# Why it is Difficult to Solve Helmholtz Problems with Classical Iterative Methods

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Collaborations with
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# A Comment on Terminology

The **indefinite Helmholtz equation** is

$$\mathcal{L}u := -(\Delta + k^2)u = f,$$

in contrast to

$$(\eta - \Delta)u = f, \quad \eta > 0$$

often also called Helmholtz equation (e.g. Zauderer, meteorology, ...), or Helmholtz equation with the good sign.

To avoid confusion, we call the latter the **positive definite Helmholtz equation**, and it can be easily solved by most iterative methods.

The subject of this talk is the indefinite Helmholtz equation.

# Problems of Preconditioned Krylov Methods

Discretization of the indefinite Helmholtz equation

$$\mathcal{L}u := -(\Delta + k^2)u = f, \quad \text{in } \Omega \subset \mathbb{R}^2$$

with appropriate boundary conditions leads to the indefinite linear system

$$A\mathbf{u} = \mathbf{f}$$

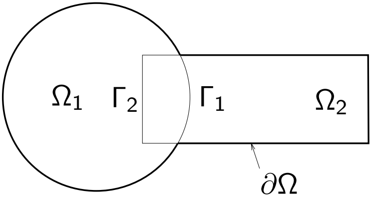
whose solution by iterative methods is difficult!

**Example:** Cavity, open on the left, with point source at the center, using the Krylov method QMR:

| | QMR | | ILU('0') | | ILU(1e-2) | |
|---|---|---|---|---|---|---|
| $k$ | it | Mflops | it | Mflops | it | Mflops |
| 5 | 197 | 120.1 | 60 | 60.4 | 22 | 28.3 |
| 10 | 737 | 1858.2 | 370 | 1489.3 | 80 | 421.4 |
| 15 | 1775 | 10185.2 | $>$ 2000 | $>$ 18133.2 | 220 | 2615.1 |
| 20 | $>$ 2000 | $>$ 20335.1 | — | — | $>$ 2000 | $>$ 42320.1 |

# Problems of Domain Decomposition Methods

Classical Schwarz method (**H.A. Schwarz (1869)**):




$$\begin{aligned} -(\Delta + k^2)u_1^{n+1} &= 0 \quad \text{in } \Omega_1 \\ u_1^{n+1} &= u_2^n \quad \text{on } \Gamma_1 \end{aligned} \qquad \begin{aligned} -(\Delta + k^2)u_2^{n+1} &= 0 \quad \text{in } \Omega_2 \\ u_2^{n+1} &= u_1^{n+1} \quad \text{on } \Gamma_2 \end{aligned}$$

solve on the disk      solve on the rectangle

Example: cavity open on two sides

| $k$ | Overlap | $10\pi$ | $20\pi$ | $40\pi$ | $80\pi$ | $160\pi$ |
|---|---|---|---|---|---|---|
| Iterative | h | div | div | div | div | div |
| Preconditioner | h | 20 | 33 | 45 | 69 | 110 |
| Iterative | fixed | div | div | div | div | div |
| Preconditioner | fixed | 16 | 23 | 43 | 86 | 155 |

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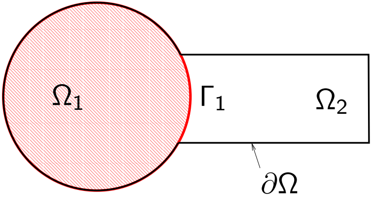


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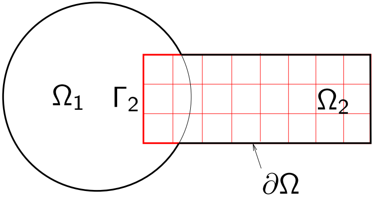


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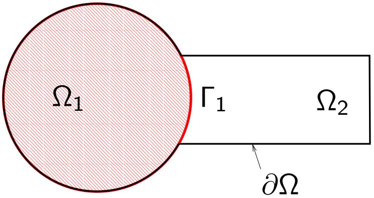


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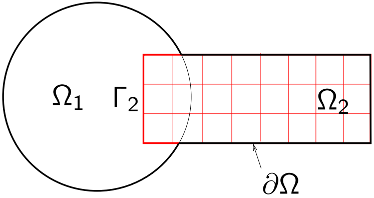


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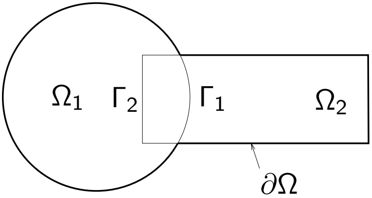


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# Problems of Multigrid



Classical multigrid algorithm for $A\mathbf{u} = \mathbf{f}$:

```
function u=Multigrid(A,f,u0);
if isSmall(A) then u=A\f else
    u=DampedJacobi(nu,A,f,u0);
    r=Restrict(f-Au);
    e=Multigrid(Ac,r,0);
    u=u+Extend(e);
    u=DampedJacobi(nu,A,f,u);
end;
```

Example: closed cavity, no resonnance

| $k$ | Smoothing steps | $2.5\pi$ | $5\pi$ | $10\pi$ | $20\pi$ |
|---|---|---|---|---|---|
| Iterative | $\nu = 2$ | 7 | div | div | div |
| Preconditioner | $\nu = 2$ | 6 | 12 | 41 | 127 |
| Iterative | $\nu = 5$ | 7 | stag | div | div |
| Preconditioner | $\nu = 5$ | 5 | 13 | 41 | 223 |
| Iterative | $\nu = 10$ | 8 | div | div | div |
| Preconditioner | $\nu = 10$ | 5 | 10 | 14 | 87 |

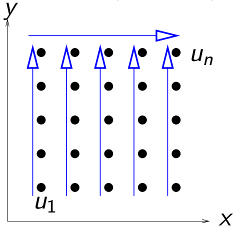
# Effective ILU Preconditioner for Helmholtz

$$A = LU, \quad L \text{ lower triangular, } U \text{ upper triangular}$$

Solution of $Au = LUu = f$ is obtained by solving

$$\begin{array}{rcll} Lv &=& f & \text{using a forward substitution} \\ Uu &=& v & \text{using a backward substitution} \end{array}$$

If $A$ is $-(\Delta + k^2)$ discretized in 2d with ordering




$L \iff$ evolution problem in the positive $x$ direction

$U \iff$ evolution problem in the negative $x$ direction

# The Analytic Incomplete LU Preconditioner

The continuous analog of block $LU$ for Helmholtz is

$$-(\Delta + k^2) = -(\partial_x + \Lambda_1)(\partial_x - \Lambda_2).$$

To solve $-(\Delta + k^2)u = -(\partial_x + \Lambda_1)(\partial_x - \Lambda_2)u = f$:

$-(\partial_x + \Lambda_1)v = f$ evolution problem in the forward $x$ direction

$(\partial_x - \Lambda_2)u = v$ evolution problem in the backward $x$ direction

At the discrete level, we obtain the block $LDL^T$ factorization

$$-\left(D_x^- + (Th - \frac{1}{h})\right)\frac{1}{h^2}T^{-1}\left(D_x^+ - (Th - \frac{1}{h})\right)$$

with the non-local operator $T$.

Approximating the non-local operator leads to the AILU (Analytic Incomplete LU) preconditioners.

# Numerical Experiments

**Same Example:** Cavity, open on the left, with point source at the center, using QMR:

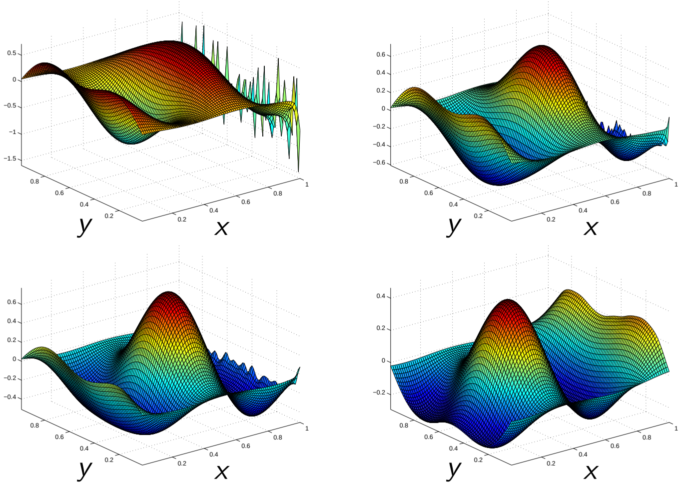
| | QMR | | ILU('0') | | ILU(1e-2) | | AILU('0') | |
|---|---|---|---|---|---|---|---|---|
| $k$ | it | Mflops | it | Mflops | it | Mflops | it | Mflops |
| 5 | 197 | 120.1 | 60 | 60.4 | 22 | 28.3 | 23 | 28.3 |
| 10 | 737 | 1858.2 | 370 | 1489.3 | 80 | 421.4 | 36 | 176.2 |
| 15 | 1775 | 10185.2 | 2000 | 18133.2 | 220 | 2615.1 | 43 | 475.9 |
| 20 | 2000 | 20335.1 | — | — | 2000 | 42320.1 | 64 | 1260.2 |
| 30 | – | – | – | – | – | – | 90 | 3984.1 |
| 40 | – | – | – | – | – | – | 135 | 10625.0 |
| 50 | – | – | – | – | – | – | 285 | 24000.4 |

See also: Frequency Filtering: Wittum (1991), Wagner (1997), Buzdin (1998), G/Nataf (2001,2002), Nataf/Achdou (2007), Nataf et al (2010)

Optimal approximation for Helmholtz: talk by B. Enquist!

# Why Does Classical Schwarz Not Work?

Error at iterations 1, 2, 3, and 8 on subdomain 1:

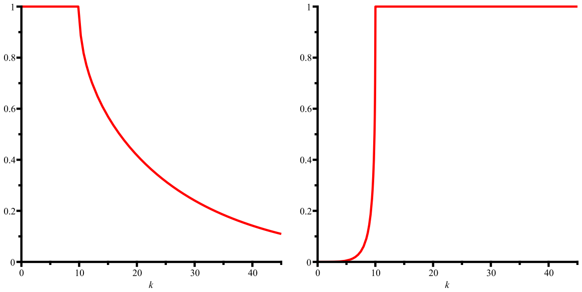


# Remedy for Domain Decomposition Methods

**B. Després (1990):** L'objectif de ce travail est, après construction d'une méthode de décomposition de domaine adaptée au problème de Helmholtz, d'en démontrer la convergence.

$$\begin{array}{rcll} -(\Delta + k^2)u_j^{n+1} & = & f, & \text{in } \Omega_j \\ (\partial_{n_j} + ik)u_j^{n+1} & = & (\partial_{n_j} + ik)u_l^n & \text{on interface } \Gamma_{jl} \end{array}$$

Convergence factor comparison:



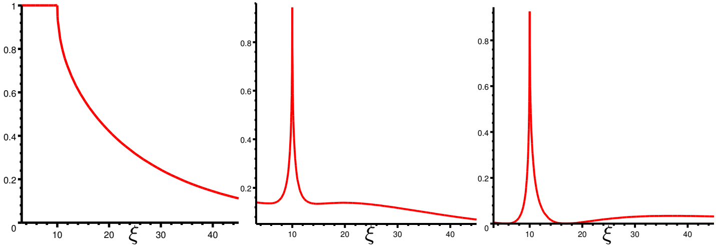

classical Schwarz Després (1990)

# Optimized Schwarz Methods for Helmholtz

$$
\begin{aligned}
-(\Delta + k^2)u_j^{n+1} &= f, && \text{in } \Omega_j \\
(\partial_{n_j} + \mathcal{S}_{jl})u_j^{n+1} &= (\partial_{n_j} + \mathcal{S}_{jl})u_l^n && \text{on interface } \Gamma_{jl}
\end{aligned}
$$

**Optimized Order Zero (OO0):** approximation of $\mathcal{S}_{jl}$ by a complex constant

**Optimized Order Two (OO2):** approximation of $\mathcal{S}_{jl}$ by a constant plus a multiple of the Laplace-Beltrami operator




classical Schwarz | OO0 Schwarz | OO2 Schwarz

# Summary of the Analytical Results

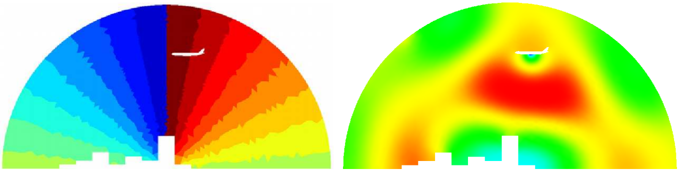
With asymptotically optimized formulas for $\mathcal{S}_{jl}$ (two-sided):

| | $k$ fixed | $k^\gamma h$ const |
|---|---|---|
| Overlap 0 | $1 - O(h^{\frac{1}{4}})$ | $1 - O(k^{\frac{1-2\gamma}{8}})$ |
| Overlap $C_L h$ | $1 - O(h^{\frac{1}{5}})$ | $\begin{cases} 1 - O(k^{-\frac{1}{8}}) & 1 \leq \gamma \leq \frac{9}{8} \\ 1 - O(k^{\frac{1-2\gamma}{10}}) & \gamma > \frac{9}{8} \end{cases}$ |
| Overlap const | $1 - const$ | $1 - O(k^{-\frac{1}{8}})$ |

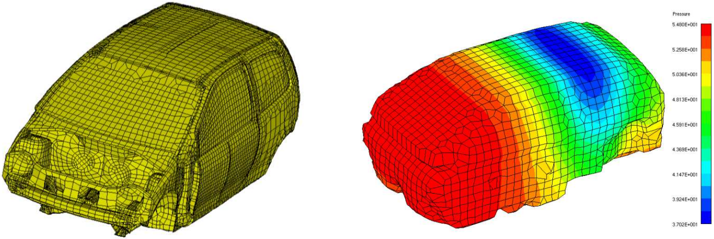
Example: cavity open on two sides, non-overlapping method

| $h$ | Iterative Optimized | Krylov Deprés | Krylov Optimized | $k$ | Krylov Deprés | Krylov Optimized |
|---|---|---|---|---|---|---|
| 1/50 | 322 | 26 | 14 | $10\pi$ | 24 | 13 |
| 1/100 | 70 | 34 | 17 | $20\pi$ | 33 | 18 |
| 1/200 | 75 | 44 | 20 | $40\pi$ | 43 | 20 |
| 1/400 | 91 | 57 | 23 | $80\pi$ | 53 | 21 |
| 1/800 | 112 | 72 | 27 | $160\pi$ | 83 | 32 |

# Examples from two Applications



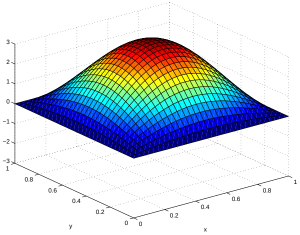
Airbus A340: reduction from 172 (Deprés) to 58 iterations.



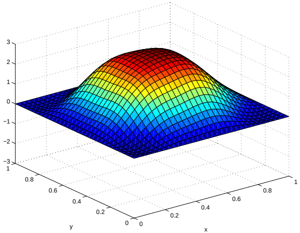

Twingo: reduction from 105 (Deprés) to 34 iterations.
(joint work with F. Magoules)

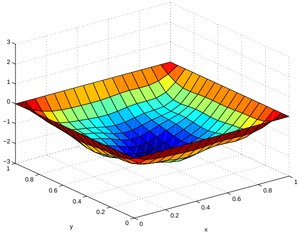
# Problems of the Coarse Grid Correction

Solution on $\Omega = (0,1) \times (0,1)$, $f = -\frac{1}{20}$, $h = \frac{1}{32}$, $k^2 = 19.7$:




Two grid cycle, Fourier smoothing, iteration 1:




error after presmoothing     coarse grid: **error to be removed**

# Observation

- while the error on the coarse grid is well resolved, the correction calculated on the coarse grid is completely wrong, it even has the wrong sign!
- Fourier analysis for error component $\varphi_1^h$:

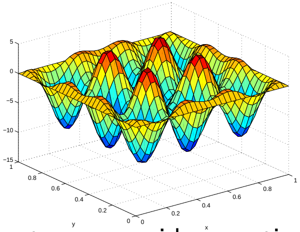
$$\varphi_1^h - I_H^h \mathbf{v}^H \approx \left(1 - \frac{\lambda_1^h}{\lambda_1^H}\right)\varphi_1^h$$

**Brandt and Ta'asan (1986):**

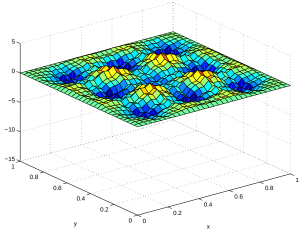
> *Usual multigrid for indefinite problems is sometimes found to be very inefficient. A strong limitation exists on the coarsest grid to be used in the process. The limitation is not so much a result of the indefiniteness itself, but of the nearness to singularity, that is, the existence of nearly zero eigenvalues. These eigenvalues are badly approximated (e.g. they may even have a different sign) on coarse grids, hence the corresponding eigenfunctions, which are usually smooth ones, cannot efficiently converge.*

# Problems of the Smoother

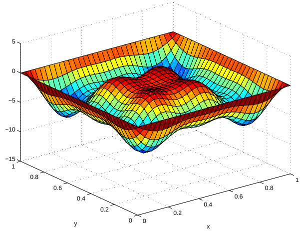
Solution for $f = -1000$, $h = \frac{1}{32}$, $k^2 = 400$




Two grid cycle, exact coarse grid correction (the exact error on the fine grid, just restricted and extended), optimally relaxed Jacobi smoother, iteration 1:




error before postsmoothing

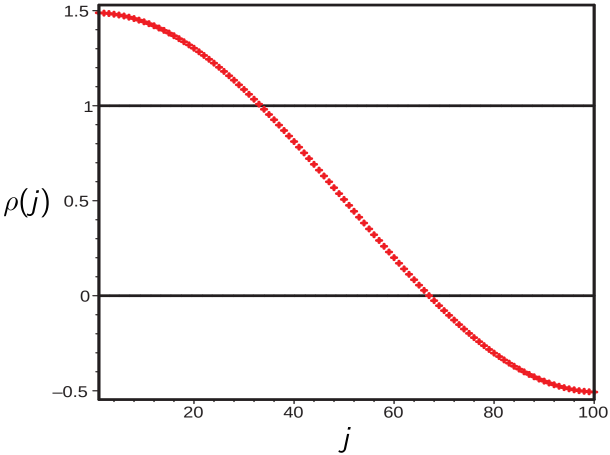



error after postsmoothing

# Fourier Analysis of the Smoother

Using an optimized relaxation parameter for Helmholtz:

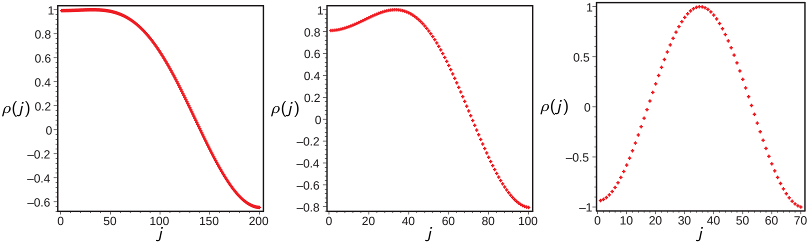
$$\omega^h \approx \frac{h^2k^2 - 2}{h^2k^2 - 3}$$



# Existing Solutions from the Literature: Smoother



**Brandt and Ta'asan (1986):**
Use Kaczmarz relaxation (Stefan Kaczmarz (1937) "Gauss Seidel applied to the normal equations")




**Elmann, Ernst, O'Leary (2001):**
Use GMRES for smoothing on the problematic levels

# Existing Solutions: Coarse Grid Correction

**Brandt and Ta'asan (1986):**
Since Kaczmarz relaxation is converging most slowly for problematic modes, one can detect them, and treat them separately. Good solution for small $k$.

**Brandt and Livshits (1997):** Wave Ray Multigrid

- Construct explicitly problematic modes using plane waves.
- On fine grids, where multigrid is effective, just use multigrid.
- On coarser grids, correct the error using a plane wave representation.

**Elmann, Ernst, O'Leary (2001):**
Use Multigrid as a preconditioner for GMRES.

# More Recent Ideas: Modified Discrete Equations

In 1d, the coarse grid correction problem can be modified to yield accurate corrections: instead of using

$$A^h\mathbf{u}^h := \frac{1}{h^2}\begin{bmatrix} 2-h^2k^2 & -1 & \\ -1 & \ddots & \ddots \end{bmatrix}\mathbf{u}^h = \mathbf{f}^h$$

we use

$$A^h\mathbf{u}^h := \frac{1}{h^2}\begin{bmatrix} 2-h^2{\color{red}k_h^2} & -1 & \\ -1 & \ddots & \ddots \end{bmatrix}\mathbf{u}^h = \mathbf{f}^h$$

with

$${\color{red}k_h^2} := \frac{1}{h^2}(2-2\cos(kh)).$$

Thus the spectral shift due to the discretization is precisely compensated by a shift of the wave-number $k_h$, and the problematic modes are all correctly treated on any coarse grid,

$$\left(1-\frac{\lambda_j^h}{\lambda_j^H}\right) \approx 0.$$

# Multi Step Jacobi Smoother

A multi step Jacobi smoother performs several Jacobi steps with different damping parameter. For example for two steps:

$$\mathbf{u}_{m+1/2} = \mathbf{u}_m + \omega_1 D^{-1}(\mathbf{b} - A\mathbf{u}_m)$$

$$\mathbf{u}_{m+1} = \mathbf{u}_{m+1/2} + \omega_2 D^{-1}(\mathbf{b} - A\mathbf{u}_{m+1/2})$$

The corresponding contraction factor for the 1d Helmholtz model problem is

$$\rho(j,\boldsymbol{\omega}) = \prod_{j=1}^{J} \left(1 - \omega_j \left(1 - \frac{2\cos(j\pi h)}{2 - h^2k^2}\right)\right)$$

For this to be a good smoother, we need to satisfy two conditions:

1. $|\rho(j,\boldsymbol{\omega})| \leq 1$ for all $j = 1, 2, \ldots, n$.
2. $\boldsymbol{\omega} = \operatorname{argmin}_{\boldsymbol{\omega}} \max_{j=n/2\ldots n} |\rho(j,\boldsymbol{\omega})|$

# Numerical Results

1d Helmholtz equation with Dirichlet BC.

10 points per wavelength, 8 pre and post smoothing steps with optimized 2 step Jacobi, relative residual reduction $10^{-6}$

| $k$ | 158.65 | 315.73 | 629.89 | 1258.21 | 2514.84 |
|---|---|---|---|---|---|
| $h$ | $2^{-8}$ | $2^{-9}$ | $2^{-10}$ | $2^{-11}$ | $2^{-12}$ |
| levels | 6 | 7 | 8 | 9 | 10 |
| iter J | 12 | 11 | 10 | 10 | 9 |
| iter G | 9 | 9 | 8 | 8 | 8 |

iter J: number of iterations when on the resonance level $O(k^2)$ 2-step optimized Jacobi steps are performed

iter G: number of iterations when on the resonance level $O(k)$ GMRES steps are performed.

# Conclusions

It is difficult to solve indefinite Helmholtz problems by iterative methods:

- Propagative modes are global over the entire domain, there is no decay in the Green's function for those modes
- These modes can be treated by modifided approximate factorization preconditioners (AILU)
- They can also be treated in domain decomposition methods by optimized transmission conditions
- For multigrid methods, both smoother and coarse grid correction create problems $\Longrightarrow$ Kaczmarz or polynomial smoothers, and waveray or shifted coarse problems

  There is no miracle: any multilevel method for Helmholz problems will have to deal with the dispersion relation problem on coarser grids.