Well-generated triangulated categories

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Let $\mathcal{A}$ be an additive category. Examples:

$$\mathcal{A} = R\text{–Mod} \qquad \text{or} \qquad \mathcal{A} = R\text{–Proj}\ .$$

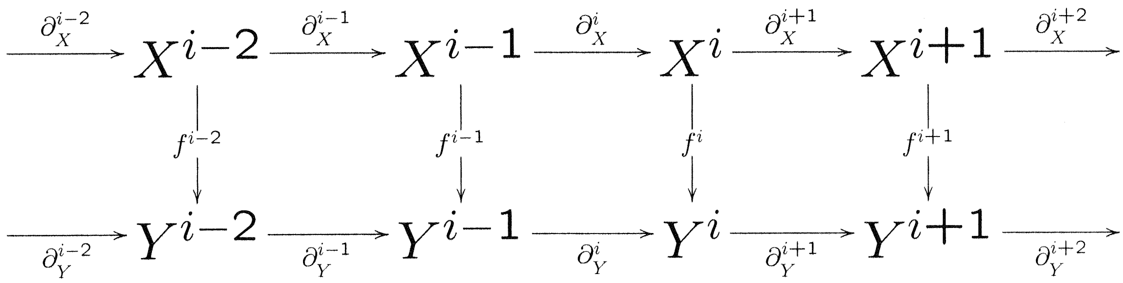
The category R–Mod is *abelian*, the category R–Proj is not.

The category $\mathbf{K}(\mathcal{A})$:

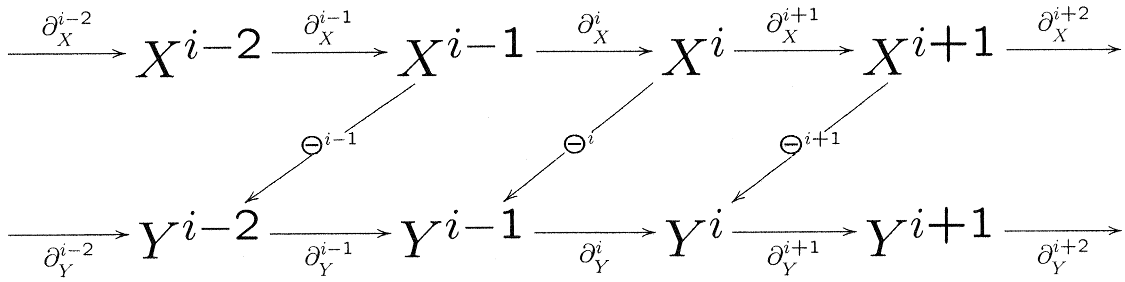
(i) Objects: cochain complexes.

(ii) Morphisms:
HOMOTOPY EQUIVALENCE CLASSES of cochain maps.

A cochain map $f : X \longrightarrow Y$ is:



Two cochain maps $f, g : X \longrightarrow Y$ are *homotopic* if there exists



with $f^i - g^i = \Theta^{i+1}\partial_X^{i+1} + \partial_Y^i \Theta^i$.

Notation: Cochain maps will be written $f : X \longrightarrow Y$, cochain homotopies $\Theta : X \Longrightarrow Y$.

$$
\begin{array}{ccccccccccc}
\longrightarrow & 0 & \longrightarrow & A & \xrightarrow{1} & A & \longrightarrow & 0 & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & A & \xrightarrow[1]{} & A & \longrightarrow & 0 & \longrightarrow
\end{array}
$$

$$
\begin{array}{ccccccccc}
\longrightarrow & 0 & \longrightarrow & A & \xrightarrow{1} & A & \longrightarrow & 0 & \longrightarrow \\
 & & \swarrow & & \swarrow{\scriptstyle 1} & & \swarrow & & \\
\longrightarrow & 0 & \longrightarrow & A & \xrightarrow[1]{} & A & \longrightarrow & 0 & \longrightarrow
\end{array}
$$

$$
\begin{array}{ccccccccccc}
\longrightarrow & X^{i-2} & \xrightarrow{\partial^{i-1}} & X^{i-1} & \xrightarrow{\partial^{i}} & X^{i} & \xrightarrow{\partial^{i+1}} & X^{i+1} & \longrightarrow \\
 & \downarrow & & \partial^{i}\downarrow & & \downarrow 1 & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & X^{i} & \xrightarrow[1]{} & X^{i} & \longrightarrow & 0 & \longrightarrow
\end{array}
$$

$$
\begin{array}{ccccccc}
X^{i-2} & \xrightarrow{\partial^{i-1}} & X^{i-1} & \xrightarrow{\partial^{i}} & X^{i} & \xrightarrow{\partial^{i+1}} & X^{i+1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X^{i-2} \oplus X^{i-1} & \longrightarrow & X^{i-1} \oplus X^{i} & \longrightarrow & X^{i} \oplus X^{i+1} & \longrightarrow & X^{i+1} \oplus X^{i+2}
\end{array}
$$

$$\nu : X \longrightarrow C(X) \ .$$

If $f : X \longrightarrow Y$ is a morphism, it factors as

$$X \xrightarrow{\begin{pmatrix} f \\ \nu \end{pmatrix}} Y \oplus C(X) \xrightarrow{(1\ \ 0)} Y \ .$$

Form the short exact sequence

$$0 \longrightarrow X \xrightarrow{\begin{pmatrix} f \\ \nu \end{pmatrix}} Y \oplus C(X) \xrightarrow{g} Z \longrightarrow 0 \ .$$

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \partial_X^{i-2}} & & \downarrow{\scriptstyle \partial_{\overline{Y}}^{i-2}} & & \downarrow{\scriptstyle \partial_Z^{i-2}} & & \\
0 & \longrightarrow & X^{i-2} & \xrightarrow{\overline{f}^{i-2}} & \overline{Y}^{i-2} & \xrightarrow{g^{i-2}} & Z^{i-2} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_X^{i-1}} & & \downarrow{\scriptstyle \partial_{\overline{Y}}^{i-1}} & & \downarrow{\scriptstyle \partial_Z^{i-1}} & & \\
0 & \longrightarrow & X^{i-1} & \xrightarrow{\overline{f}^{i-1}} & \overline{Y}^{i-1} & \xrightarrow{g^{i-1}} & Z^{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_X^{i}} & & \downarrow{\scriptstyle \partial_{\overline{Y}}^{i}} & & \downarrow{\scriptstyle \partial_Z^{i}} & & \\
0 & \longrightarrow & X^{i} & \xrightarrow{\overline{f}^{i}} & \overline{Y}^{i} & \xrightarrow{g^{i}} & Z^{i} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_X^{i+1}} & & \downarrow{\scriptstyle \partial_{\overline{Y}}^{i+1}} & & \downarrow{\scriptstyle \partial_Z^{i+1}} & & \\
0 & \longrightarrow & X^{i+1} & \xrightarrow{\overline{f}^{i+1}} & \overline{Y}^{i+1} & \xrightarrow{g^{i+1}} & Z^{i+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_X^{i+2}} & & \downarrow{\scriptstyle \partial_{\overline{Y}}^{i+2}} & & \downarrow{\scriptstyle \partial_Z^{i+2}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

If W is the complex

$$\xrightarrow{\partial^{i-2}} W^{i-2} \xrightarrow{\partial^{i-1}} W^{i-1} \xrightarrow{\partial^{i}} W^{i} \xrightarrow{\partial^{i+1}} W^{i+1} \xrightarrow{\partial^{i+2}}$$

Define ΣW to be

$$\xrightarrow{-\partial^{i-1}} W^{i-1} \xrightarrow{-\partial^{i}} W^{i} \xrightarrow{-\partial^{i+1}} W^{i+1} \xrightarrow{-\partial^{i+2}} W^{i+2} \xrightarrow{-\partial^{i+3}}$$

For each i, the map $g^i : \overline{Y}^i \longrightarrow Z^i$ is a split epimorphism; choose a splitting $\Theta^i : Z^i \longrightarrow \overline{Y}^i$.

$$\begin{array}{ccccccccc}
\xrightarrow{\partial_Z^{i-2}} & Z^{i-2} & \xrightarrow{\partial_Z^{i-1}} & Z^{i-1} & \xrightarrow{\partial_Z^{i}} & Z^{i} & \xrightarrow{\partial_Z^{i+1}} & Z^{i+1} & \xrightarrow{\partial_Z^{i+2}} \\
 & & \swarrow{\scriptstyle \Theta^{i-1}} & & \swarrow{\scriptstyle \Theta^{i}} & & \swarrow{\scriptstyle \Theta^{i+1}} & & \\
\xrightarrow[-\partial_{\overline{Y}}^{i-1}]{} & \overline{Y}^{i-1} & \xrightarrow[-\partial_{\overline{Y}}^{i}]{} & \overline{Y}^{i} & \xrightarrow[-\partial_{\overline{Y}}^{i+1}]{} & \overline{Y}^{i+1} & \xrightarrow[-\partial_{\overline{Y}}^{i+2}]{} & \overline{Y}^{i+2} & \xrightarrow[-\partial_{\overline{Y}}^{i+3}]{} \\
 & \downarrow{\scriptstyle g^{i-1}} & & \downarrow{\scriptstyle g^{i}} & & \downarrow{\scriptstyle g^{i+1}} & & \downarrow{\scriptstyle g^{i+2}} & \\
\xrightarrow[-\partial_Z^{i-1}]{} & Z^{i-1} & \xrightarrow[-\partial_Z^{i}]{} & Z^{i} & \xrightarrow[-\partial_Z^{i+1}]{} & Z^{i+1} & \xrightarrow[-\partial_Z^{i+2}]{} & Z^{i+2} & \xrightarrow[-\partial_Z^{i+3}]{}
\end{array}$$

We obtain a cochain map $H : Z \longrightarrow \Sigma\overline{Y}$, and the composite

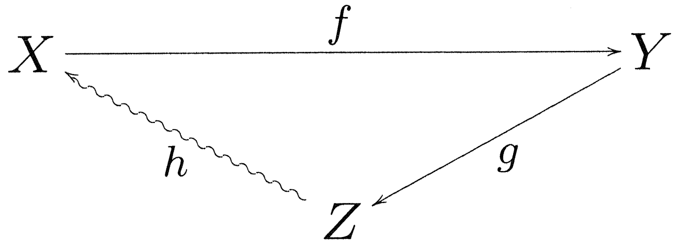
$$Z \xrightarrow{H} \Sigma\overline{Y} \xrightarrow{\Sigma g} \Sigma Z$$

vanishes. Hence H factors through

$$h : Z \longrightarrow \Sigma X \ .$$

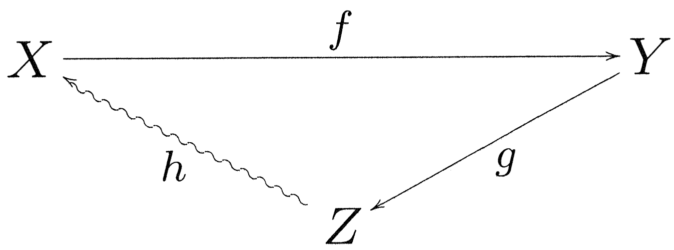
We assemble this to

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \ .$$



AXIOMS:

TR1: Every isomorph of a triangle is a triangle. The sequence $X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X$ is a triangle. Every morphism $f : X \longrightarrow Y$ can be completed to a triangle



TR2: $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle iff $Y \xrightarrow{-g} Z \xrightarrow{-h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ is.

TR3 and TR4: Given a commutative diagram where the rows are triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \alpha\downarrow & & \downarrow\beta & & & & \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma X' \end{array}$$

there is a morphism $\gamma : Z \longrightarrow Z'$ rendering commutative the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \alpha\downarrow & & \downarrow\beta & & \downarrow\gamma & & \downarrow\Sigma\alpha \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma X' . \end{array}$$

Furthermore, γ can be chosen so that

$$X' \oplus Y \xrightarrow{\begin{pmatrix} f' & \beta \\ 0 & -g \end{pmatrix}} Y' \oplus Z \xrightarrow{\begin{pmatrix} g' & \gamma \\ 0 & -h \end{pmatrix}} Z' \oplus \Sigma X \xrightarrow{\begin{pmatrix} h' & \Sigma\alpha \\ 0 & -\Sigma f \end{pmatrix}} \Sigma X' \oplus \Sigma Y$$

is a triangle.

Definition. Let $\mathcal{S}$, $\mathcal{T}$ be triangulated categories. A "triangulated" or "exact" functor $F : \mathcal{S} \longrightarrow \mathcal{T}$ is a functor preserving the structure.

Examples: $\mathbf{K}(\mathcal{A})$ is triangulated. If $\mathcal{A}$ is abelian we set

$$\mathbf{A}(\mathcal{A}) = \{\text{Acyclics}\} \ ;$$

then the inclusion $\mathbf{A}(\mathcal{A}) \longrightarrow \mathbf{K}(\mathcal{A})$ is triangulated. Define

$$\mathbf{D}(\mathcal{A}) \quad = \quad \frac{\mathbf{K}(\mathcal{A})}{\mathbf{A}(\mathcal{A})} \ .$$

The natural projection

$$\mathbf{K}(\mathcal{A}) \xrightarrow{\pi} \mathbf{D}(\mathcal{A})$$

is triangulated.

Definition. A functor $H : \mathcal{T} \longrightarrow \mathcal{A}$ is *homological* if

(i) $\mathcal{T}$ is triangulated.

(ii) $\mathcal{A}$ is abelian.

(iii) For every triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$, the sequence $H(X) \longrightarrow H(Y) \longrightarrow H(Z)$ is exact.

If we set $H^n(W) = H(\Sigma^n W)$ then we obtain

$$\longrightarrow H^{-1}(Z) \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \longrightarrow H^1(X) \longrightarrow$$

Example: all representable functors are homological.

TR5: coproducts exist in $\mathcal{T}$.

Let $\mathcal{T}$ be a [TR5] triangulated category.

Definition 1: $k \in \mathcal{T}$ is compact $\Longleftrightarrow$ any map

$$k \longrightarrow \coprod_{\lambda \in \Lambda} t_\lambda$$

factors as

$$\begin{array}{ccc} k & \longrightarrow & \coprod_{\lambda \in \Lambda} t_\lambda \\ & \searrow & \nearrow \\ & \coprod_{i=1}^{n} t_{\lambda_i} & \end{array}$$

Definition 2: $\mathcal{T}^c \subset \mathcal{T}$ is the full subcategory of all the compacts.

Definition 3: $\mathcal{T}$ is compactly generated $\Longleftrightarrow \exists$ a set $G \subset \mathcal{T}^c$ so that any non-zero object $t \in \mathcal{T}$ has a non-zero map $g \longrightarrow t$, $g \in G$.

Let $\mathcal{T}$ be a [TR5] triangulated category, and let $G \subset \mathcal{T}^c$ be as in Definition 3.

Theorem 1: If $\mathcal{S} \subset \mathcal{T}^c$ contains G and is closed under triangles and retracts, then $\mathcal{S} = \mathcal{T}^c$.

Theorem 2: If $\mathcal{S} \subset \mathcal{T}$ contains G and is closed under triangles are coproducts, then $\mathcal{S} = \mathcal{T}$.

Theorem 3: $\mathcal{T}$ satisfies Brown representability. This means: a functor $H : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{A}b$ is representable iff

(i) H is homological.

(ii) H respects products: that is

$$H\left(\coprod_{\lambda\in\Lambda} t_\lambda\right) \quad = \quad \prod_{\lambda\in\Lambda} H(t_\lambda).$$

Suppose $\mathcal{R} \hookrightarrow \mathcal{S}$ are [TR5] triangulated categories. Assume that the inclusion $\mathcal{R} \hookrightarrow \mathcal{S}$ is triangulated, fully faithful and respects coproducts. Form the quotient $\mathcal{T} = \mathcal{S}/\mathcal{R}$. Easy to show: $\mathcal{T}$ is also a [TR5] triangulated category. We have a diagram

$$\begin{array}{ccccc} \mathcal{R}^c & & \mathcal{S}^c & & \mathcal{T}^c \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathcal{R} & \longrightarrow & \mathcal{S} & \xrightarrow[\pi]{} & \mathcal{T} \end{array}$$

Theorem 4: Let $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T} = \mathcal{S}/\mathcal{R}$ be as above. Assume further that

(i) There is an $H \subset \mathcal{S}^c$ as in Definition 3.

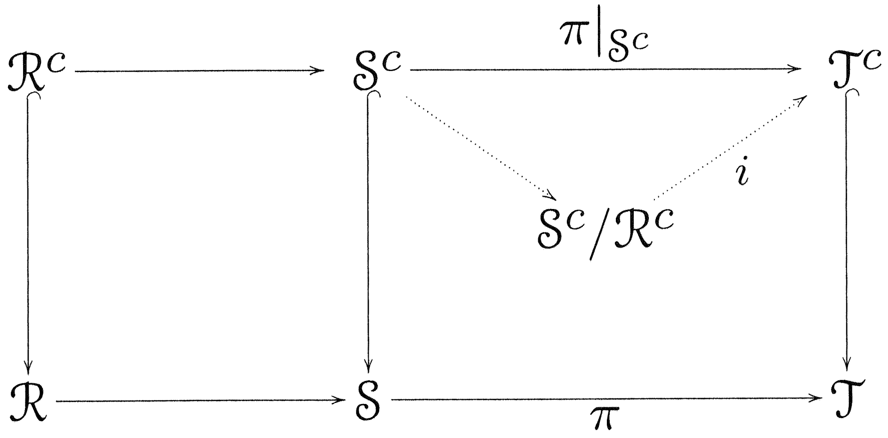
(ii) There is a $G \subset \mathcal{R} \cap \mathcal{S}^c$ which "generates" $\mathcal{R}$ as in Definition 3.

Then:

- The functors $\mathcal{R} \longrightarrow \mathcal{S}$ and $\pi : \mathcal{S} \longrightarrow \mathcal{T}$ take compacts to compacts. In other words, we get a diagram

$$
\begin{array}{ccccc}
\mathcal{R}^c & \dashrightarrow & \mathcal{S}^c & \overset{\pi|_{\mathcal{S}^c}}{\dashrightarrow} & \mathcal{T}^c \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{R} & \longrightarrow & \mathcal{S} & \underset{\pi}{\longrightarrow} & \mathcal{T}
\end{array}
$$

which factors further as



- In the diagram above, the functor i is fully faithful, and, up to splitting idempotents, every object is in the image of i. Precisely: for every object $t \in \mathcal{T}^c$ there exist objects $t' \in \mathcal{T}^c$, $s \in \mathcal{S}^c$ and an isomorphism $t \oplus t' \cong i(s)$.

Example: Let X be a noetherian, quasi-projective scheme. Put $\mathcal{T} = \mathbf{D}(\text{qc}/X)$, the (unbounded) derived category of cochain complexes of quasi-coherent sheaves on X.

$\mathcal{T}$ satisfies [TR5]; arbitrary direct sums of unbounded cochain complexes of quasi-coherent sheaves exist.

Easy to show: any bounded complex of vector bundles is compact. That is,

$$\mathbf{D}^b(\text{Vect}/X) \subset \mathcal{T}^c .$$

Let $\mathcal{L}$ be an ample line bundle on X ($\mathcal{L}$ exists because X is assumed quasi-projective). Define

$$G = \{\Sigma^n \mathcal{L}^m \mid m, n \in \mathbb{Z}\} \ .$$

Clearly $G \subset \mathbf{D}^b(\text{Vect}/X) \subset \mathcal{T}^c$. A small exercise in algebraic geometry gives that G satisfies the hypothesis of Definition 3.

Corollary 1 *Let $\mathcal{T} = \mathbf{D}(\mathrm{qc}/X)$. The subcategory $\mathcal{T}^c$, that is the full subcategory of compact objects in $\mathcal{T}$, is precisely $\mathbf{D}^b(\mathrm{Vect}/X)$.*

Proof: We know the inclusion $\mathbf{D}^b(\mathrm{Vect}/X) \subset \mathcal{T}^c$. On the other hand, $\mathbf{D}^b(\mathrm{Vect}/X)$ contains G, is triangulated, and contains any direct summand of any of its objects. Theorem 1 now tells us that $\mathbf{D}^b(\mathrm{Vect}/X) = \mathcal{T}^c$. $\square$

Corollary 2 (=Grothendieck duality). *Let $f : X \longrightarrow Y$ be a morphism of noetherian schemes, and assume X is quasi-projective. Then the map*

$$Rf_* : \mathbf{D}(\mathsf{qc}/X) \longrightarrow \mathbf{D}(\mathsf{qc}/Y)$$

has a right adjoint $f^!$.

Proof: For any objects $x \in \mathbf{D}(\mathsf{qc}/X)$ and $y \in \mathbf{D}(\mathsf{qc}/Y)$, we can consider the abelian group

$$\mathrm{Hom}_{\mathbf{D}(\mathsf{qc}/Y)}(Rf_*x, y).$$

Fix y, and view this as a functor in x. This is a homological functor $\mathbf{D}(\mathsf{qc}/X)^{\mathsf{op}} \longrightarrow \mathcal{A}b$ respecting products, hence representable by Theorem 3. Thus

$$\mathrm{Hom}_{\mathbf{D}(\mathsf{qc}/Y)}(Rf_*x, y) = \mathrm{Hom}_{\mathbf{D}(\mathsf{qc}/X)}(x, f^!y).$$

□

$$
\begin{array}{lcl}
X & = & \text{quasi-projective, noetherian scheme,} \\
U & = & \text{Zariski open subset } U \subset X, \\
Z & = & X - U.
\end{array}
$$

Let

$$
\begin{array}{rcl}
\mathcal{S} & = & \mathbf{D}(\mathsf{qc}/X) \\
\mathcal{T} & = & \mathbf{D}(\mathsf{qc}/U) \\
\pi : \mathcal{S} \longrightarrow \mathcal{T} & = & \text{restriction to } U.
\end{array}
$$

Put $\mathcal{R} = \ker(\pi)$; it is the subcategory

$$
\mathcal{R} \quad = \quad \mathbf{D}_Z(\mathsf{qc}/X) \qquad \subset \qquad \mathbf{D}(\mathsf{qc}/X)
$$

of all complexes whose restriction to U is acyclic.

Easy to show: the natural map $\mathcal{S}/\mathcal{R} \longrightarrow \mathcal{T}$ is an equivalence. Furthermore, the technical conditions of Theorem 4 are satisfied.

Theorem 4 applies. The general diagram of Theorem 4

$$
\begin{array}{ccccc}
\mathcal{R}^c & \longrightarrow & \mathcal{S}^c & \xrightarrow{\ \pi|_{\mathcal{S}^c}\ } & \mathcal{T}^c \\
 & & & \searrow \quad \nearrow_{i} & \\
 & & & \mathcal{S}^c/\mathcal{R}^c & \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{R} & \longrightarrow & \mathcal{S} & \xrightarrow[\ \pi\ ]{} & \mathcal{T}
\end{array}
$$

becomes, in our special case,

$$
\begin{array}{ccccc}
\mathbf{D}^b_Z(\mathsf{Vect}/X) & \longrightarrow & \mathbf{D}^b(\mathsf{Vect}/X) & \xrightarrow{\ \pi|_{\mathbf{D}^b(\mathsf{Vect}/X)}\ } & \mathbf{D}^b(\mathsf{Vect}/U) \\
 & & & \searrow \quad \nearrow_{i} & \\
 & & & \dfrac{\mathbf{D}^b(\mathsf{Vect}/X)}{\mathbf{D}^b_Z(\mathsf{Vect}/X)} & \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathbf{D}_Z(\mathsf{qc}/X) & \longrightarrow & \mathbf{D}(\mathsf{qc}/X) & \xrightarrow[\ \pi\ ]{} & \mathbf{D}(\mathsf{qc}/U)
\end{array}
$$

Theorem 4 says that the functor i is fully faithful, and is just an idempotent completion; every object of $D^b(\mathsf{Vect}/U)$ is a direct summand of an object in the image of i.

Interesting derived categories that are not compactly generated:

Proposition 3 *Let X be a connected, non-compact manifold of dimension ≥ 1. Let $\mathbf{D}(\mathsf{Ab}/X)$ be the derived category of all sheaves of abelian groups on X. The only compact object in $\mathbf{D}(\mathsf{Ab}/X)$ is the zero object.*

Let $\mathcal{T}$ be a [TR5] triangulated category, let α be a regular cardinal.

Definition 1': Let $\mathcal{G} \subset \mathcal{T}$ be a triangulated subcategory. $\mathcal{G}$ is α–compact if, for any

$$g \longrightarrow \coprod_{\lambda \in \Lambda} t_\lambda ,$$

there exists $\Lambda' \subset \Lambda$ with $\#\Lambda' < \alpha$ and

$$\begin{array}{ccccc} g & & \longrightarrow & & \displaystyle\coprod_{\lambda\in\Lambda} t_\lambda \\ & \searrow & & \nearrow & \\ \displaystyle\coprod_{\lambda\in\Lambda'} g_\lambda & & \xrightarrow{\;\coprod_{\lambda\in\Lambda'} f_\lambda\;} & & \displaystyle\coprod_{\lambda\in\Lambda'} t_\lambda \end{array}$$

Definition–Theorem 2': $\mathcal{T}^\alpha \subset \mathcal{T}$ is the maximal α–compact $\mathcal{G}$.

Definition 3': $\mathcal{T}$ is α–compactly generated $\Longleftrightarrow$ $\exists$ a small α–compact $\mathcal{G} \subset \mathcal{T}$ so that any non-zero object $t \in \mathcal{T}$ has a non-zero map $g \longrightarrow t$, $g \in \mathcal{G}$.

Let $\mathcal{T}$ be a [TR5] triangulated category, and let $\mathcal{G} \subset \mathcal{T}$ be as in Definition 3'.

Theorem 1': If $\mathcal{S} \subset \mathcal{T}^\alpha$ contains $\mathcal{G}$ and is closed under triangles, (retracts) and α–coproducts, then $\mathcal{S} = \mathcal{T}^\alpha$.

Theorem 2': If $\mathcal{S} \subset \mathcal{T}$ contains $\mathcal{G}$ and is closed under triangles are coproducts, then $\mathcal{S} = \mathcal{T}$.

Theorem 3': $\mathcal{T}$ satisfies Brown representability. This means: a functor $H : \mathcal{T}^{\text{op}} \longrightarrow \mathcal{A}b$ is representable iff

(i) H is homological.

(ii) H respects products: that is

$$H\left(\coprod_{\lambda\in\Lambda} t_\lambda\right) \quad = \quad \prod_{\lambda\in\Lambda} H(t_\lambda).$$

Suppose $\mathcal{R} \hookrightarrow \mathcal{S}$ are [TR5] triangulated categories. Assume that the inclusion $\mathcal{R} \hookrightarrow \mathcal{S}$ is triangulated, fully faithful and respects coproducts. Form the quotient $\mathcal{T} = \mathcal{S}/\mathcal{R}$. Easy to show: $\mathcal{T}$ is also a [TR5] triangulated category. We have a diagram

$$\begin{array}{ccccc}
\mathcal{R}^\alpha & & \mathcal{S}^\alpha & & \mathcal{T}^\alpha \\
\big\downarrow\cap & & \big\downarrow\cap & & \big\downarrow\cap \\
\mathcal{R} & \longrightarrow & \mathcal{S} & \xrightarrow[\pi]{} & \mathcal{T}
\end{array}$$

Theorem 4': Let $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T} = \mathcal{S}/\mathcal{R}$ be as above. Assume further that

(i) There is an α–compact $\mathcal{H} \subset \mathcal{S}$ as in Definition 3'.

(ii) There is a $\mathcal{G} \subset \mathcal{R} \cap \mathcal{S}^\alpha$ which is α–compact in $\mathcal{R}$ and "generates" $\mathcal{R}$ as in Definition 3'.

Then:

- The functors $\mathcal{R} \longrightarrow \mathcal{S}$ and $\pi : \mathcal{S} \longrightarrow \mathcal{T}$ take α–compacts to α–compacts. In other words, we get a diagram

$$\begin{CD}
\mathcal{R}^\alpha @>>> \mathcal{S}^\alpha @>{\pi|_{\mathcal{S}^\alpha}}>> \mathcal{T}^\alpha \\
@VVV @VVV @VVV \\
\mathcal{R} @>>> \mathcal{S} @>>{\pi}> \mathcal{T}
\end{CD}$$

which factors further as

$$
\begin{array}{ccccc}
\mathcal{R}^\alpha & \longrightarrow & \mathcal{S}^\alpha & \xrightarrow{\pi|_{\mathcal{S}^\alpha}} & \mathcal{T}^\alpha \\
& & & \searrow \quad \mathcal{S}^\alpha/\mathcal{R}^\alpha \quad \nearrow_{i} & \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{R} & \longrightarrow & \mathcal{S} & \xrightarrow[\pi]{} & \mathcal{T}
\end{array}
$$

- In the diagram above, the functor i is fully faithful, and (up to splitting idempotents) every object is in the image of i. If $\alpha > \aleph_0$ we can say more: the map i is an equivalence.

What happens as we change α?

(i) If $\alpha < \beta$ then $\mathcal{T}^\alpha \subset \mathcal{T}^\beta$.

(ii) If $\alpha < \beta$ and $\mathcal{T}$ is α–compactly generated, then it is also β–compactly generated. We call $\mathcal{T}$ WELL GENERATED if it is α–compactly generated for some α.

(iii) Assume that $\mathcal{T}$ is well generated. Then

$$\mathcal{T} = \bigcup_\alpha \mathcal{T}^\alpha .$$