QUIVER HECKE ALGEBRAS and CANONICAL BASIS

(M. VARAGNOLO)

Joint work with E. VASSEROT.

PLAN of the TALK

1. Quiver Hecke algebras (QHA)
2. Induction and projectives
3. QHA and categorification of quantum gps
4. QHA as convolution algebras.

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1. (I, H) a finite quiver

I = set of vertices

H = set of arrows

h_{ij} = n° arrows from i to j

$\rightsquigarrow$ Cartan matrix $A = (a_{ij})_{i,j \in I}$

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -h_{ij} - h_{ji} & \text{otherwise} \end{cases}$$

Fix $m > 0$ $\quad \nu = \sum_{i \in I} \nu_i \cdot i \in \mathbb{N} I$ s.t. $\sum_{i \in I} \nu_i = m$

$$I_\nu := \{ (i_1, \dots, i_m) \mid \begin{array}{l} i_k \in I \\ \sum_{k \in 1,..,m} i_k = \nu \end{array} \}$$

K any field of char. 0

$$F(\nu) = \bigoplus_{\underline{i} \in I_\nu} F_{\underline{i}} = \bigoplus_{\underline{i} \in I_\nu} K[x_{\underline{i}}(1), \dots, x_{\underline{i}}(m)] \circlearrowleft \mathfrak{S}_m$$

$$\mathfrak{S}_m \ni w \; \underbrace{f(x_{\underline{i}}(1), \ \ , x_{\underline{i}}(m))}_{\in F_{\underline{i}}} = \underbrace{f(x_{w(\underline{i})}(w(1)), \dots, x_{w(\underline{i})}(w(m)))}_{\in F_{w(\underline{i})}}$$

Khovanov and Lauda, Rouquier

$R(\nu) =$ QHA $=$ subalgebra of $\mathrm{End}_K(F(\nu))$

generated by $1_{\underline{i}}$, $x_{\underline{i}}(k)$, $\sigma_{\underline{i}}(\ell)$

$\underline{i} \in I_\nu$; $k = 1, \ , m$; $\ell = 1, \dots, m-1$

$1_{\underline{i}}$: projection $\bigoplus_{\underline{i} \in I_\nu} F_{\underline{i}} \longrightarrow F_{\underline{i}}$

$x_{\underline{i}}(k)$: 0 on $F_{\underline{i}'}$

mult. by $x_{\underline{i}}(k)$ on $F_{\underline{i}}$

$$\sigma_{\underline{i}}(\ell) = \begin{cases} 0 & \text{on } F_{\underline{i}} \text{ if } s_\ell(\underline{i})=\underline{i} \qquad f \mapsto \dfrac{\overbrace{s_\ell f - f}^{\in F_{\underline{i}}}}{x_{\underline{i}}(\ell+1)-x_{\underline{i}}(\ell)} \quad \text{(Demazure)} \\ \text{on } F_{\underline{i}} \text{ if } s_\ell(\underline{i})\neq\underline{i} = \underline{i}' & f \mapsto \underbrace{\left(x_{\underline{i}'}(\ell+1)-x_{\underline{i}'}(\ell)\right)^{h_{i_\ell, i_{\ell+1}}} s_\ell f}_{\in F_{s_\ell(\underline{i})}} \end{cases}$$

Properties of $R(\nu)$ [Kh-L]

- (PBW) $R(\nu)$ is a free $F(\nu)$ mod. of rk $m!$
- $Z(R(\nu)) = F(\nu)^{\mathfrak{S}_m} \simeq K[x_1, \ldots, x_m]^{\mathfrak{S}_\nu}$ $\quad \mathfrak{S}_\nu := \prod_i \mathfrak{S}_{\nu_i}$
- $R(\nu)$ is free over its center of rank $(m!)^2$
- $R(\nu)$ is $\mathbb{Z}$ graded

$$\deg 1_{\underline{i}} = 0 \qquad \deg x_{\underline{i}}(k) = 2 \qquad \deg \sigma_{\underline{i}}(\ell) = \begin{cases} -2 & s_\ell(\underline{i})=\underline{i} \\ -a_{i_\ell i_{\ell+1}} & s_\ell(\underline{i})\neq\underline{i} \end{cases}$$

Fond. exemple

$I = \{i\}$ $\quad \nu = mi \quad I_\nu = \{(i,\dots,i)\} \quad F(\nu) = K[x_1, \dots, x_m]$

$R(mi)$ = subalg of $\mathrm{End}(K[x_1,\dots,x_m])$ generated by

mult. by x_k (x_k) and Demazure op. (σ_e)

$\rightsquigarrow$ nil affine Hecke algebras

a presentation is

$$\sigma_e^2 = 0 \qquad \sigma_e \sigma_{e+1} \sigma_e = \sigma_{e+1} \sigma_e \sigma_{e+1} \qquad \sigma_e \sigma_k = \sigma_k \sigma_e \text{ if } |k-e| > 1$$

$$[x_k, x_r] = 0 \qquad [\sigma_e, x_k] = 0 \text{ if } |k-e| > 1$$

$$\sigma_e x_e - x_{e+1} \sigma_e = 1$$

Using the basis of Schubert polynomials

$$\rightsquigarrow R(mi) = \mathrm{End}_{K[x_1,\dots,x_m]^{\mathfrak{S}_m}}(K[x_1,\dots,x_m])$$

Note: Kunneth formula $\Rightarrow$

$$R(mi) \simeq \left(H_*^{Gl_m}(\mathcal{F} \times \mathcal{F}), \text{conv. prod.}\right)$$

We will see after that we can always realize the QHA as a convolution algebra.

Note that $\sigma_{w_0}(x_1^{m-1} \cdots x_{m-1}) = 1$

$\uparrow$ longest element of σ_m

$\Rightarrow \quad e_{i,m} := \sigma_{w_0} x_1^{m-1} \cdots x_{m-1}$ is an idempotent

(2) We have an embedding of algebras

$$R(\nu) \otimes R(\nu') \hookrightarrow R(\nu+\nu')$$

$$1_{\underline{i}} \otimes 1_{\underline{i}'} \mapsto 1_{\underline{i}\underline{i}'}$$

$$x_{\underline{i}}(k) \otimes 1_{\underline{i}'} \mapsto x_{\underline{i}\underline{i}'}(k)$$

$$1_{\underline{i}} \otimes x_{\underline{i}}(k') \mapsto x_{\underline{i}\underline{i}'}(m+k)$$

similar for $\sigma_{\underline{i}}(e)$

note $\quad 1 \otimes 1 \mapsto 1_{\nu,\nu'}$ an idempotent

$\rightsquigarrow$ induction functor

f.g + graded

$$R(\nu)\text{-mod} \times R(\nu')\text{-mod} \to R(\nu+\nu')\text{-mod}$$

$$M \otimes M' \longmapsto R(\nu+\nu')1_{\nu,\nu'} \otimes_{R(\nu)\otimes R(\nu')} M \otimes M'$$

which sends projectives to projectives.

$$\Rightarrow \quad K(R) = \bigoplus_{\nu \in \mathbb{N}I} K_0(R(\nu))$$

$\leftarrow$ Grothendieck ring of $R(\nu)$-proj $\leftarrow$ f.s. graded, proj is a ring

NB! $\mathcal{A} = \mathbb{Z}[q, q^{-1}] \curvearrowright K(R)$

q acts by shift of degree.

Define $P_{\underline{i}} = R(\nu) 1_{\underline{i}}$ $\quad P_{\underline{i}} \in R(\nu)\text{-proj}$

Consider $Y_\nu = \{ (i_1^{(a_1)}, \dots, i_r^{(a_r)}) \mid \sum_k a_k i_k = \nu \}$

note $I_\nu \subseteq Y_\nu$ $\quad$ exemple $\quad i \longrightarrow j$ $\quad \nu = 2i + j$

$I_\nu = \{ (iij), (iji), (jii) \}$ and

$Y_\nu = I_\nu \cup \{ (i^{(2)}, j), (j, i^{(2)}) \}$

$\forall \underline{y} \in Y_\nu$ $\quad$ define $\quad P_{\underline{y}} = R(\nu) 1_{\underline{y}} \overset{\text{shift}}{\left[\sum \frac{a_i(a_i - 1)}{2} \right]}$

where $1_{\underline{y}}$ is defined by

$$R(a_1 i_1) \otimes \cdots \otimes R(a_r i_r) \hookrightarrow R(\nu)$$

$$e_{i_1, a_1} \otimes \cdots \otimes e_{i_r, a_r} \longmapsto 1_{\underline{y}}$$

Then $[P_{\underline{y}}] \in K_0(R(\nu))$

and in $K(R)$ we have

$$[P_{\underline{y}}][P_{\underline{y}'}] = [P_{\underline{y}\underline{y}'}]$$

(3)

$$(I, H) \rightsquigarrow f = \frac{\text{free ass } K(q) \text{ alg. generated by } \theta_i \ (i \in I)}{q\text{-Serre}}$$

${}_{\mathcal{A}}f$ = $\mathcal{A}$-integer form generated by q-divided powers $\theta_i^{(a)}$

$$\underline{y} \in Y_\nu \Rightarrow \theta_{\underline{y}} := \theta_{i_1}^{(a_1)} \cdots \theta_{i_r}^{(a_r)}$$

Theorem (Khovanov and Lauda)

There exists an algebra isomorphism

$$\gamma : {}_{\mathcal{A}}f \longrightarrow K(R)$$

$$\theta_{\underline{y}} \longmapsto [P_{\underline{y}}]$$

□

Conj (Kh-L)

$\gamma(\{q^d b \mid b \in \mathcal{B} = \underset{\text{basis of } {}_{q}f}{\text{Lusztig canonical}}\}) = \mathbb{Z}$ basis of ind. proj.

Main result : proof of this conjecture.

Need the geometric realization of ${}_{\mathcal{A}}f$ given by Lusztig

(I, H) $\rightsquigarrow$ $V = \bigoplus_{i \in I} V_i$ graded vect space $/ \mathbb{C}$

ν of graded dim ν

$$G_V = \prod_{i \in I} GL(V_i)$$

$$E_V = \bigoplus_{h \in H} \mathrm{Hom}(V_{h'}, V_{h''}) \qquad h' \xrightarrow{h} h''$$

$\forall\, \underline{y} \in Y_\nu$ 2 smooth varieties

$\mathcal{F}_{\underline{y}}$ = flags of type $\underline{y}$ in V $= \{ \phi = V^0 \subset V^1 \subset \cdots \subset V^m = V \mid \dim V^e / V^{e-1} = a_e i_e \}$

$\widetilde{\mathcal{F}}_{\underline{y}} = \{ (x, \phi) \mid \phi \in \mathcal{F}_{\underline{y}},\ x \in E_V$ ϕ is x-stable $\}$

$$\pi_{\underline{y}} : \widetilde{\mathcal{F}}_{\underline{y}} \to E_V$$

$$(\phi, x) \mapsto x$$

$L_{\underline{y}} \in D_{G_V}(E_V)$ bound. derived category of complexes of G_V eq. sheaves of k-vect. spaces on E_V

$\mathcal{L}_{\underline{y}} := \pi_{\underline{y}!}(k)[\dim \tilde{\mathcal{F}}_{\underline{y}}]$

$\mathcal{P}_V$ = simple perverse sheaves in $D_{G_V}(E_V)$ $\mathcal{L}$ s.t. $\exists\, r \in \mathbb{Z}$

$\exists\, \underline{i} \in I_\nu$ and $\mathcal{L}[r] \leq_{\oplus} \mathcal{L}_{\underline{i}}$

Q_V = full subcategory of $D_{G_V}(E_V)$ whose objects

are finite direct sums of shift of elem. in $\mathcal{P}_V$.

$K(Q_V)$ = ab. gp with a generator for each isoclass of el. of Q_V and $[\mathcal{L}]+[\mathcal{L}']=[\mathcal{L}'']$

if $\mathcal{L}\oplus\mathcal{L}' \simeq \mathcal{L}''$

$K(Q_V)$ depends only on ν

$$K(Q) := \bigoplus_{\nu \in \mathbb{N}I} K(Q_\nu)$$

It is a ring with respect to the convolution product

defined by Lusztig

For this product we have $[\mathcal{L}_{\underline{y}}][\mathcal{L}_{\underline{y}'}] = [\mathcal{L}_{\underline{y}\underline{y}'}]$

<u>Theorem</u> (Lusztig)

$\exists$ algebra isomorphism $\lambda : K(Q) \to {}_{\mathcal{A}}f$

$[\mathcal{L}_{\underline{y}}] \mapsto \theta_{\underline{y}}$

Moreover $\mathcal{B} = \{ \lambda([\mathcal{L}]) \mid \mathcal{L} \in \mathcal{P}_V \}$
$\shortparallel$
canonical basis

$\square$

So now we have the following situation

$\gamma : {}_{\mathcal{A}}f \longrightarrow K(R)$

$\theta_{\underline{y}} \mapsto [P_{\underline{y}}]$

$\lambda : K(Q) \longrightarrow {}_{\mathcal{A}}f$

$[\mathcal{L}_{\underline{y}}] \mapsto \theta_{\underline{y}}$

$[\mathcal{L}] \mapsto b_{\mathcal{L}}$

$\cap$
$\mathcal{P}_V$

Pbm : Link
between $K(R)$ and $K(Q)$

Suppose you have a functor

$\Upsilon : Q_V \to R(\nu)\text{-proj}$

$\mathcal{L}_{\underline{y}} \mapsto \Upsilon\mathcal{L}_{\underline{y}} = P_{\underline{y}}$ in $K(R)$

$\mathcal{P}_V \ni \mathcal{L} \mapsto \Upsilon_{\mathcal{L}}$ indecomposable (proj)

Then we are done.

$$\overset{\cup}{\mathcal{L}} = \mathrm{Ext}^*_{D_{G_V}(E_V)}(\mathcal{L}_V, \mathcal{L}) = \text{an } \mathrm{Ext}^*_{D_{G_V}(E_V)}(\mathcal{L}_V, \mathcal{L}_V)$$

$$\mathrm{Ext}^*_{D_{G_V}(E_V)}(\mathcal{L}_V, \mathcal{L}) = \bigoplus_{\underline{i} \in I_\nu} \mathcal{L}_{\underline{i}}$$

module (via Yoneda product)

'Standard' (Ginzburg's book in the non equivariant case):

$$\mathrm{Ext}^*_{D_{G_V}(E_V)}(\mathcal{L}_V, \mathcal{L}_V) \simeq \left(H_*^{G_V}(Z_V, k), \text{ conv. product}\right)$$

where $Z_V = \tilde{\mathcal{F}}_V \times_{E_V} \tilde{\mathcal{F}}_V = \bigsqcup_{\underline{i}, \underline{i}'} \tilde{\mathcal{F}}_{\underline{i}} \times_{E_V} \tilde{\mathcal{F}}_{\underline{i}'}$ is

a Steinberg type variety.

(4)

<u>Thm</u> $H_*^{G_V}(Z_V, k) \simeq R(\nu)$

Strategy to prove this result:

- find candidates for generators $1_{\underline{i}}$, $x_{\underline{i}}(k)$, $\sigma_{\underline{i}}(\ell)$ of $R(\nu)$ in $H_*^{G_V}(Z_V, k)$

- notice that there is a natural faithful rep.

$$H_*^{G_V}(Z_V, k) \curvearrowright H_*^{G_V}(\tilde{\mathcal{F}}_V, k) \simeq F(\nu)$$

since $\forall \underline{i} \in I_\nu$, $\mathcal{F}_{\underline{i}} \simeq \prod_{i \in I} \underbrace{\mathcal{F}(V_i)}$

flag variety of $GL(V_i)$

- there are only finitely many fixed points for the action of a fixed Torus $T \subset G_V$. Using the localization thm we can compute this faithful rep. in the basis of fixed points

- write 'generators' in the basis of T-fixed points

$\rightsquigarrow$ same action $\Rightarrow R(\nu) \overset{\varphi}{\hookrightarrow} H_*^{G_V}(Z_V, k)$

- use another argument for surjectivity:

$$\varphi\Big(\bigoplus_{x \leq w} F(\nu)\sigma(x)\Big) = H_*^{G_V}\Big(\bigcup_{x \leq w} Z_V^x, k\Big)$$

$\sigma(x)$: PBW basis

Z_V^x: some closed sub. of Z_V

$x, w \in \mathfrak{S}_m$

$Z_V = \bigcup_x Z_V^x$ defined later

(use induction on the length of w).

Let me just explain which are the elements in

$H_*^{G_V}(Z_V, k)$ which correspond to generators for

$R(\nu)$.

First an analogy. Case of H = degenerate affine Hecke alg. of Type A

$$H \underset{\text{vec. sp.}}{\simeq} k\mathfrak{S}_m \otimes k[x_1, \dots, x_m]$$

$$+ \quad s_i x_i - x_{i+1} s_i = 1$$

$\mathcal{F}$ = flag variety of GL_m

$T^*\mathcal{F}$, $\quad Z = T^*\mathcal{F} \times_{\mathcal{N}} T^*\mathcal{F} = \bigsqcup Z_{\mathcal{O}}$

$\mathcal{O} \subseteq \mathcal{F}\times\mathcal{F}$ G-orbits, $G = GL_m$; $Z_{\mathcal{O}}$ conormal to the orbit $\mathcal{O}$

different people (e.g. Lusztig) give an alg. iso

$$H \simeq H_*^G(Z, \mathbb{C}), \text{conv.}$$

$$x_k \mapsto c_1(\mathcal{O}_{Z_\Delta}(k))$$

pull back of the k-th tautl. bundle on $\mathcal{F}$ via $Z_\Delta \to \tilde{\mathcal{F}} \to \mathcal{F}$

$$s_i \mapsto [\overline{Z}_{\mathcal{O}_{s_i}}]$$

orbit of $(1, s_i)$

In our case $\mathcal{F} = \mathcal{F}(V = \bigoplus_{i \in I} V_i)$

$G = GL(V) \supset G_V$ as a levi

We have
$$\mathcal{F}_V = \mathcal{F}^{Z(G_V)}$$
$$\mathcal{F}_V = \bigcup_i \mathcal{F}_i$$

the connected components of $\mathcal{F}_V$ are parametrized by $I_\nu \simeq \mathfrak{S}_m / \mathfrak{S}_\nu$

$\mathfrak{S}_m \simeq$ Weyl gp of (G, T)
$\mathfrak{S}_\nu =$ Weyl gp of (G_V, T) for some fixed torus T

G-orbit in $\mathcal{F} \times \mathcal{F}$ are parametrized by $\mathfrak{S}_m : \mathcal{O}^w$

$$\mathcal{O}^w_V = \mathcal{O}^w \cap (\mathcal{F}_V \times \mathcal{F}_V)$$

$$q : Z_V \to \mathcal{F}_V \times \mathcal{F}_V$$
$$(x, \phi, \phi') \mapsto (\phi, \phi')$$

$$Z_V^w = \overline{q^{-1}(\mathcal{O}_V^w)} \qquad Z = \bigcup_w Z_V^w$$

Then

$$1_{\underline{i}} \mapsto [\underbrace{\overline{Z}^e_v \cap Z_{\underline{i}\,\underline{i}}}_{Z^e_{\underline{i},\underline{i}}}]$$

$$y_{i\underline{i}}(k) \mapsto c_1\left(\mathcal{O}_{Z^e_{\underline{i},\underline{i}}}(k)\right)$$

pull back via $Z^e_{\underline{i}\,\underline{i}} \to \tilde{\mathcal{F}}_{\underline{i}} \to \mathcal{F}_{\underline{i}}$ (o)

$$\sigma_{\underline{i}}(\ell) \mapsto [Z^{s_e}_{s_\ell(\underline{i}),\underline{i}}] = Z_v^{s_e} \cap Z_{s_\ell(\underline{i}),\underline{i}}$$

(o) nb! $\tilde{\mathcal{F}}_i$ smooth $\Rightarrow$ coh "=" hom