad \omega_n = (\cos\frac{2\pi n}{p}, \sin\frac{2\pi n}{p})$$

Theorem (h-version: Moiola, Cessenat & Després)

Let K be a shape regular element with diameter h. Let $p = 2\mu + 1$. Then there exists $v \in W(2\mu + 1)$ s.t.

$$\|u - v\|_{j,k,K} \le C_p h^{\mu-j+1}\|u\|_{\mu+1,k,K}, \qquad 0 \le j \le \mu + 1$$

where $\|v\|^2_{j,k,K} = \sum_{m=0}^{j} k^{2(j-m)}|v|^2_{H^m(K)}$.

Remarks:

- Extension to 3D possible
- analogous results for cylindrical waves

Approximation properties of systems of plane waves II

Theorem (p-version, exponential convergence)

Let $\Omega \subset \mathbb{R}^2$, $\Omega' \subset\subset \Omega$. *Then:*

$$\inf_{v \in W(p)} \|u - v\|_{H^1(\Omega')} \leq C e^{-bp/\log p},$$

Theorem (p-version, algebraic conv.)

Let Ω *be star shaped with respect to a ball and satisfy an exterior cone condition with angle* $\lambda\pi$. *Let* $u \in H^k(\Omega)$, $k \geq 1$. *Then:*

$$\inf_{v \in W(p)} \|u - v\|_{H^1(\Omega)} \leq C \left(\frac{\log^2(p+2)}{p+2} \right)^{\lambda(k-1)}.$$

Remarks:

- simultaneously h and p-explicit bounds possible (2D, Moiola)
- extension to 3D possible (Moiola)

approximation methods using special ansatz functions

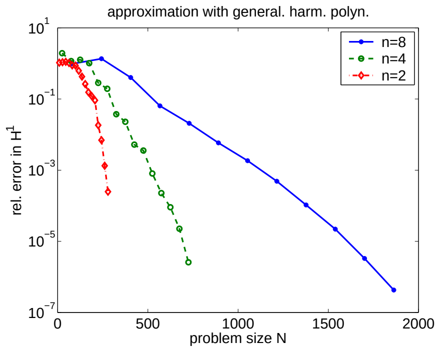
1. partition-of-unity methods (Babuška & Melenk, Bettes & Laghrouche, Astley, etc.) employ "standard" variational formulation and construct H^1-conforming ansatz spaces based on the chosen ansatz function
2. Least squares method: approximate with special fcts elementwise and penalize jumps across interelement boundaries (Trefftz, Stojek, Monk & Wang, Betcke, Desmet)
3. Discontinuous enrichement method (Farhat et al.): approximate with plane waves elementwise and enforce interelement continuity by a Lagrange multiplier
4. ultra weak formulation: (Cessenat & Després, Monk & Huttunen, Hiptmair & Moiola & Perugia, Feng et al.) DG-like variational formulation that is only posed on the "skeleton"; solution defined as L-harmonic extension into the elements

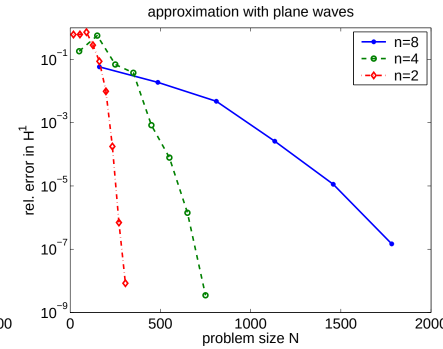
$$-\Delta u - k^2 u = 0 \quad \text{on } \Omega = (0,1)^2, \qquad \partial_n u + \mathbf{i} k u = g, \quad \text{on } \partial\Omega$$

exact solution: $u(x,y) = e^{\mathbf{i}k(\cos\theta, \sin\theta)\cdot(x,y)}$, $\theta = \frac{\pi}{16}$, $k = 32.$

partition of unity: bilinears φ_i on uniform $n \times n$ grid

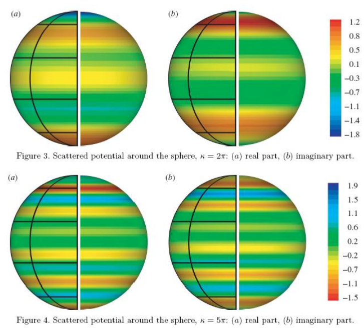
Note: $\dim V(p) = 2p+1, \quad \dim W(p) = p$





performance of PUM: scattering by a sphere

- scattering by a sphere B_1 (radius 1) of an incident plane wave
- sound hard b.c. on $\Gamma = \partial B_1$ (i.e., Neumann b.c.)
- computational domain: ball of diameter $1 + 4\lambda$ ($\lambda = 2\pi/k$)
- b.c. $\partial_n u^s + (\frac{1}{r} - \mathbf{i}\kappa) u^s = 0$ on outer boundary
- mesh: 4 layers in rad. dir., 8×5 elem./layers; $\to 160$ elem.; 170 nodes



k	number waves per node	DOF	$L^2(\Gamma)$ error	DOF per wave length
π	58	9860	0.1%	2.95
2π	58	9860	0.8%	2.66
3π	58	9860	2.1%	2.43
4π	98	16660	0.9%	2.67
5π	98	16660	2.7%	2.48

taken from: Perrey-Debain, Laghrouche, Bettess, Trevelyan, '03

interelement continuity by penalty (Least Squares)

model problem:

$-\Delta u - k^2 u = 0$ on Ω, $\quad \partial_n u - \mathbf{i}ku = g$ on Γ,

notation:

$\mathcal{T}$ = mesh, $\quad \mathcal{E}$ = internal edges, $\quad \mathcal{E}^\Gamma$ = edges on Γ,

approximation space:

$V_N \subset \{v \in L^2(\Omega) \,|\, \left(-\Delta v - k^2 v\right)|_K = 0 \quad \forall K \in \mathcal{T}\}$

Cost functional

$$J(u) := \sum_{e\in\mathcal{E}^I} k^2 \|[u]\|^2_{L^2(e)} + \|[\nabla u]\|^2_{L^2(e)} + \sum_{e\in\mathcal{E}^\Gamma} \|\partial_n u - \mathbf{i}ku - g\|^2_{L^2(e)}$$

numerical method: minimize J over V_N.

1. existence and uniqueness of minimizer $u_N \in V_N$ is guaranteed
2. consistency: if exact solution u is sufficiently smooth, then $J(u) = 0$.

interelement continuity by penalty: error estimates

- $\rightarrow$ can get bounds for $J(u_N)$ from a elementwise approximation properties of V_N given above for plane waves:

$$J(u_N) \le \inf_{v\in V_N} J(v) = \inf_{v\in V_N} \sum_{e\in\mathcal{E}^I} k^2\|[u-v]\|^2_{L^2(e)} + \|[\nabla_h(u-v)]\|^2_{L^2(e)} + \sum_{e\in\mathcal{E}^\Gamma} \|\partial_n(u-v) - \mathbf{i}k(u-v)\|^2_{L^2(e)}$$

- extract L^2-error estimates from $J(u_N)$:

Theorem (Monk & Wang)

Let Ω be convex. Let $\mathcal{T}$ be a quasi-uniform mesh with mesh size h. Let $u - u_N$ satisfy the homogeneous Helmholtz equation elementwise. Then:

$$\|u - u_N\|^2_{L^2(\Omega)} \le C_{\Omega,k} h^{-1} J(u_N)$$

proof: duality argument (later) numerics

Lagrange multiplier technique for interelement continuity

original problem:

find $u \in H^1(\Omega)$ s.t. $a(u,v) = l(v) \qquad \forall v \in H^1(\Omega)$

notation: $\mathcal{T}$ = mesh, $\quad \mathcal{E}$ = set of internal edges/faces

spaces: $X = \{u \in L^2(\Omega) \,|\, u|_K \in H^1(K) \quad \forall K \in \mathcal{T}\}$,

$$M = \prod_{E\in\mathcal{E}} \left(H^{1/2}(E)\right)',$$

define $b(u,\mu) = \sum_{E\in\mathcal{E}} \langle [u], \mu\rangle$

define $a_{\mathcal{T}}(u,\mu) = \sum_{K\in\mathcal{T}} a_K(u,v)$, $\quad a_K(u,v) = \int_K \nabla u\cdot\nabla\overline{v} - k^2 u\overline{v} \pm \mathbf{i}k\int_{\partial K\cap\partial\Omega} u\overline{v}$

Let $X_N \subset X$, $M_N \subset M$: Find $(u_N, \lambda_N) \in X_N \times M_N$ s.t.

$$a_{\mathcal{T}}(u_N, v) + b(v, \lambda_N) = l(v) \quad \forall v \in X_N$$
$$b(u_N, \mu) = 0 \quad \forall \mu \in M_N$$

"Discontinuous enrichement method" of Farhat et al. (IJNME '06)

Ansatz space for solution u:

$$X_N := \prod_{K\in\mathcal{T}} W_K,$$
$$W_K := \operatorname{span}\{e^{\mathbf{i}k\mathbf{d}_n\cdot\mathbf{x}} \,|\, n = 1,\ldots,N_u\}$$

Ansatz space for Lagrange multiplier

$$M_N := \prod_{E\in\mathcal{E}} \widetilde{W}_E,$$
$$\widetilde{W}_E := \operatorname{span}\{e^{\mathbf{i}kc_n\omega_n\cdot\mathbf{t}} \,|\, n = 1,\ldots,N_\lambda\}$$

where the parameters c_n are between 0.4 and 0.8 and are obtained from a numerical study of a test problem

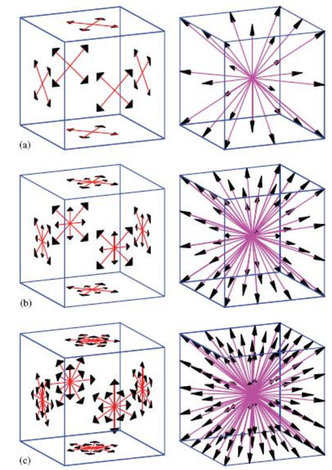


Figure 1. 3D DGMH elements: directions of the Lagrange multipliers (left), directions of the element basis functions (right): (a) DGMH-26-4; (b) DGMH-56-8; and (c) DGMH-98-12.

performance of DEM: scattering by a sphere

- scattering by a sphere B_1 (radius 1) of an incident plane wave
- sound hard b.c. on $\Gamma = \partial B_1$ (i.e., Neumann b.c.)
- computational domain: ball of diameter 2
- b.c. $\partial_n u^s - \mathbf{i}\kappa u^s = 0$ on outer boundary

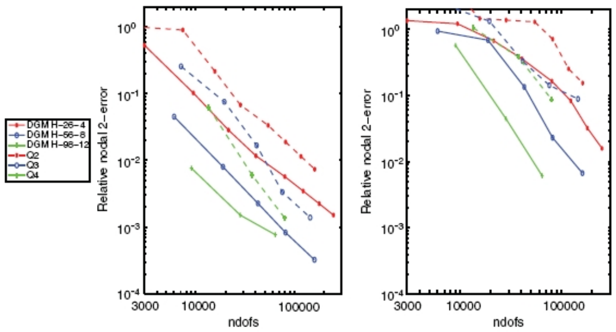


Figure 6. Convergence of the Galerkin and DGM elements for the problem of sound-hard scattering by a sphere: $R_1 = 1$, $R_2 = 2$, $kR_1 = 12$ (left) and $kR_1 = 24$ (right).

legend: dashed lines = standard Q_2, Q_3, Q_4 elements; solid lines = new elements; taken from Tezaur & Farhat, IJNME 06

DG approach: continuous flux formulation

model problem: $-\Delta u - k^2 u = 0$ on Ω, $\partial_n u + \mathbf{i}ku = g$ on $\partial\Omega$

reformulation as a first order system by setting $\boldsymbol{\sigma} := \nabla u/(\mathbf{i}k)$:

$$\begin{aligned} \mathbf{i}k\boldsymbol{\sigma} &= \nabla u && \text{on } \Omega \\ \mathbf{i}ku - \nabla\cdot\boldsymbol{\sigma} &= 0 && \text{on } \Omega \\ \mathbf{i}k\boldsymbol{\sigma}\cdot\mathbf{n} + \mathbf{i}ku &= g && \text{on } \partial\Omega \end{aligned}$$

weak elementwise formulation: for every $K \in \mathcal{T}$ there holds

$$\begin{aligned} \int_K \mathbf{i}k\boldsymbol{\sigma}\cdot\overline{\boldsymbol{\tau}} + \int_K u\nabla\cdot\overline{\boldsymbol{\tau}} - \int_{\partial K} u\overline{\boldsymbol{\tau}}\cdot\mathbf{n} &= 0 && \forall \boldsymbol{\tau} \in H(\mathrm{div}, K) \\ \int_K \mathbf{i}ku\overline{v} + \int_K \boldsymbol{\sigma}\cdot\nabla\overline{v} - \int_{\partial K} \boldsymbol{\sigma}\cdot\mathbf{n}\overline{v} &= 0 && \forall v \in H^1(K) \end{aligned}$$

DG: discrete flux formulation

discretization:

- $H^1(K) \to V_N(K)$, $u \to u_N$ (elementwise)
- $H(\mathrm{div}, K) \to \boldsymbol{\Sigma}_N(K)$ $\boldsymbol{\sigma} \to \boldsymbol{\sigma}_N$ (elementwise)
- (multivalued) traces u and $\boldsymbol{\sigma}$ on the skeleton are replaced with (single-valued) numerical fluxes $\widehat{u}_N$, $\widehat{\boldsymbol{\sigma}}_N$

discrete formulation (flux formulation)

$$\begin{aligned} \int_K \mathbf{i}k\boldsymbol{\sigma}_N\cdot\overline{\boldsymbol{\tau}} + \int_K u_N\nabla\cdot\overline{\boldsymbol{\tau}} - \int_{\partial K} \widehat{u}_N\overline{\boldsymbol{\tau}}\cdot\mathbf{n} &= 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_N(K) \\ \int_K \mathbf{i}ku_N\overline{v} + \int_K \boldsymbol{\sigma}_N\cdot\nabla\overline{v} - \int_{\partial K} \widehat{\boldsymbol{\sigma}}_N\cdot\mathbf{n}\overline{v} &= 0 \quad \forall v \in V_N(K) \end{aligned}$$

DG: from the flux formulation back to the primal formulation

elimination of the variable $\boldsymbol{\sigma}_N$ by

1. requiring $\nabla V_N(K) \subset \boldsymbol{\Sigma}_N(K)$ for all $K \in \mathcal{T}$
2. selecting test fct $\boldsymbol{\tau} = \nabla v$ and integrating by parts gives

$$\int_K \nabla u_N\nabla\overline{v} - k^2 u_N\overline{v} - \int_{\partial K}(u_N - \widehat{u}_N)\partial_n\overline{v} - \mathbf{i}k\widehat{\boldsymbol{\sigma}}_N\cdot\mathbf{n} = 0 \qquad \forall K \in \mathcal{T}$$

Since V_N consists of discontinuous functions, this is equivalent to:

DG formulation

Find $u_N \in V_N$ s.t. for all $v \in V_N$

$$\sum_{K\in\mathcal{T}} \int_K \nabla u_N\cdot\nabla\overline{v} - k^2 u_N\overline{v} + \int_{\partial K}(\widehat{u}_N - u_N)\nabla\overline{v}\cdot\mathbf{n} - \int_{\partial K}\mathbf{i}k\widehat{\boldsymbol{\sigma}}_N\cdot\mathbf{n}\overline{v} = 0$$

DG: special choices of fluxes

- for interior edges

$$\widehat{\boldsymbol{\sigma}}_N = \frac{1}{\mathbf{i}k}\{\!\{\nabla_h u\}\!\} - \alpha[\![u_N]\!]$$

$$\widehat{u}_N = \{\!\{u_N\}\!\} - \beta\frac{1}{\mathbf{i}k}[\![\nabla_h u_N]\!]$$

- for boundary edges

$$\widehat{\boldsymbol{\sigma}}_N = \frac{1}{\mathbf{i}k}\nabla_h u_N - \frac{1-\delta}{\mathbf{i}k}\left(\nabla_h u_N + \mathbf{i}k u_N \mathbf{n} - g\mathbf{n}\right)$$

$$\widehat{u}_N = u_N - \frac{\delta}{\mathbf{i}k}\left(\nabla_h u \cdot \mathbf{n} + \mathbf{i}k u_N - g\right)$$

1. $\alpha = \beta = \delta = 1/2$: UWVF (Cessenat/Després, Monk et al.)
2. $\alpha = O(p/(kh\log p))$, $\beta = O((kh\log p)/p)$, $\delta = O((kh\log p)/p)$: Hiptmair/Moiola/Perugia

- if $V_N(K)$ = space of elementwise solutions of homogeneous Helmholtz eqn $\rightarrow$ volume contribution can be made to vanish by further integration by parts

DG formulation

find $u_N \in V_N$ s.t. $A_N(u_N, v) = \mathbf{i}\frac{1}{k}\int_\Gamma \delta g \partial_n \overline{v} + \int_\Gamma (1-\delta) g\overline{v} \quad \forall v \in V_N$

$$\begin{aligned} A_N(u,v) = & \int_{\mathcal{E}^I}\{\!\{u\}\!\}[\![\nabla_h\overline{v}]\!] + \mathbf{i}\frac{1}{k}\int_{\mathcal{E}^I}\beta[\![\nabla_h u]\!][\![\nabla_h\overline{v}]\!] \\ & - \int_{\mathcal{E}^I}\{\!\{\nabla_h u\}\!\}[\![\overline{v}]\!] + \mathbf{i}k\int_{\mathcal{E}^I}\alpha[\![u]\!][\![\overline{v}]\!] \\ & + \int_\Gamma (1-\delta)u\partial_n\overline{v} + \mathbf{i}\frac{1}{k}\int_\Gamma \delta\partial_n u\partial_n\overline{v} \\ & - \int_\Gamma \delta\partial_n u\overline{v} + \mathbf{i}k\int_\Gamma(1-\delta)u\overline{v} \end{aligned}$$

coercivity

$$\alpha,\beta,\delta > 0 \quad \Longrightarrow \quad \operatorname{Im} A(u,u) > 0 \quad \forall 0 \neq u \in V_N$$

convergence theory: coercivity properties I

$$\begin{aligned} \|u\|^2_{DG} & := \frac{1}{k}\|\beta^{1/2}[\![\nabla_h u]\!]\|^2_{L^2(\mathcal{E}^I)} + \|\alpha^{1/2}[\![u]\!]\|^2_{L^2(\mathcal{E}^I)} \\ & \quad + \frac{1}{k}\|\delta^{1/2}\partial_n u\|^2_{L^2(\Gamma)} + k\|(1-\delta)u\|^2_{L^2(\Gamma)} \\ \|u\|^2_{DG,+} & := \|u\|^2_{DG} + k\|\beta^{-1/2}\{\!\{u\}\!\}\|^2_{L^2(\mathcal{E}^I)} \\ & \quad + k^{-1}\|\alpha^{-1/2}\{\!\{u\}\!\}\|^2_{L^2(\mathcal{E}^I)} + k\|\delta^{-1/2}u\|^2_{L^2(\Gamma)} \end{aligned}$$

Theorem (Buffa/Monk, Hiptmair/Moiola/Perugia)

$$\begin{aligned} \operatorname{Im} A(u,u) & = \|u\|^2_{DG} && \forall u \in V_N \\ |A(u,v)| & \leq C\|u\|_{DG}\|v\|_{DG,+} && \forall u,v \in V_N \end{aligned}$$

In particular, therefore, the DG method is quasi-optimal in $\|\cdot\|_{DG}$.

convergence theory: L^2–estimates

$$\begin{aligned} \|e\|_{L^2(\Omega)} & = \sup_{\varphi\in L^2(\Omega)}\frac{(e,\varphi)_{L^2(\Omega)}}{\|\varphi\|_{L^2(\Omega)}} \qquad \text{(adj. problem } -\Delta v - k^2 v = \varphi) \\ & = \sup_\varphi \frac{\sum_{K\in\mathcal{T}}(e, -\Delta v - k^2 v)_{L^2(K)}}{\|\varphi\|_{L^2}} \\ & \leq \sup_\varphi \frac{\|e\|_{DG}}{\|\varphi\|_{L^2}}\left(\sum_{K\in\mathcal{T}} k\|\beta^{-1/2}v\|^2_{L^2(\partial K)} + k^{-1}\|\alpha^{-1/2}\nabla v\|^2_{L^2(\partial K)}\right)^{1/2} \\ & \lesssim C\left(\frac{1}{\sqrt{kh}} + \sqrt{kh}\right)\|e\|_{DG} \end{aligned}$$

- $-\Delta u - k^2 e = 0$ elementwise
- trace estimates (elementwise) and quasi-uniformity of mesh
- α, β, δ = chosen constants $\neq 0$ (independent of h, k, p)
- Ω convex, in order to use *a priori* estimates

$$k\|v\|_{L^2} + |v|_{H^1} + k^{-1}|v|_{H^2} \leq C\|\varphi\|_{L^2}$$

Theorem (Monk/Buffa, Hiptmair/Moiola/Perugia)

- *use p plane waves elementwise, $p \geq 2s+1$*
- *Let kh be bounded*
- *for the L^2-estimate: $\mathcal{T}_N$ quasi-uniform and $\Omega \subset \mathbb{R}^2$ convex*

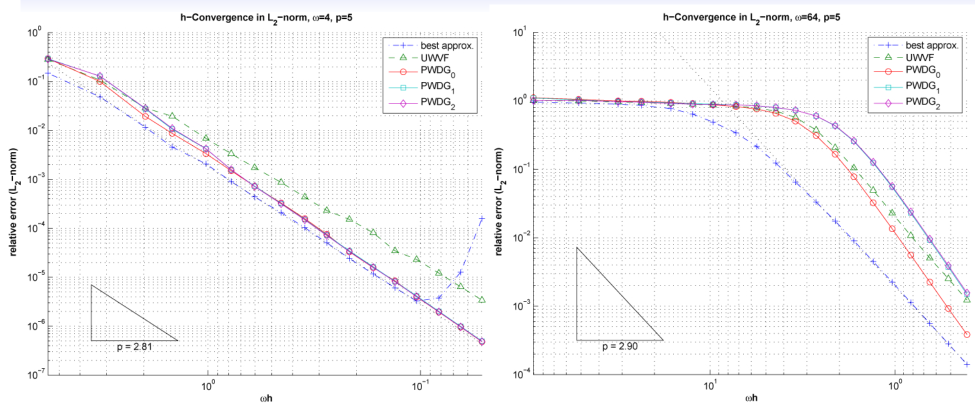
Then, for α, β, δ constant (independent of h, k, p):

$$\|u-u_N\|_{DG} \leq Ck^{-1/2}h^{s-1/2}\left(\frac{\log p}{p}\right)^{s-1/2}\|u\|_{s+1,k,\Omega}$$

$$k\|u-u_N\|_{L^2(\Omega)} \leq Ch^{s-1}\left(\frac{\log p}{p}\right)^{s-1/2}\|u\|_{s+1,k,\Omega}$$

where $\|u\|^2_{s,k,\Omega} = \sum_{j=0}^{s} k^{2(s-j)}|u|^2_{H^j(\Omega)}$

remark: for Hiptmair/Moiola/Perugia choice of α, β, δ:
optimal rates in $\|\cdot\|_{DG}$ but same convergence result in L^2.



h-version performance (smooth sol.): left: $k=4$, right: $k=64$

geometry: $\Omega = (0,1)^2$, exact solution: $H_0^{(1)}(k|x-x_0|)$, $x_0 = (-1/4, 0)^\top$
5 plane waves per element

source: Gittelson/Hiptmair/Perugia '08

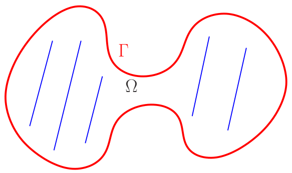
summary for volume-based methods

- standard hp-FEM:
  - quasi-optimality can be achieved with a fixed number of a DOF per wavelength, if high order methods are used
  - proof relies on the fact that one has a Gårding and k-explicit regularity estimates for the adjoint problem
- nonstandard approximation spaces:
  - significant progress has been made to understand the approximation properties of these spaces
  - stability: available (so far) only for discretizations for which coericivity can be shown

exterior Dirichlet problem (sound soft scattering)

$$\begin{aligned} -\Delta u - k^2 u &= 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ u &= g \quad \text{on } \Gamma := \partial\Omega, \end{aligned}$$

Sommerfeld radiation condition at ∞



- $d \in \{2,3\}$
- Γ analytic

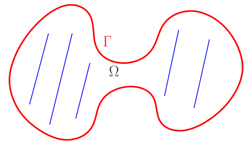
reformulations as 2nd kind BIEs

1. "Brakhage-Werner": find φ s.t. $A\varphi = g$
2. "Burton-Miller": $\partial_n u$ solves $A'\partial_n u = f$ (f given in terms of g)

fact: A and $A' : L^2(\Gamma) \to L^2(\Gamma)$ boundedly invertible

potential operators

$$G_k(z) := \begin{cases} -\frac{e^{\mathbf{i}k|z|}}{2\mathbf{i}k} & d = 1, \\ \frac{\mathbf{i}}{4} H_0^{(1)}(k\|z\|) & d = 2, \\ \frac{e^{\mathbf{i}k\|z\|}}{4\pi\|z\|} & d = 3. \end{cases}$$



$$\begin{aligned} \text{Newton potential } \mathcal{N}_k(f) &:= \int_{\mathbb{R}^d} G_k(x-y) f(y)\, dy \\ \text{single layer potential } \widetilde{V}_k(f) &:= \int_\Gamma G_k(x-y) f(y)\, ds_y \\ \text{double layer potential } \widetilde{K}_k(f) &:= \int_\Gamma \mathbf{n}(y) \cdot \nabla_y G_k(x-y) f(y)\, ds_y. \end{aligned}$$

facts: $\widetilde{V}_k f$ and $\widetilde{K}_k f$ satisfy

- the (homogeneous) Helmholtz equation piecewise
- the Sommerfeld radiation condition at ∞

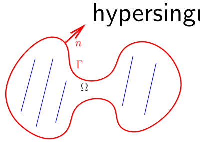
the BIOs V_k, K_k, K'_k, D_k

the 4 operators

$$V_k,\ K_k,\ K'_k,\ D_k : \text{ functions on } \Gamma \ \to\ \text{functions on } \Gamma$$

are defined by taking traces:

single layer V_k:	$V_k\varphi := \gamma_0^{ext}\widetilde{V}_k\varphi$
double layer K_k:	$(\frac{1}{2} + K_k)\varphi := \gamma_0^{ext}\widetilde{K}_k\varphi$
adjoint double layer K_k:	$(-\frac{1}{2} + K'_k)\varphi := \gamma_1^{ext}\widetilde{V}_k\varphi$
hypersingular op. D_k:	$D_k\varphi := -\gamma_1^{ext}\widetilde{K}_k\varphi$



$\gamma_1^{ext} u := (n \cdot \nabla u)|_\Gamma$

jump relations:

$$[\widetilde{V}_k\varphi] = 0, \qquad [\partial_n \widetilde{V}_k\varphi] = -\varphi$$
$$[\widetilde{K}_k\varphi] = \varphi, \qquad [\partial_n \widetilde{K}_k\varphi] = 0.$$

representation formula and Calderón identities

representation formula/Green's identity

Let u solve the homogeneous Helmholtz eqn in $\mathbb{R}^d \setminus \overline{\Omega}$ (and Sommerfeld radiation condition). Then:

$$u(x) = (\widetilde{K}_k \gamma_0^{ext} u)(x) - (\widetilde{V}_k \gamma_1^{ext} u)(x) \qquad x \in \mathbb{R}^d \setminus \overline{\Omega}$$

taking the trace γ_0^{ext} and the conormal trace γ_1^{ext} on Γ leads to

Calderón identities

$$\begin{aligned} \gamma_0^{ext} u &= \left(\frac{1}{2}\operatorname{Id} + K_k\right)\gamma_0^{ext} u \qquad - V_k \gamma_1^{ext} u \\ \gamma_1^{ext} u &= -D_k \gamma_0^{ext} u \qquad + \left(\frac{1}{2}\operatorname{Id} - K'_k\right)\gamma_1^{ext} u \end{aligned}$$

indirect methods

Ansatz: the solution of the Dirichlet problem is sought as a potential

- (first attempt): $u = \widetilde{V}_k \varphi$ for an unknown density φ. $\to$ BIE

$$V_k \varphi = g \quad \text{on } \Gamma$$

However: V_k not injective for some k

- (second attempt) $u = \widetilde{K}_k \varphi$. Again no good solvability theory for all k
- (combined field ansatz) $u = (\mathbf{i}\eta \widetilde{V}_k + \widetilde{K}_k)\varphi$ for some parameter $\eta \in \mathbb{R} \setminus \{0\}$. $\to$

$$g = \gamma_0^{ext} u = \mathbf{i}\eta V_k \varphi + \left(\frac{1}{2} + K_k\right)\varphi =: A\varphi$$

Brakhage-Werner

$A : L^2(\Gamma) \to L^2(\Gamma)$ is boundedly invertible for every $\eta \in \mathbb{R} \setminus \{0\}$

direct methods

starting point: Calderón projector:

$$\begin{aligned} \gamma_0^{ext} u &= \left(\frac{1}{2}\operatorname{Id} + K_k\right)\gamma_0^{ext} u - V_k \gamma_1^{ext} u \qquad | \cdot (-\mathbf{i}\eta) \\ \gamma_1^{ext} u &= -D_k \gamma_0^{ext} u + \left(\frac{1}{2}\operatorname{Id} - K_k'\right)\gamma_1^{ext} u \end{aligned}$$

linear combination yields

$$\left[\mathbf{i}\eta\left(\frac{1}{2} - K_k\right) - D_k\right]\gamma_0^{ext} u = \left[\mathbf{i}\eta V_k + \left(\frac{1}{2} + K_k'\right)\right]\gamma_1^{ext} u =: A' \gamma_1^{ext} u.$$

Dirichlet problem: given $\gamma_0^{ext} u = g$, solve for $\gamma_1^{ext} u$.
Representation formula gives u:

$$u = -\widetilde{V}_k \gamma_1^{ext} u + \widetilde{K}_k \gamma_0^{ext} u.$$

the BIOs A and A'

- $A = \frac{1}{2} + K_k + \mathbf{i}\eta V_k, \qquad A' = \frac{1}{2} + K_k' + \mathbf{i}\eta V_k$
- coupling parameter η with $|\eta| \sim k$.
- Γ smooth $\Longrightarrow$ A and A' are compact perturbations of $\frac{1}{2}\operatorname{Id}$ $\to$ Fredholm theory available

question

how does k enter the mapping properties of A, A' and their inverses A^{-1}, $(A')^{-1}$?

Galerkin discretizations

- given $(V_N)_{N\in\mathbb{N}} \subset L^2(\Gamma)$

  find $\varphi_N \in V_N$ s.t. $\quad \langle A\varphi_N, v\rangle_{L^2(\Gamma)} = \langle f, v\rangle_{L^2(\Gamma)} \qquad \forall v \in V_N$

- asymptotic quasioptimality: $\exists N_0$ s.t. $\forall N \geq N_0$

$$\|\varphi - \varphi_N\|_{L^2(\Gamma)} \leq 2 \inf_{v \in V_N} \|\varphi - v\|_{L^2(\Gamma)}$$

- question: how does N_0 depend on k?

hp-BEM spaces $S^{p,0}(\mathcal{T}_h)$

- $\mathcal{T}_h$ = mesh on Γ, mesh width h
- element maps analytic (+ suitable scaling properties)
- $S^{p,0}(\mathcal{T}_h) \subset L^2(\Gamma)$
- $S^{p,0}(\mathcal{T}_h)$ = piecewise (mapped) polynomials of degree p

Theorem (Quasi-optimality of hp-BEM)

Assumption:

- *(adjoint) well-posedness:* $\|(A')^{-1}\|_{L^2 \leftarrow L^2} \leq C k^\alpha$

Then: $\exists\, c_1, c_2 = c_2(\alpha)$ *independent of* k *s.t. the*

- *scale resolution condition* $\frac{kh}{p} \leq c_1$ *and* $p \geq c_2 \log k$

implies

$$\|\varphi - \varphi_N\|_{L^2(\Gamma)} \leq 2 \inf_{v \in S^{p,0}(\mathcal{T}_h)} \|\varphi - v\|_{L^2(\Gamma)}$$

Corollary

Selecting $p = O(\log k)$ *and* $h \sim \frac{p}{k}$ *leads to quasi-optimality for an* hp*-BEM space of dimension* $N \sim k^{d-1}$.

remarks on assumption of well-posedness

Assumption of well-posedness

for some $\alpha \in \mathbb{R}$ there holds

$$\|(A')^{-1}\|_{L^2 \leftarrow L^2} \leq C k^\alpha$$

- $\alpha = 0$ for star shaped domains (Chandler-Wilde & Monk)
- often observed in practice
- $\|(A')^{-1}\|_{L^2 \leftarrow L^2} \geq C e^{\gamma k_m}$: for certain trapping domains and $k_m \to \infty$ (Betcke/Chandler-Wilde/Graham/Langdon/Lindner)

possible to show:

$$\|\varphi - \varphi_N\|_{L^2(\Gamma)} \leq (1 + \varepsilon_{h,p}) \inf_{v \in S^{p,0}(\mathcal{T}_h)} \|\varphi - v\|_{L^2(\Gamma)}$$

where $\varepsilon_{h,p} \to 0$ if $\frac{kh}{p} \to 0$ (and $p \gtrsim \log k$)

regularity through decomposition

- idea: decompose operators into a
  - part with k-independent bounds
  - part with smoothing properties and k-explicit bounds
- example:

$$A^{-1} = A_1 + \mathcal{A}_1$$

  - A_1 order zero operator; k-independent bounds for $\|A_1\|$
  - $\mathcal{A}_1$ maps into space of analytic functions
- example:

$$A = \frac{1}{2} + K_k + \mathbf{i}\eta V_k = \frac{1}{2} + K_0 + \quad R \quad + \mathcal{A}$$

  - $\mathcal{A}$: maps into space of analytic functions; k-explicit bounds
  - R: "small", order -1, k-explicit bounds

decomposition of V_k

Theorem

Let Γ *be analytic and choose* $q \in (0,1)$. *Then:*

$$V_k = V_0 + \mathcal{S}_V + \mathcal{A}_V$$

where

(i) $\mathcal{S}_V : L^2(\Gamma) \to H^3(\Gamma)$ *and*

$$\|\mathcal{S}_V\|_{L^2 \leftarrow L^2} \lesssim q k^{-1}, \quad \|\mathcal{S}_V\|_{H^1 \leftarrow L^2} \lesssim q, \quad \|\mathcal{S}_V\|_{H^3 \leftarrow L^2} \lesssim k^2$$

(ii) $\mathcal{A}_V : L^2(\Gamma) \to$ *space of analytic functions and*

$$\|\nabla^n \mathcal{A}_V \varphi\|_{L^2(\Gamma)} \lesssim k^{3/2} \max\{k, n\}^n \gamma^n \|\varphi\|_{H^{-3/2}(\Gamma)} \quad \forall n \in \mathbb{N}_0$$

analogous result for K_k

decomposition of $\widetilde{V}_k$

- study: $\widetilde{V}_k - \chi\widetilde{V}_0$ (χ = smooth cut-off fct, $\chi \equiv 1$ near Γ)
- given $\varphi \in H^{-1/2}(\Gamma)$ set $u := \widetilde{V}_k\varphi$ and $u_0 := \widetilde{V}_0\varphi \in H^1(B_R)$.
- Then $\delta := u - \chi u_0 = \widetilde{V}_k\varphi - \chi\widetilde{V}_0\varphi$ solves

$$-\Delta\delta - k^2\delta = k^2 u_0\chi + 2\nabla\chi\cdot\nabla u_0 + u_0\Delta\chi =: f$$
$$[\delta] = 0 \qquad [\partial_n\delta] = 0 \quad \text{on } \Gamma, \quad \delta \text{ satisfies radiation condition}$$

- $\rightarrow \delta = \mathcal{N}_k(f) = \mathcal{N}_k(H_{\eta k}f) + \mathcal{N}_k(L_{\eta k}f)$, where $H_{\eta k}$ and $L_{\eta k}$ are the high and low pass filters, $\eta > 1$.
- $L_{\eta k}f$ analytic $\Longrightarrow \mathcal{N}_k(L_{\eta k}f)$ analytic
- $\mathcal{N}_k(H_{\eta k}f)$ is
  - an element of $H^3(B_R)$ (since $f \in H^1(B_R)$)
  - small in $H^1(B_R)$ for large $\eta > 1$:

$$|\mathcal{N}_k(H_{\eta k}f)|_{H^1(\mathbb{R}^d)} \le Ck^{-1}\|H_{\eta k}f\|_{L^2(\mathbb{R}^d)} \le C\frac{1}{\eta k^2}\|f\|_{H^1(\mathbb{R}^d)} \le C\frac{1}{\eta}\|\varphi\|_{H^{-1/2}(\Gamma)}.$$

Theorem (decomposition of $\widetilde{V}_k$)

Let Γ be analytic, $s \ge -1$, and choose $q \in (0,1)$. Then:

$$\widetilde{V}_k = \chi\widetilde{V}_0 + \widetilde{S}_V + \widetilde{\mathcal{A}}_V$$

where

(i) $\widetilde{S}_V : H^{-1/2+s}(\Gamma) \to H^2(B_R) \cap H^{3+s}(B_R\setminus\Gamma)$ *and*

$$\|\widetilde{S}_V\varphi\|_{H^{s'}(B_R\setminus\Gamma)} \lesssim q^2(q/k)^{1+s-s'}\|\varphi\|_{H^{-1/2+s}(\Gamma)}, \qquad 0 \le s' \le 3+s$$

(ii) $\widetilde{\mathcal{A}}_V : H^{-1/2+s}(\Gamma) \to$ *space of p.w. analytic functions and*

$$\|\nabla^n\widetilde{\mathcal{A}}_V\varphi\|_{L^2(B_R\setminus\Gamma)} \lesssim \gamma^n k\max\{k,n\}^n\|\varphi\|_{H^{-3/2}(\Gamma)} \quad \forall n \in \mathbb{N}_0$$

decomposition of A^{-1}

Theorem

Let Γ be analytic.

Assume: $\|A^{-1}\|_{L^2\leftarrow L^2} \le Ck^\alpha$ *for some* $\alpha \ge 0$

Then: $A^{-1} = A_1 + \mathcal{A}_1$

where

(i) $A_1 : L^2(\Gamma) \to L^2(\Gamma)$ *with* $\|A_1\|_{L^2\leftarrow L^2} \le C$ *independent of k*

(ii) $\mathcal{A}_1 : L^2(\Gamma) \to$ *space of analytic functions with*

$$\|\nabla^n\mathcal{A}_1\varphi\|_{L^2(\Gamma)} \le Ck^\beta\max\{k,n\}^n\|\varphi\|_{L^2(\Gamma)} \quad \forall n \in \mathbb{N}_0$$

for suitable $\beta \ge 0$.

Remark: Analogous decomposition for $(A')^{-1}$

sketch of the proof

- the operators V_k and K_k can be decomposed as $V_k = V_0 + R_V + \mathcal{A}_V$ and $K_k = K_0 + R_K + \mathcal{A}_K$, where

$$\|R_V\|_{L^2\leftarrow L^2} \le qk^{-1}, \qquad \|R_K\|_{L^2\leftarrow L^2} \le q.$$

- hence, decompose $A = \frac{1}{2} + K_k + \mathbf{i}\eta V_k$ as (use $\eta = O(k)$)

$$A = \frac{1}{2} + K_k + \mathbf{i}\eta V_k = \left(\frac{1}{2} + K_0 + \mathbf{i}V_0\right) + R + \mathcal{A}$$
$$=: (A_0 + R) + \mathcal{A} =: \widehat{A}_0 + \mathcal{A}$$

where

1. $A_0 : L^2(\Gamma) \to L^2(\Gamma)$ is boundedly invertible
2. $\|R\|_{L^2\leftarrow L^2} \le q$ with arbitrary $q \in (0,1)$
3. $\mathcal{A}$ maps into a space of analytic functions
4. $\widehat{A}_0 : L^2(\Gamma) \to L^2(\Gamma)$ is boundedly invertible (with norm independent of k)

sketch of the proof, II

$$A = \widehat{A}_0 + \mathcal{A}$$

- $\widehat{A}_0 : L^2(\Gamma) \to L^2(\Gamma)$ is boundedly invertible
- $\mathcal{A}$ maps into a space of analytic functions
- Then
$$A^{-1} = \widehat{A}_0^{-1} - A^{-1}\mathcal{A}\widehat{A}_0^{-1} =: A_1 + \mathcal{A}_1$$
is the desired decomposition of A^{-1} if we can show that A^{-1} maps analytic functions to analytic functions
- more specifically:
  - structure of $\mathcal{A}$: $\mathcal{A}$ is constructed by taking traces of potentials $\to \mathcal{A}\varphi = [z]$ for a piecewise analytic function (depending on φ)
  - $\to$ will need that A^{-1} maps traces of jumps of piecewise analytic functions to jumps of piecewise analytic functions

Theorem (analytic data)

Let $\partial\Omega$ be analytic. Let f be the jump of a piecewise analytic function. Let $\varphi \in L^2(\Gamma)$ solve

$$(\frac{1}{2} + K_k + \mathbf{i}\eta V_k)\varphi = A\varphi = f$$

Then $\varphi = [u]$ for a piecewise analytic function u.

ideas of the proof:

1. define $u = \widetilde{K}_k\varphi + \mathbf{i}\eta\widetilde{V}_k\varphi$.
2. jump conditions: $\varphi = [u]$.
3. $\gamma_0^{ext} u = (\frac{1}{2} + K_k + \mathbf{i}\eta V_k)\varphi = f \to$ get bounds for u on $B_R \setminus \overline{\Omega}$
4. $[u] = \varphi$ and $[\partial_n u] = \mathbf{i}\eta\varphi$ implies $[\partial_n u] + \mathbf{i}\eta[u] = 0$.
5. once $u|_{\mathbb{R}^d\setminus\Omega}$ is known, we have an elliptic equation in Ω with Robin boundary data $\to$ estimates for $u|_\Omega$.

set-up of numerical examples

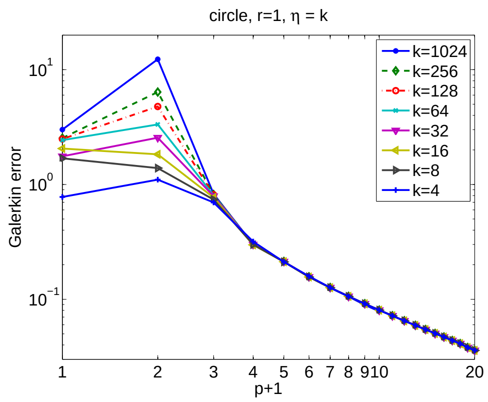
- $A' = \frac{1}{2} + K_k' + \mathbf{i}kV_k$
- mesh $\mathcal{T}_h$ is quasi-uniform and $h \sim 1/k$
- for fixed mesh $\mathcal{T}_h$, degree p ranges from 1 to 14.
- Galerkin projector $P_{h,p} : L^2(\Gamma) \to S^{p,0}(\mathcal{T}_h)$
- approximate quasi-optimality constant

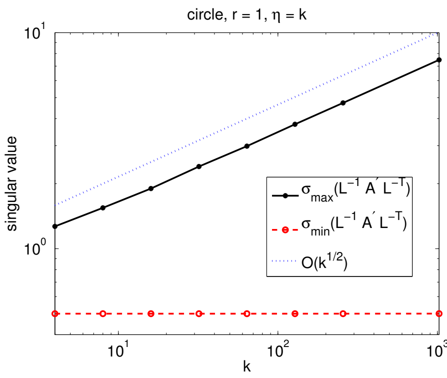
$$\|\operatorname{Id} - P_{h,p}\|_{L^2\leftarrow L^2} \approx \sup_{v\in S^{20,0}(\mathcal{T}_h)} \frac{\|(\operatorname{Id} - P_{h,p})v\|_{L^2}}{\|v\|_{L^2}}$$

- indications for $\|A'\|_{L^2\leftarrow L^2}$ and $\|(A')^{-1}\|_{L^2\leftarrow L^2}$
- usually $\eta = k$ (some computations: $\eta = 1$)
- recall scale resolution condition:

$$\frac{kh}{p} \text{ small} \qquad \text{and} \qquad p \geq c\log k.$$

circle (radius 1)



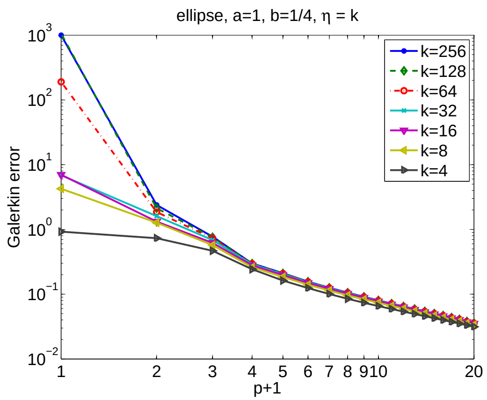


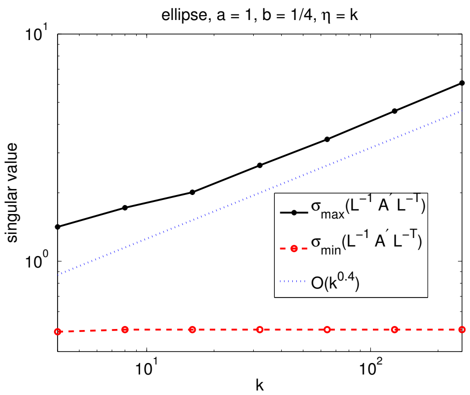
- Number of elements $N = k$
- Galerkin Error $= \sqrt{\|\operatorname{Id} - P_{h,p}\|^2 - 1}$

- quasioptimality constant:

$$C_{opt} = \sqrt{1 + \text{Galerkin Error}^2}$$

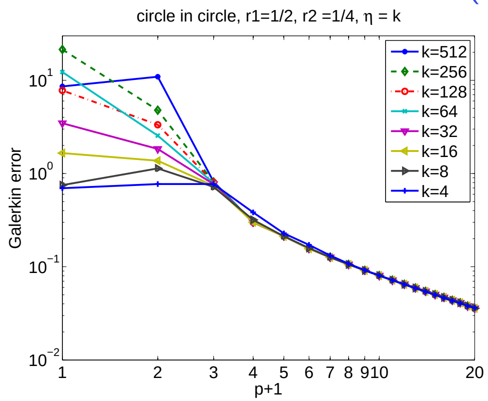
ellipse (semi-axes 1, $1/4$)

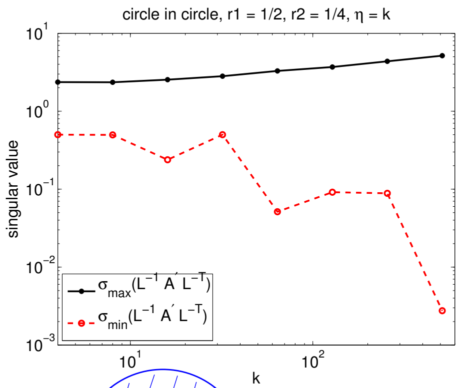


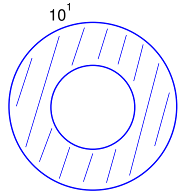


- Number of elements $N = k$
- Galerkin Error $= \sqrt{\|\,\mathrm{Id} - P_{h,p}\|^2 - 1}$

circle in circle (radii: $1/2$, $1/4$)

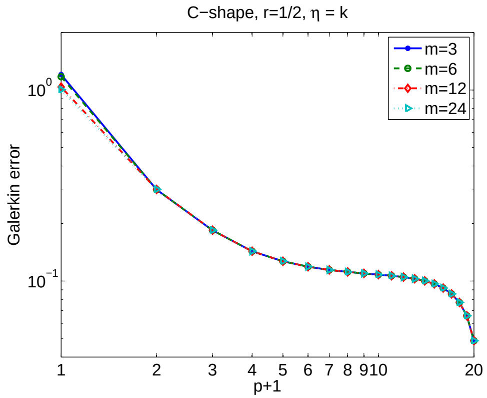


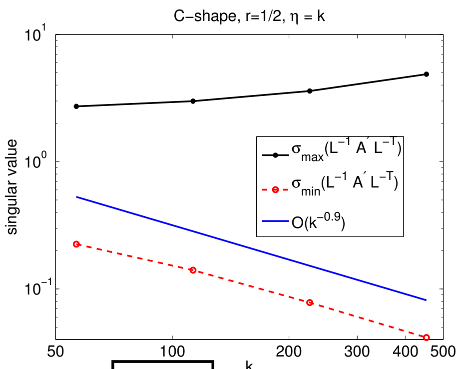


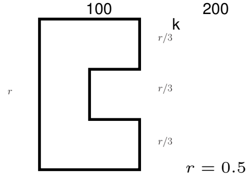


- Number of elements $N = 2k$
- Galerkin Error $= \sqrt{\|\,\mathrm{Id} - P_{h,p}\|^2 - 1}$

C-shaped domain ($\eta = k$)

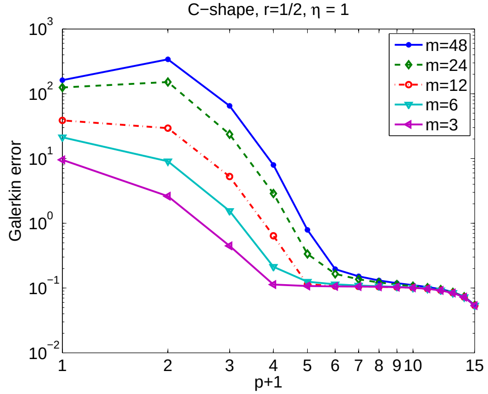


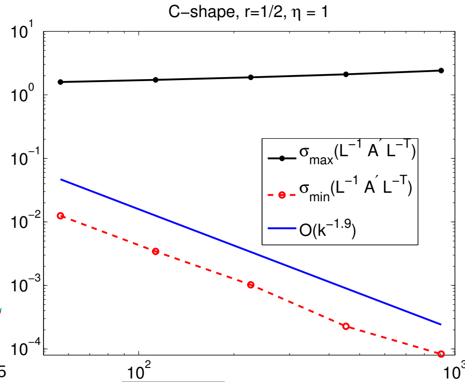


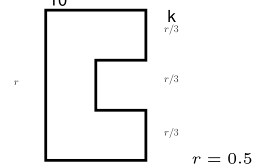


- Number of elements $N = 20m$
- $k = 3\pi m/r$

C-shaped domain ($\eta = 1$)

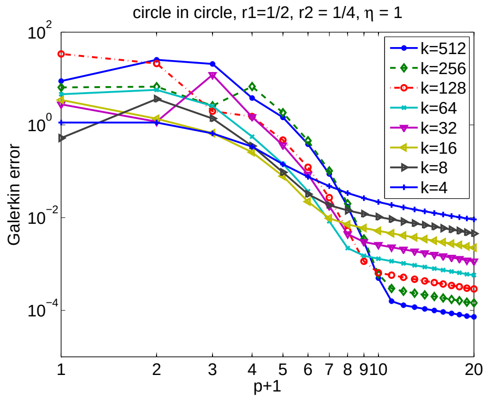


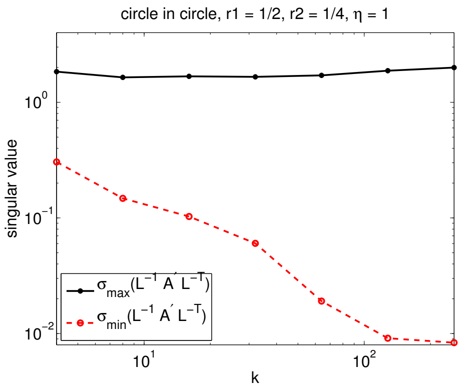




- Number of elements $N = 20m$
- $k = 3\pi m/r$

circle in circle (radii $1/2$, $1/4$, $\eta = 1$)





- Number of elements $N = 2k$

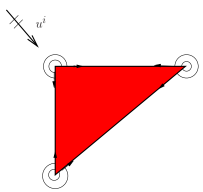
Conclusions for classical hp-BEM

- decomposition of A and A^{-1} into
  - parts with k-independent bounds
  - parts with smoothing properties and k-explicit bounds
- hp-BEM quasi-optimal with k-independent constant if

$$\frac{kh}{p} \text{ is sufficiently small} \qquad \text{and } p \geq c \log k$$

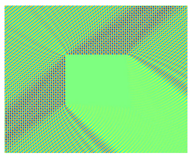
- quasi-optimality for problem size $N = O(k^{d-1})$ possible (select $p = O(\log k)$ and $h = O(p/k)$)
- often observe quasi-optimality already for "$\frac{kh}{p}$ small"
- caveat: the continuous problem needs to be well-posed, i.e., $\|A^{-1}\|$ grows only polynomially in k

non-standard BEM: a sound soft scattering problem



$\Omega =$ polygonal obstacle

$u^i(\mathbf{x}) := \exp(\mathbf{i}k\, \mathbf{d} \cdot \mathbf{x}), \qquad |\mathbf{d}| = 1$



$$\begin{aligned} -\Delta u - k^2 u &= 0 && \text{in } \mathbb{R}^d \setminus \overline{\Omega} \\ u &= 0 && \text{on } \partial\Omega \\ u^s &:= u - u^i && \text{satisfies} (\partial_r - \mathbf{i}k) u^s = o(r^{-(d-1)/2}), \end{aligned}$$

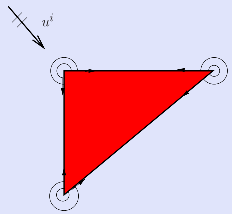
goal: determine $\partial_n u$ on $\partial\Omega$

question: find space V_N from which $\partial_n u$ can be approximated well

multiscale approximation spaces

Kirchhoff approximation of $\partial_n u$:

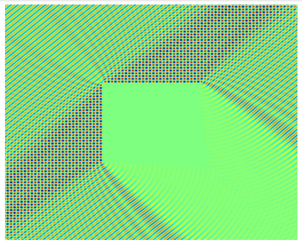
$$\partial_n u \sim \Psi := \begin{cases} 2\partial_n u^i & \text{on lit side} \\ 0 & \text{on shadow side} \end{cases}$$



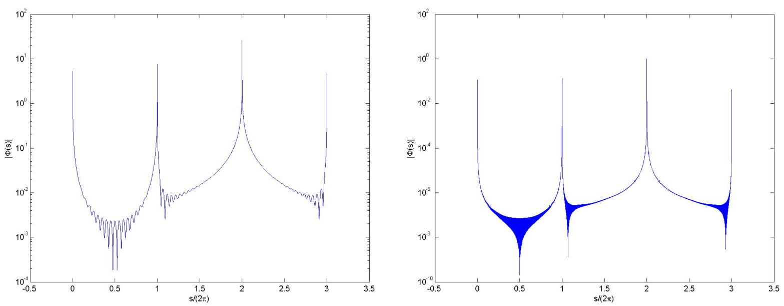
Ansatz

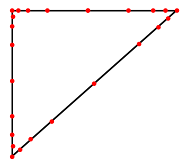
$$\partial_n u = \Psi + k\phi$$

for a function ϕ to be determined



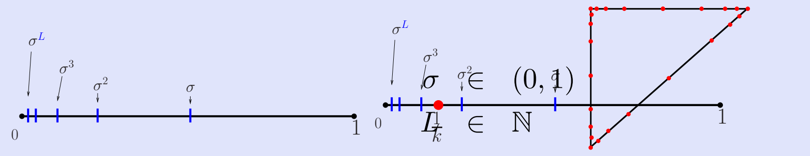
$\partial_n u$ for $k = 10$ and $k = 10240$





- sharp gradient at corners
- highly oscillatory
- piecewise smooth

geometric mesh $\mathcal{T}_L$ with L layers



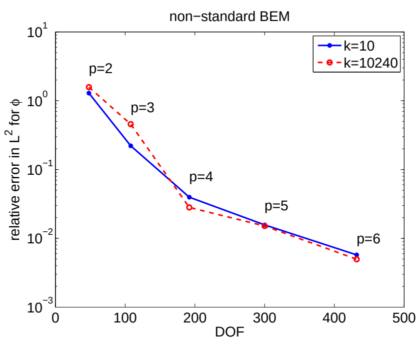
Theorem

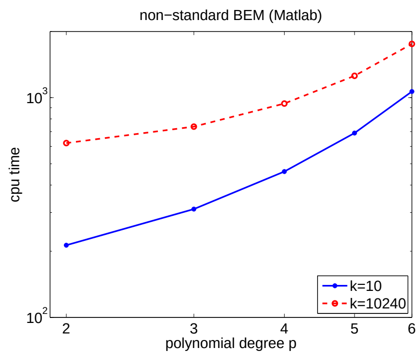
$$V_N^+ := \exp(\mathbf{i}ks) \times \textit{p.w. polynomials of degree } p \textit{ on } \mathcal{T}_L,$$
$$V_N^- := \exp(-\mathbf{i}ks) \times \textit{p.w. polynomials of degree } p \textit{ on } \mathcal{T}_L,$$
$$p \sim L \gtrsim \log k.$$

Then: $V_N := V_N^+ + V_N^-$ *satisfies*

$$\inf_{v \in V_N} \|\phi - v\|_{L^2(\partial\Omega)} \lesssim k^{-1/2} e^{-bp},$$
$$\dim V_N \sim p^2$$

nonstandard BEM: $k = 10$ and $k = 10240$





composite Filon quadrature

- k-robust exponential convergence (absolute error)
- cost of quadrature formula independent of k

conclusions for nonstandard BEM

- possible to design special approximation spaces that incorporate both the oscillatory nature of the solution and the corner singularities
- approximation properties are only weakly dependent on k
- possible to design (in 2D) exponentially convergent quadrature rule to set up BEM stiffness matrix with work depending only weakly on k