Finite Volume/DG Schemes Based on Constrained Minimization Function Recovery

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London Mathematical Society Durham Symposium

Computational Linear Algebra for Partial Differential Equations

JULY 15, 2008

† Work performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344.

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The Equations of Gas Dynamics

The equation of gas dynamics in Eulerian coordinates: ε

$$\begin{aligned}
\text{conservation of mass}: \quad & \frac{\partial \varrho}{\partial t} &&= -\mathbf{div}\,(\varrho\,\mathbf{v}),\\
\text{conservation of momentum}: \quad & \frac{\partial (\varrho\,\mathbf{v})}{\partial t} &&= -\nabla p - \sum_{j=1}^{d} \frac{\partial(\varrho v_j\,\mathbf{v})}{\partial x_j},\\
\text{conservation of energy}: \quad & \frac{\partial (\varrho\,E)}{\partial t} &&= -\mathbf{div}\,\left((\varrho\,E + p)\mathbf{v}\right).
\end{aligned}$$

$$E = \varepsilon + \frac{1}{2}\,|\mathbf{v}|^2 \qquad = \text{ internal energy } + \text{ kinetic energy.}$$

Equation of state:

$$p = p(\varrho,\ \varepsilon) = \mathrm{EOS}(\varrho,\ \varepsilon),$$

For example (polytropic ideal gas):

$$p = (\gamma - 1)\ \varrho\varepsilon,\ \text{ for a } \gamma > 1.$$

Lagrangian Coordinates

Let

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x},\ t), \quad \mathbf{x} = (x_i) \in \mathbb{R}^d,\ d = 2 \text{ or } 3,$$

with initial condition

$$\mathbf{x}|_{t=0} = \boldsymbol{\xi}.$$

By definition, the pair $(\boldsymbol{\xi},\ t)$ is called Lagrangian coordinates associated with the velocity field $\mathbf{v}$. Let

$$J(\boldsymbol{\xi},\ t) = \det \left(\frac{\partial x_i(\boldsymbol{\xi},\ t)}{\partial \xi_j} \right).$$

Define, $\overline{\varphi}(\boldsymbol{\xi},\ t) = \varphi(\mathbf{x}(\boldsymbol{\xi},\ t),\ t)$. Then

$$\frac{\partial\left(\overline{\varphi} J\right)}{\partial t} = J \left(\overline{\frac{\partial \varphi}{\partial t}} + \overline{\mathbf{div}\ \ (\varphi\ \mathbf{v})} \right).$$

Integral Form of Equations

Conservation of mass:

$$\frac{\partial}{\partial t} \int\limits_{V(t)} \varrho \, d\,\mathbf{x} = 0.$$

Conservation of momentum:

$$\frac{\partial}{\partial t} \int\limits_{V(t)} \varrho \mathbf{v} \, d\,\mathbf{x} = - \int\limits_{V(t)} \nabla p \, d\,\mathbf{x}.$$

Conservation of total energy:

$$\frac{\partial}{\partial t} \int\limits_{V(t)} \varrho \, E \, d\mathbf{x} = - \int\limits_{V(t)} \mathbf{div} \; (p \, \mathbf{v}) \, d\mathbf{x}.$$

Conservation of Momentum: a General Integral Form

Let $\mathbf{x}^{\underline{\alpha}} = x_1^{\alpha_1} \ldots x_d^{\alpha_d}$. Then

$$\frac{\partial}{\partial t} \int\limits_{V(t)} \varrho\, \mathbf{x}^{\underline{\alpha}} \mathbf{v}\, d\,\mathbf{x} = - \int\limits_{V(t)} \mathbf{x}^{\underline{\alpha}}\, \nabla p\, d\,\mathbf{x}.$$

Proof:

$$\begin{aligned}
\frac{\partial}{\partial t} \int\limits_{V(t)} \mathbf{x}^{\underline{\alpha}}\, \varrho \mathbf{v}\, d\,\mathbf{x} &= \frac{\partial}{\partial t} \int\limits_{V(0)} \boldsymbol{\xi}^{\underline{\alpha}}\, \overline{\varrho \mathbf{v}}\, J\, d\,\boldsymbol{\xi} \\
&= \int\limits_{V(0)} \boldsymbol{\xi}^{\underline{\alpha}}\, \frac{\partial\, \overline{\varrho \mathbf{v}}\, J}{\partial t}\, d\,\boldsymbol{\xi} \\
&= \int\limits_{V(t)} \mathbf{x}^{\underline{\alpha}} \left(\sum_{j=1}^{d} \frac{\partial \varrho v_j\, \mathbf{v}}{\partial x_j} + \frac{\partial \varrho \mathbf{v}}{\partial t} \right) d\,\mathbf{x} \\
&= - \int\limits_{V(t)} \mathbf{x}^{\underline{\alpha}}\, \nabla p\, d\,\mathbf{x}.
\end{aligned}$$

Conservative Finite Volume Schemes

The integral form is the basis for constructing conservative finite volume schemes and also for DG schemes (for $\underline{\alpha} \neq 0$).

Consider conservation of momentum equation

$$\frac{\partial}{\partial t} \int\limits_{V(t)} \varrho \mathbf{v} \, d\,\mathbf{x} = - \int\limits_{V(t)} \nabla p \, d\,\mathbf{x}.$$

Use time discretization $t_{n+1} = t_n + \Delta t$ and let $V_n = V(t_n)$.

We have, (for an explicit scheme)

$$\frac{1}{\Delta t} \left(\int\limits_{V_{n+1}} \varrho \mathbf{v} \, d\,\mathbf{x} - \int\limits_{V_n} \varrho \mathbf{v} \, d\,\mathbf{x} \right) = - \int\limits_{V_n} \nabla p_h \, d\,\mathbf{x}.$$

Here, p_h is a finite element approximation of p (to be specified).

Conservative Finite Volume Schemes

Let $m(V) = \int_V \varrho \, d\mathbf{x}$ be the mass associated with a volume V. From the conservation of mass equation

$$\frac{\partial}{\partial t} \int_{V(t)} \varrho \, d\,\mathbf{x} = 0,$$

we have that the mass is constant, i.e.,

$$m(V_n) = \int_{V(t_n)} \varrho \, d\,\mathbf{x} = \int_{V(t_{n+1})} \varrho \, d\,\mathbf{x} = m(V_{n+1}).$$

We approximate ϱ at $t = t_n$ with discontinuous piecewise constants:

$$\varrho_n = m(V_n)/|V_n| = m(V)/|V_n|.$$

A FV Conservation of Momentum Equation

Then the following FV scheme is straightforward:

$$m(V)\,\frac{1}{\Delta t}\,\left(\frac{1}{|V_{n+1}|}\int\limits_{V_{n+1}} \mathbf{v}_{n+1}\;d\,\mathbf{x} - \frac{1}{|V_n|}\int\limits_{V_n} \mathbf{v}_n\;d\,\mathbf{x}\right) = -\int\limits_{V_n} \nabla p_h\;d\,\mathbf{x}.$$

It is clear that we can compute the average values

$$\frac{1}{|V_{n+1}|}\int\limits_{V_{n+1}} \mathbf{v}_{n+1}\;d\,\mathbf{x} = \frac{1}{m(V)}\int\limits_{V_{n+1}} \varrho\;\mathbf{v}_{n+1}\;d\mathbf{x},$$

without knowing the actual approximation $\mathbf{v}_h$ to $\mathbf{v}_{n+1}$.

Smooth Function Recovery From Averages

Thus the problem of function recovery arises:
Given the (weighted) average values

$$\frac{1}{m(V)} \int\limits_{V_{n+1}} \varrho \mathbf{v}_{n+1} \, d\, \mathbf{x},$$

construct a smooth function $\mathbf{v}_h$ (that has the prescribed averages) to be used in the approximation of conservation of energy equation

$$\frac{1}{\Delta t} \left(\int\limits_{V_{n+1}} \varrho\, \varepsilon \, d\mathbf{x} - \int\limits_{V_n} \varrho\, \varepsilon \, d\mathbf{x} \right) = - \int\limits_{V_{n+1}} p_h \, \mathbf{div} \;\; \mathbf{v}_h \, d\mathbf{x}.$$

We formulate one function recovery procedure based on minimizing certain energy functional subject to some constraints.

Constrained Total Variation (TV) Function Recovery

We need a second finite element mesh $\mathcal{T}_h$,
a refinement of the primal (FV or finite element) mesh $\mathcal{T}_H$.

The accuracy of the scheme is determined by $\mathcal{T}_H$.

The TV function recovery reads:

Find a finite element function $\mathbf{v}_h$ with minimal total variation

$$\mathbf{J}_{TV}(\mathbf{v}_h) = \int_\Omega |\nabla \mathbf{v}_h| \, d\mathbf{x} \mapsto \min,$$

with prescribed integral moments for all $V = V_{n+1} \in \mathcal{T}_H$

$$\int_V \varrho \, \mathbf{v}_h \, d\mathbf{x}$$

Constrained Total Variation (TV) Function Recovery

Consider now the conservation of energy equation (for V as an union of elements from $\mathcal{T}_h$):

$$\int\limits_{V_{n+1}} \varrho_{n+1} E_{n+1}\, d\mathbf{x} = \int\limits_{V_n} \varrho_n E_n\, d\mathbf{x}_n - \Delta t \int\limits_{\partial V_{n+1}} p_h \mathbf{v}_h \cdot \mathbf{n}\, d\sigma.$$

From physical consideration (nonnegative internal energy), splitting $E = \varepsilon + \frac{1}{2}|\mathbf{v}|^2$, gives

$$0 \le \left(\int\limits_{V_{n+1}} \varrho\varepsilon\, d\mathbf{x} = \right) \int\limits_{V_n} \varrho_n E_n\, d\mathbf{x}_n - \Delta t \int\limits_{\partial V_{n+1}} p_h \mathbf{v}_h \cdot \mathbf{n}\, d\sigma - \frac{1}{2} \int\limits_{V_{n+1}} \varrho_{n+1} |\mathbf{v}_h|^2\, d\mathbf{x}.$$

This is a quadratic inequality constraint for $\mathbf{v}_h = \mathbf{v}_{n+1}$ imposed on any $V = V_{n+1} \in \mathcal{T}_H$ (viewed as a fine-grid, $\mathcal{T}_h$, domain), if ϱ_{n+1} and p_h are considered given.

Constrained Energy Minimization Function Recovery

Similar problem can be formulated for p_h. Find a finite element function p_h such that

$$J_{TV}(p_h) = \int\limits_{\Omega} |\nabla p_h| \, d\mathbf{x} \mapsto \min,$$

subject to the equality constraints (for all $V \in T_H$) using the E.O.S.:

$$\frac{1}{|V|} \int\limits_{V} p_h \, d\mathbf{x} = \overline{p} \equiv \frac{\gamma - 1}{|V|} \int\limits_{V} \varrho\varepsilon \, d\mathbf{x}.$$

Note that the quadratic inequality for $\mathbf{v}_h$ implies that $\overline{p} \geq 0$.

Function Recovery as Regularized "Interpolation"

The equality constraints can be imposed (approximately) via the Rudin-Osher-Fatemi noise removal functional (Physica D, 1992):

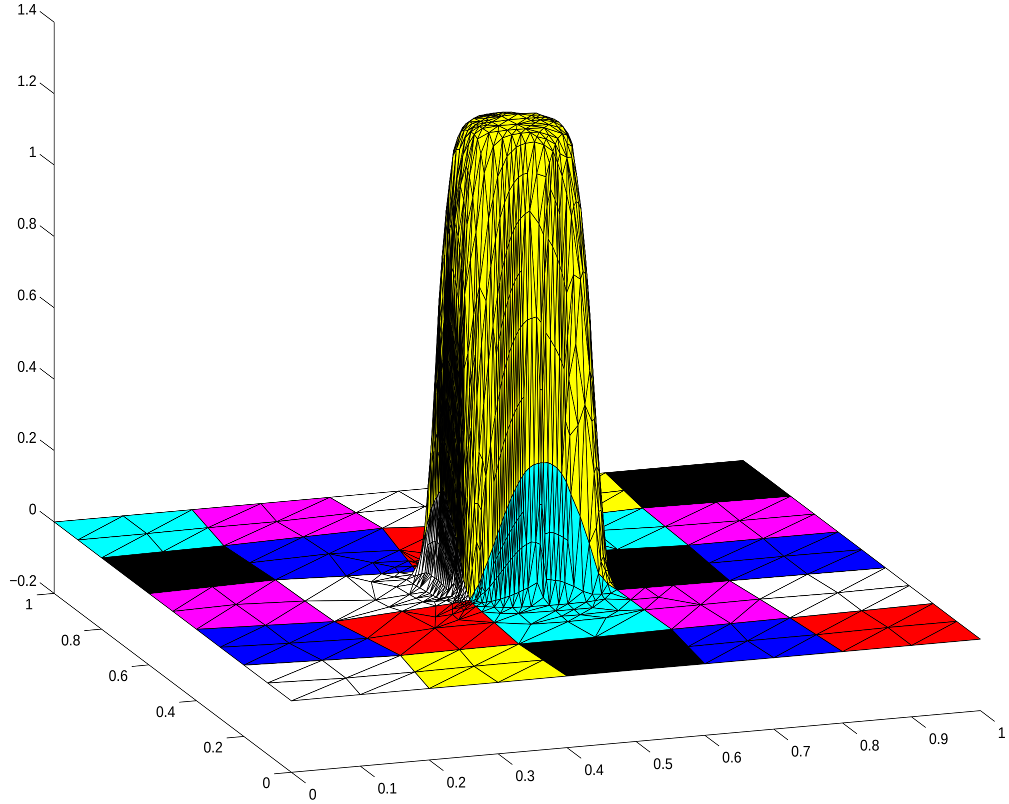
$$J_{ROF}(p_h) = \|p_h - \overline{p}\|_0^2 + \epsilon \int_{\Omega} |\nabla p_h| \, d\mathbf{x} \mapsto \min\,.$$

The purpose of the recovery procedure is to construct a smooth function (with prescribed averages) so that its derivatives ($\mathbf{grad}$ and/or $\mathbf{div}$) can be used to close-up the overall FV/DG scheme. That is,

- we first have a sort of "interpolation" procedure (from averages construct a function), and then
- perform "numerical differentiation" (use $\mathbf{grad}$ or $\mathbf{div}$).

This is an ill–posed problem. Hence the need of regularization, which is provided by the TV-functional.

Non-oscillatory TV Function Recovery



TV recovery of a piecewise constant function.

The Overall FV Scheme

We have a primal (moving) mesh $\mathcal{T}_H$. In the recovery procedures, we need a dynamically constructed mesh $\mathcal{T}_h$ that is a refinement of $\mathcal{T}_H$.

Algorithm 1 (Conservative FV scheme)

- *To move the mesh, find a finite element function* $\mathbf{x}_h$ *such that*

$$\|\mathbf{x}_h - (\mathbf{x}_n + \Delta t\ \mathbf{v}_n)\|_0^2 + \epsilon \int_{\Omega_n} |\nabla \mathbf{x}_h| \mapsto \min .$$

Then, $\mathbf{x}_{n+1}$ *equals* $\mathbf{x}_h$ *restricted to the vertices of* $\mathcal{T}_H$ *(at* $t = t_n$*) and defines the vertices of the moved* $\mathcal{T}_H$ *at time* $t = t_{n+1}$*. Thus, we can compute the volumes* $|V|$ *for any* $V = V_{n+1} \in \mathcal{T}_H$*. We can then compute*

$$\varrho_{n+1} = \frac{m(V)}{|V_{n+1}|}, \qquad \overline{\mathbf{v}}_{n+1} = \frac{1}{m(V)} \left[\int_{V_n} \varrho_n \mathbf{v}_n \, d\mathbf{x}_n - \Delta t \int_{V_{n+1}} \nabla p_h \, d\mathbf{x}_{n+1} \right].$$

The Overall Scheme

- Solve the constrained energy minimization problems for $\mathbf{v}_h \in \mathbf{S}_h$ and $p_h \in S_h$ (vector and scalar H^1–conforming finite element spaces):

$$\mathbf{J}_{ROF}(\mathbf{v}_h) = \|\mathbf{v}_h - \overline{\mathbf{v}}_{n+1}\|^2_{0,\,\varrho_{n+1}} + \epsilon \int\limits_{\Omega_{n+1}} |\nabla \mathbf{v}_h| \, d\mathbf{x}_{n+1} \mapsto \min,$$

$$J_{ROF}(p_h) = \|p_h - \overline{p}_{n+1}\|^2_0 + \epsilon \int\limits_{\Omega_{n+1}} |\nabla p_h| \, d\mathbf{x}_{n+1} \mapsto \min,$$

subject to the quadratic inequality constraints for any $V = V_{n+1} \in \mathcal{T}_H$

$$-\frac{1}{2} \int\limits_{V_{n+1}} \varrho_{n+1} \, |\mathbf{v}_h|^2 \, d\mathbf{x} - \Delta t \int\limits_{\partial V_{n+1}} p_h \mathbf{v}_h \cdot \mathbf{n} \, d\sigma + \int\limits_{V_n} \varrho E_n \, d\mathbf{x}_n \geq 0.$$

- From the E.O.S., compute $\overline{p}_{n+1} = \frac{\gamma-1}{|V_{n+1}|} \int\limits_{V_{n+1}} \varrho_{n+1} \varepsilon_{n+1} \, d\mathbf{x}_{n+1} =$

$$\frac{\gamma-1}{|V_{n+1}|} \left[\int\limits_{V_n} \varrho_n E_n \, d\mathbf{x}_n - \Delta t \int\limits_{\partial V_{n+1}} p_h \mathbf{v}_h \cdot \mathbf{n} \, d\sigma - \frac{1}{2} \int\limits_{V_{n+1}} \varrho_{n+1} \, |\mathbf{v}_h|^2 \, d\mathbf{x} \right] \geq 0.$$

Computational Issues

The nonlinear TV functional is non-elliptic. In practice, we approximate it with a nonlinear elliptic one:

$$|\nabla p_h| \approx \begin{cases} \frac{1}{|\nabla p_h|}|\nabla p_h|^2, & \text{if } |\nabla p_h| \geq \delta, \\ \frac{1}{\delta}|\nabla p_h|^2, & \text{if } |\nabla p_h| < \delta, \end{cases} = g_\delta(|\nabla p_h|)\, |\nabla p_h|^2,$$

for a mesh–dependent tolerance δ. The approximation to the ROF functional gives rise to a quadratic (matrix–vector) functional

$$\mathcal{J}(\mathbf{v}) \equiv \frac{1}{2}\mathbf{v}^T(M + \epsilon A(\mathbf{v}))\mathbf{v} - \mathbf{v}^T\mathbf{b} \mapsto \min .$$

M is the mass–matrix and A comes from the non–linear elliptic form

$$a(u,\ \varphi) = \int_\Omega g_\delta(|\nabla u|)\ \nabla u \cdot \nabla \varphi \, d\mathbf{x}.$$

Computational Issues

The overall minimization procedure is based on monotone Gauss–Seidel iterations within Picard linearization. That is, for a current iterate $\mathbf{v}$ we perform a loop over all indices i. At every step i, based on the unit coordinate vector $\mathbf{e}_i$, we solve 1D quadratic minimization problem:

$$\mathcal{J}(\mathbf{v} + t\mathbf{e}_i) \mapsto \min, \; t \in \mathbb{R},$$

subject to the quadratic inequality constraints. The set of constraints provides a set of intervals where $t \in \mathbb{R}$ can vary. All the intervals contain the origin. Thus the intersection of all intervals is non–empty.

In summary, each 1D minimization step involves finding minimum of a (scalar) quadratic functional over a (scalar) interval. This ensures the monotonicity of the process.
One monotone Gauss–Seidel loop is completed after all indices i are visited.

Entropy

Introducing the fluxes

$$\mathbf{f}_i = p \begin{bmatrix} 0 \\ \mathbf{e}_i \\ 0 \end{bmatrix} + v_i \begin{bmatrix} \varrho \\ \varrho \mathbf{v} \\ \varrho E + p \end{bmatrix}, \quad \mathbf{e}_i \in \mathbb{R}^d,$$

the original Euler equations take the vector form

$$\frac{\partial \widehat{\boldsymbol{\eta}}}{\partial t} + \sum_{j=1}^{d} \frac{\partial \mathbf{f}_j}{\partial x_j} = 0.$$

Here $\widehat{\boldsymbol{\eta}} = (\eta_k)_{k=0}^{d+1}$ is the vector of the conserved variables:

$$\eta_0 = \varrho, \boldsymbol{\eta} = (\eta_k)_{k=1}^{d} = \varrho \, \mathbf{v}, \text{ and } \eta_{d+1} = \varrho E,$$

Entropy

The E.O.S. gives

$$p = (\gamma - 1)\varrho(E - \frac{1}{2}|\mathbf{v}|^2) = (\gamma - 1)(\eta_{d+1} - \frac{1}{2}|\boldsymbol{\eta}|^2/\eta_0).$$

Thus, in terms of the conserved variables (η_k)

$$\mathbf{f}_i = (\gamma - 1)(\eta_{d+1} - \frac{1}{2}|\boldsymbol{\eta}|^2/\eta_0) \begin{bmatrix} 0 \\ \mathbf{e}_i \\ 0 \end{bmatrix} + \begin{bmatrix} \eta_i \\ \frac{\eta_i}{\eta_0}\boldsymbol{\eta} \\ \frac{\eta_i}{\eta_0}\left(\gamma\eta_{d+1} - \frac{\gamma-1}{2}\frac{|\boldsymbol{\eta}|^2}{\eta_0}\right) \end{bmatrix}.$$

Entropy

The entropy function is

$$U = U(\widehat{\boldsymbol{\eta}}) = U(\eta_0, \boldsymbol{\eta},\ \eta_{d+1}) \quad = -\varrho \log\left(\frac{\varepsilon}{\varrho^{\gamma-1}}\right)$$
$$= -\eta_0 \log\left(\frac{\eta_{d+1} - \frac{1}{2}\frac{|\boldsymbol{\eta}|^2}{\eta_0}}{\eta_0^{\gamma}}\right).$$

The entropy fluxes are

$$F_j = U v_j = U \frac{\eta_j}{\eta_0}.$$

The following relations hold, for any $k = 0,\ \ldots, d+1$ and $j = 1,\ \ldots,\ d$,

$$\nabla_{\widehat{\boldsymbol{\eta}}} U \cdot \frac{\partial \mathbf{f}_j}{\partial \eta_k} = \frac{\partial F_j}{\partial \eta_k}.$$

Entropy Inequality

This property shows that the original (vector) conservation law

$$\frac{\partial \widehat{\boldsymbol{\eta}}}{\partial t} + \sum_{j=1}^{d} \frac{\partial \mathbf{f}_j}{\partial x_j} = 0,$$

implies the (scalar) conservation law (assuming enough smoothness)

$$\frac{\partial U}{\partial t} + \sum_{j=1}^{d} \frac{\partial F_j}{\partial x_j} = 0.$$

Convexity of U and a limit in $\epsilon \mapsto 0$ of an elliptically perturbed system leads to the entropy inequality $(F_j = U v_j)$

$$\frac{\partial U}{\partial t} + \sum_{j=1}^{d} \frac{\partial (v_j U)}{\partial x_j} = \frac{\partial U}{\partial t} + \mathbf{div}\,(U\mathbf{v}) \leq 0.$$

Entropy Inequality

Since $U = -\varrho s$, $s : e^s = \frac{E - \frac{1}{2}|\mathbf{v}|^2}{\varrho^{\gamma-1}}$, the entropy inequality reads

$$\frac{\partial(-\varrho s)}{\partial t} + \mathbf{div}\,(-\varrho s \mathbf{v}) \leq 0.$$

In Lagrangian coordinates, it takes the form:

$$\frac{\partial}{\partial t} \int\limits_{V(t)} \varrho s \, d\mathbf{x} \geq 0.$$

In practice, we can use the inequality (since the mass is constant)

$$\frac{1}{\int_{V_{n+1}} \varrho \, dx} \int_{V_{n+1}} \varrho \; s \; d\mathbf{x} \geq \frac{1}{\int_{V_n} \varrho \, dx} \int_{V_n} \varrho s \; d\mathbf{x}.$$

Discrete Entropy Inequality

That is, the average value of s increases:

$$\overline{s}_{n+1} \geq \overline{s}_n.$$

We may as well assume that the average value of

$$e^s = \frac{E - \frac{1}{2}|\mathbf{v}|^2}{\varrho^{\gamma-1}} = \frac{\varepsilon}{\varrho^{\gamma-1}}$$

increases. Hence, for the average internal energy $\overline{\varepsilon} = \frac{1}{m(V)} \int\limits_V \varrho\varepsilon \, d\mathbf{x}$,

the following discrete entropy inequality holds:

$$\overline{\varepsilon}_{n+1} \geq \left(\frac{\varrho_{n+1}}{\varrho_n}\right)^{\gamma-1} \overline{\varepsilon}_n = \left(\frac{|V_n|}{|V_{n+1}|}\right)^{\gamma-1} \overline{\varepsilon}_n.$$

Discrete Entropy Inequality

Thus in the recovery procedure, we can use the stronger inequality

$$-\frac{1}{2}\int\limits_{V_{n+1}} \varrho_{n+1}\,|\mathbf{v}_h|^2\,d\mathbf{x} - \Delta t \int\limits_{\partial V_{n+1}} p_h \mathbf{v}_h \cdot \mathbf{n}\,d\sigma + \int\limits_{V_n} \varrho_n E_n\,d\mathbf{x}_n$$

$$\left(= \int\limits_{V_{n+1}} \varrho\varepsilon_{n+1}\,d\mathbf{x}\right) \geq \left(\frac{|V_n|}{|V_{n+1}|}\right)^{\gamma-1} \int\limits_{V_n} \varrho_n \varepsilon_n\,d\mathbf{x}.$$

This inequality poses the challenge to find a feasible $\mathbf{v}_h$ that satisfies

all the quadratic inequality constraints (for all $V \in \mathcal{T}_H$).

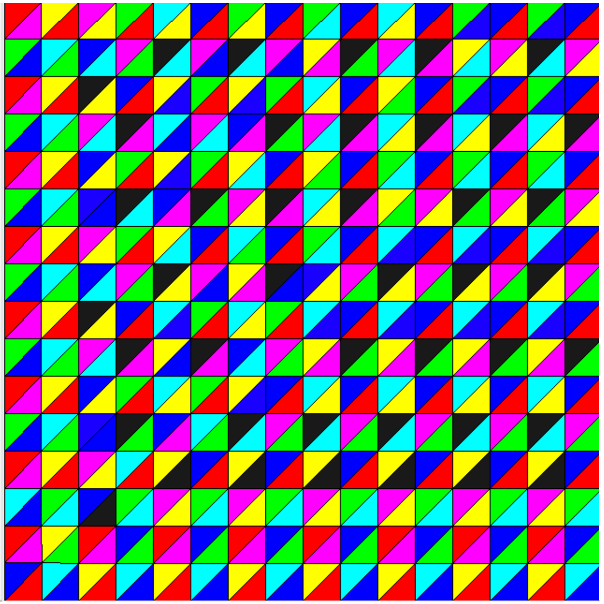
Note that the simpler inequalities (with zero on the r.h.s.) are satisfied with $\mathbf{v}_h = 0$.

Numerical Illustration

At $t = 0$, $p \approx 0$ outside a single volume (square) $V \in \mathcal{T}_H$ and p is equal to a constant on V such that the total energy $\int\limits_\Omega \rho E \, d\mathbf{x} = 1$. Also, $\mathbf{v} = 0$ and $\rho = 1$ initially. We keep $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$ for $t \geq 0$.

The tests show conversion of internal energy into kinetic and vice-versa.

Numerical Illustration



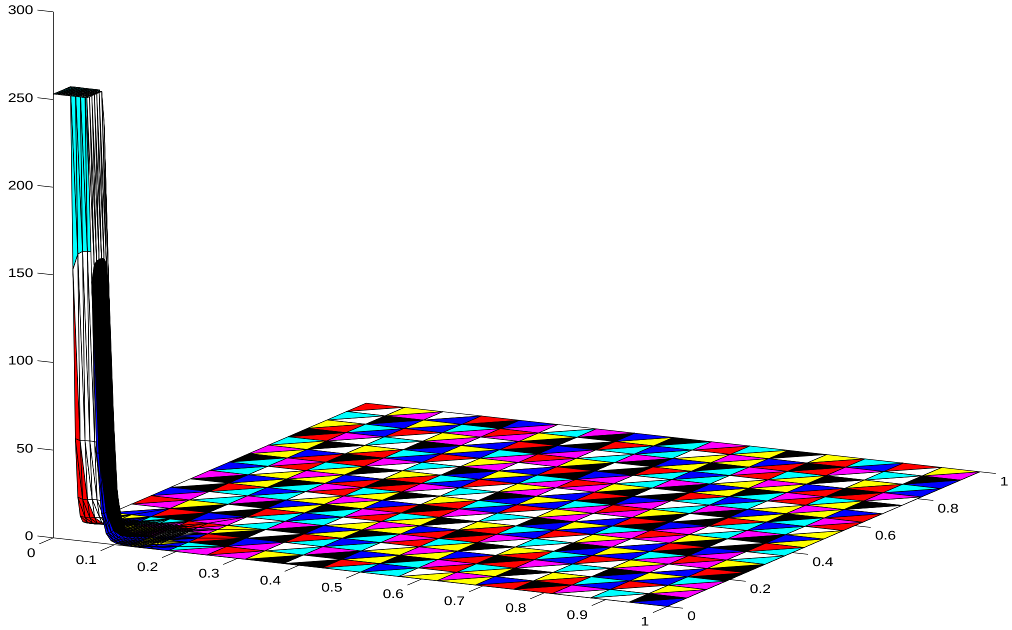


Figure 1: Initial mesh and recovered pressure.

Numerical Illustration

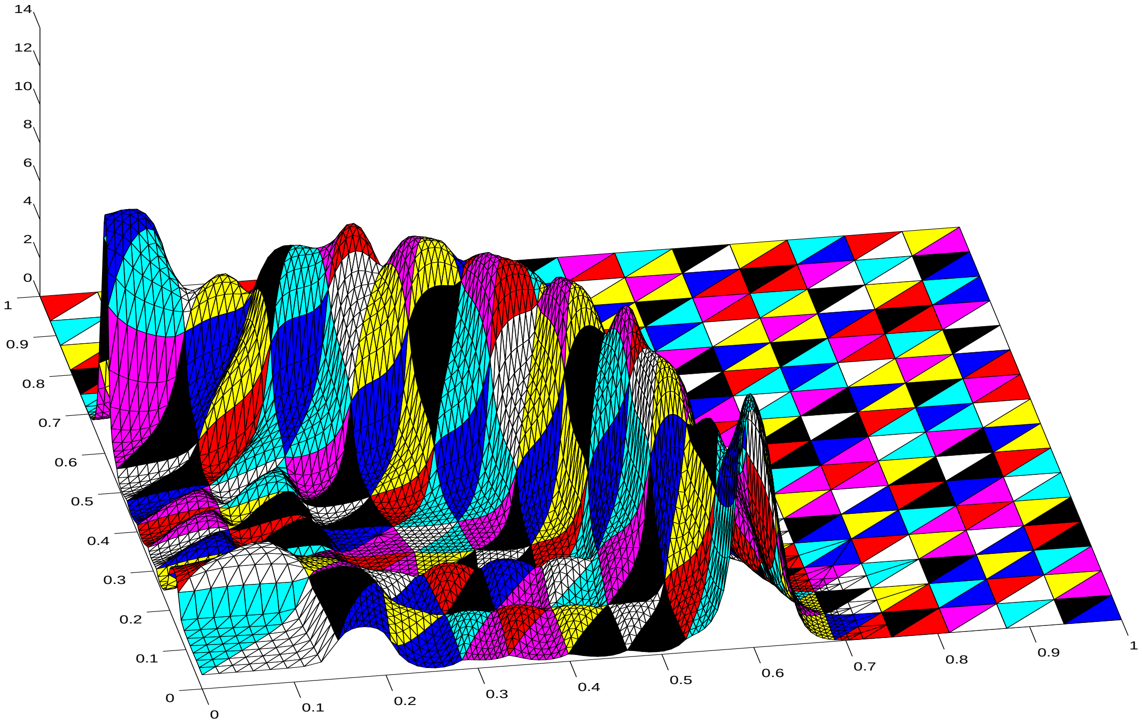


Figure 2: Recovered pressure at time $t = 0.0994$.

Numerical Illustration

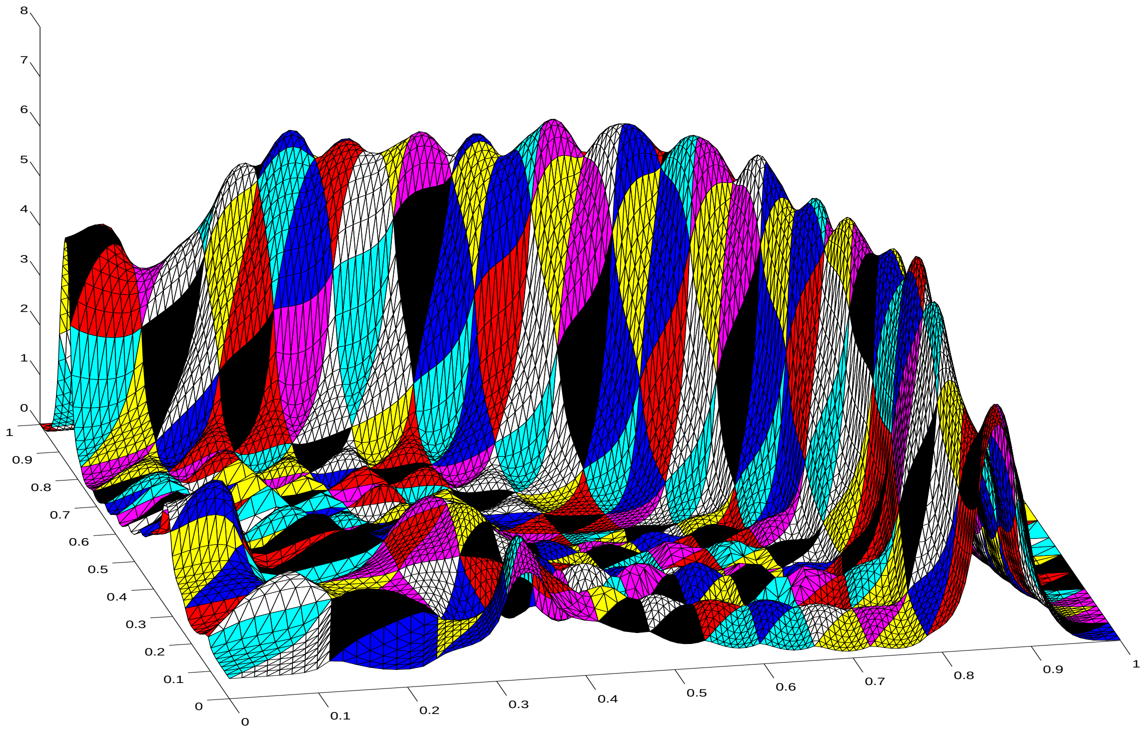


Figure 3: Recovered pressure at time $t = 0.200$.

Numerical Illustration

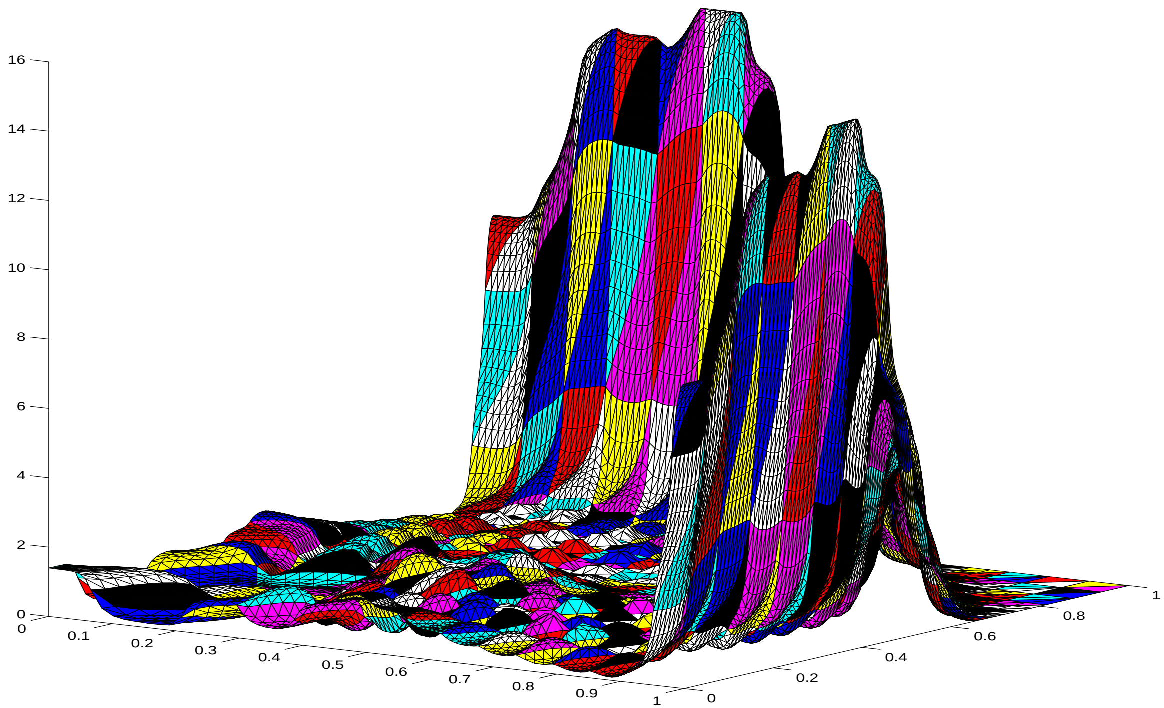


Figure 4: Recovered pressure at time $t = 0.289$.

Numerical Illustration

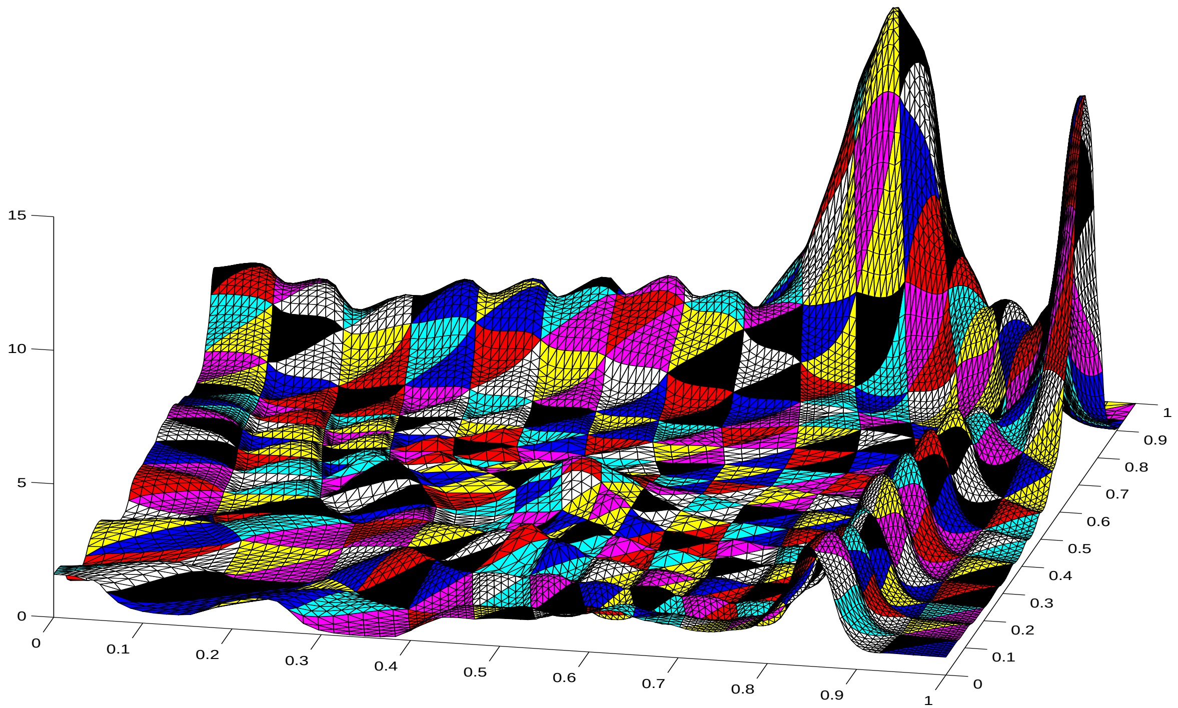


Figure 5: Recovered pressure at time $t = 0.352$.

Numerical Illustration

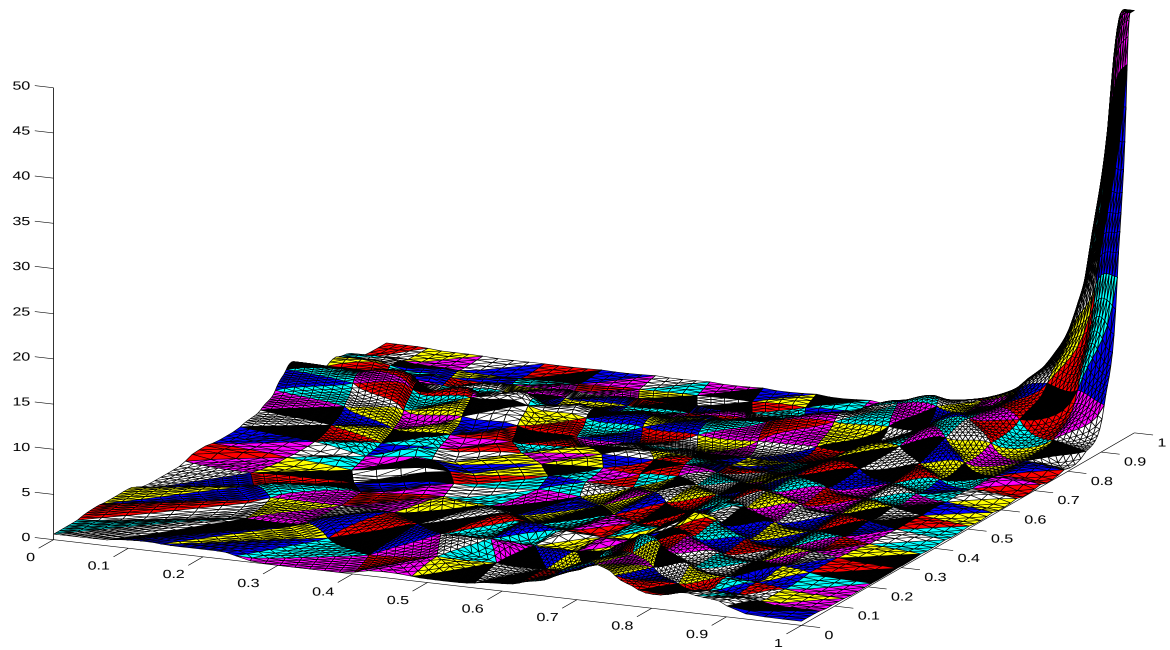


Figure 6: Recovered pressure at time $t = 0.430$.

Numerical Illustration

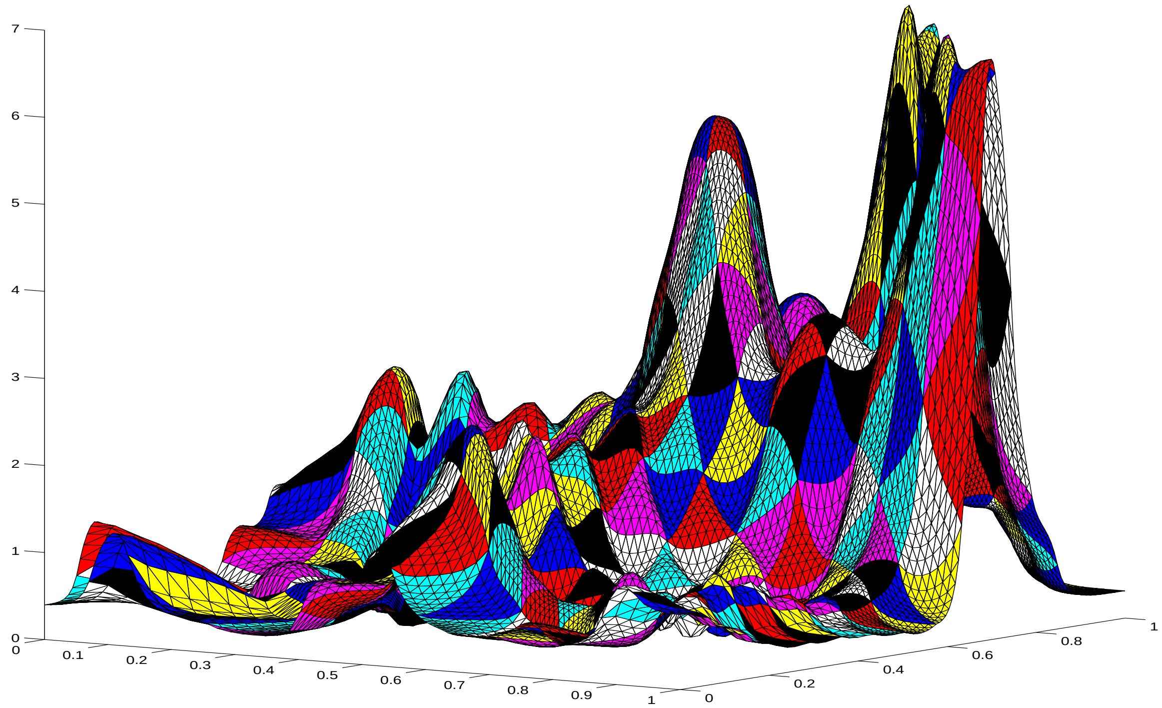


Figure 7: Recovered pressure at time $t = 0.534$.

Numerical Illustration

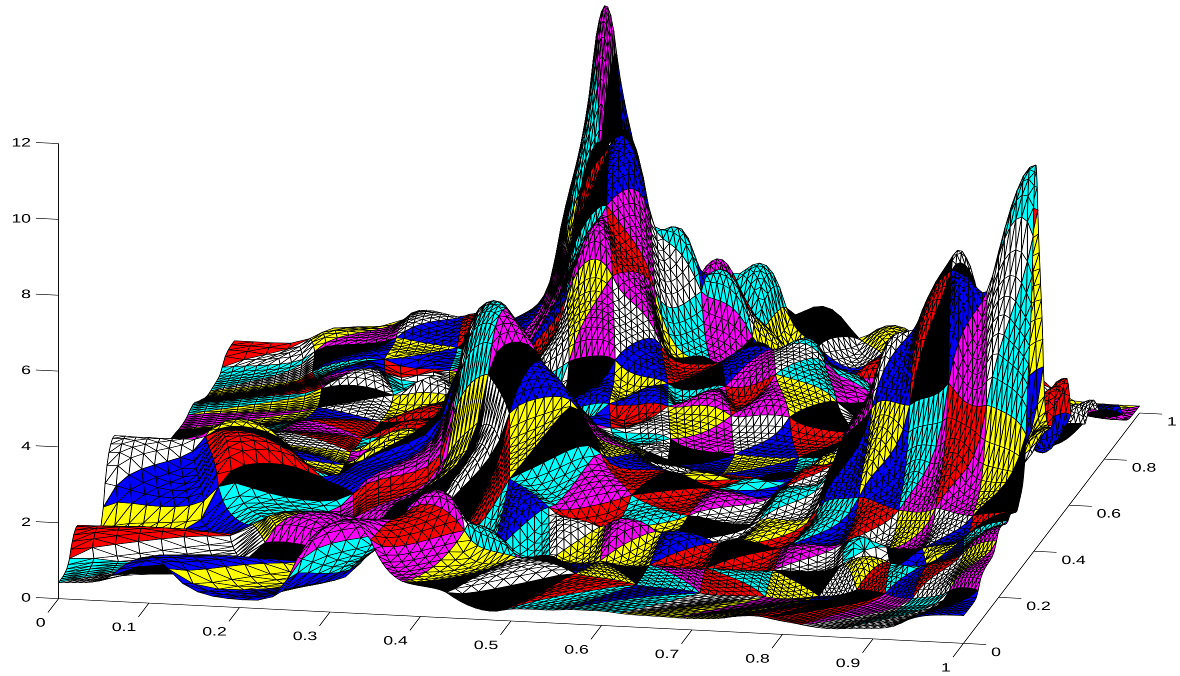


Figure 8: Recovered pressure at time $t = 0.639$.

Numerical Illustration

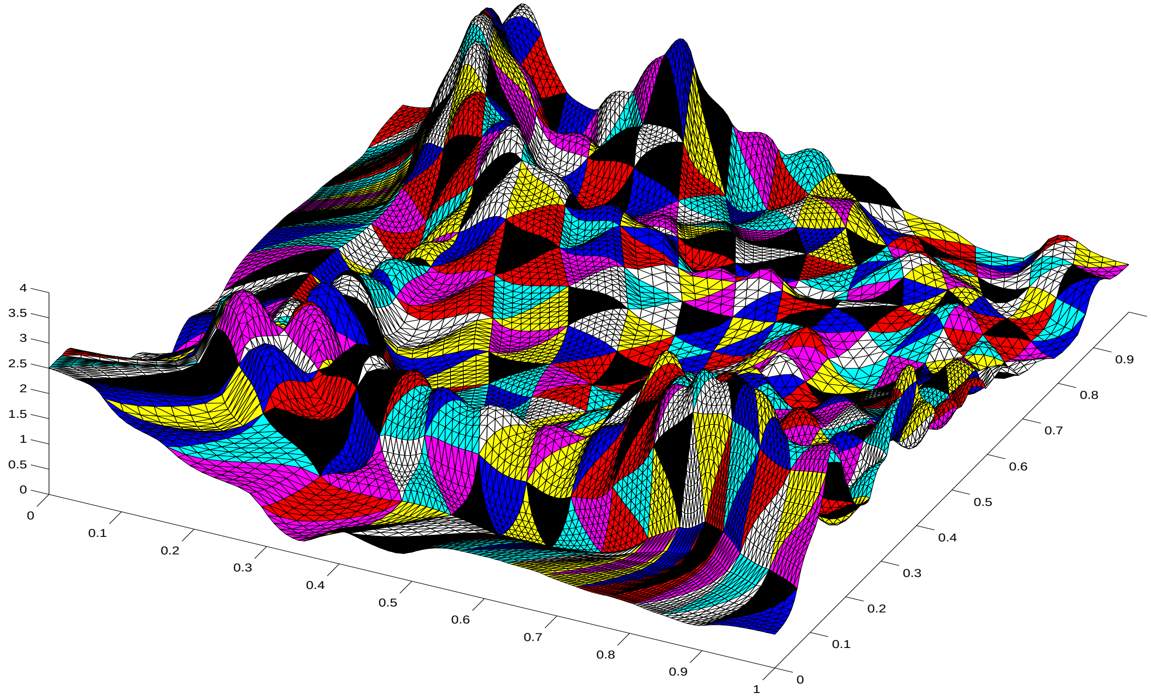


Figure 9: Recovered pressure at time $t = 0.754$.

Numerical Illustration

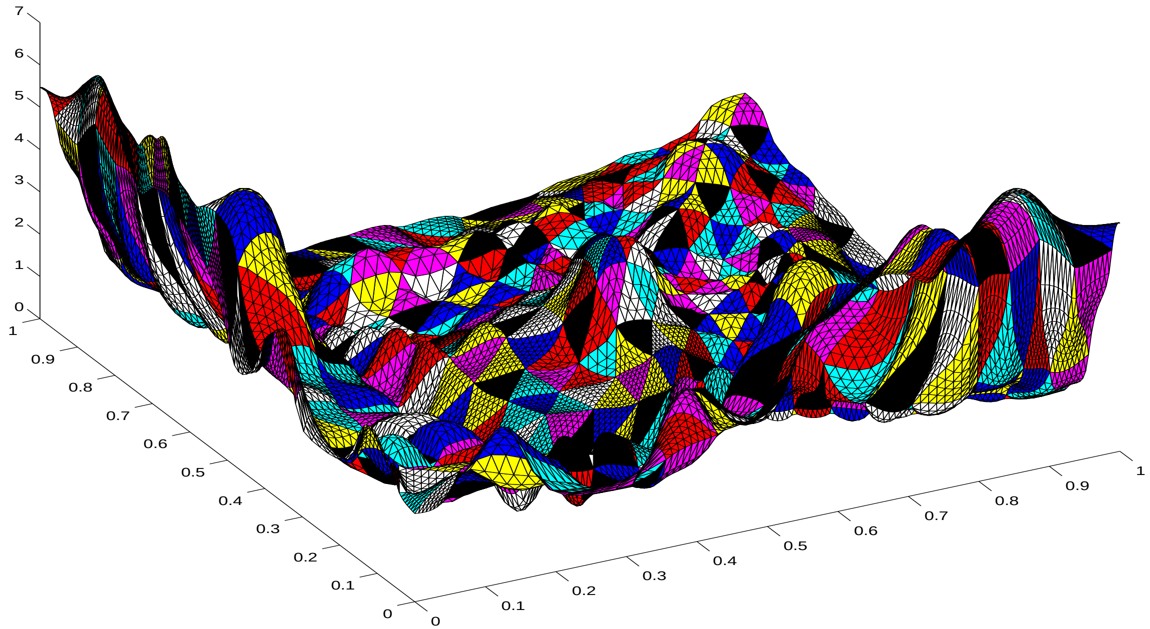


Figure 10: Recovered pressure at time $t = 0.857$.

Numerical Illustration

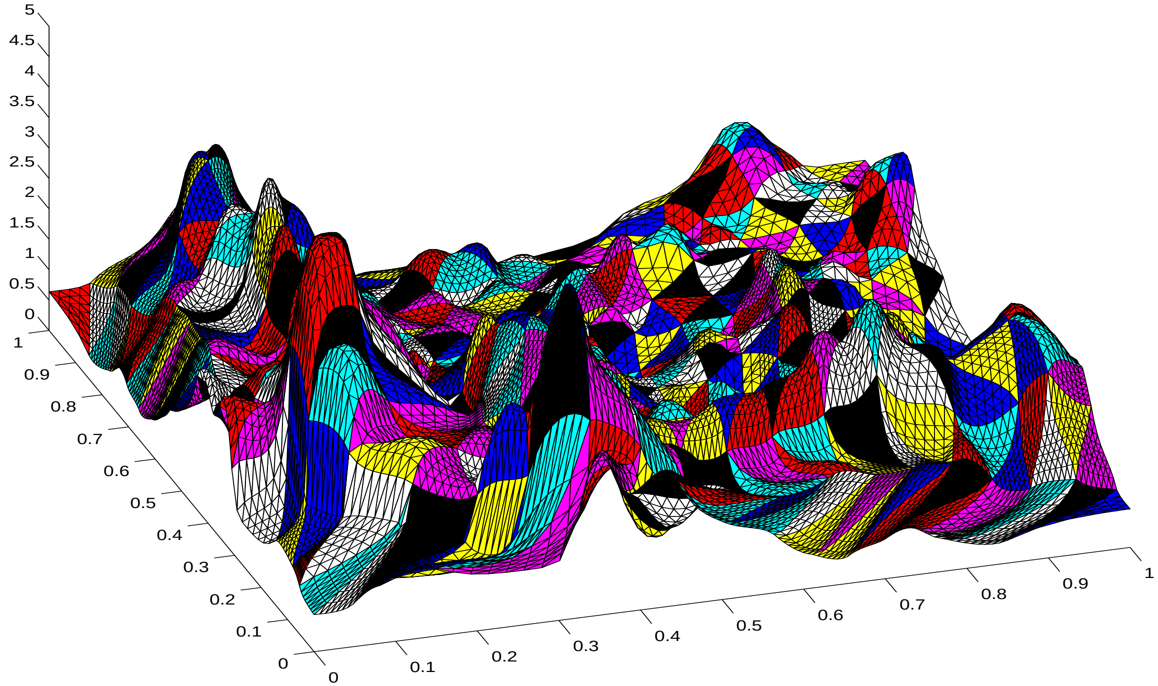


Figure 11: Recovered pressure at time $t = 0.969$.

Numerical Illustration

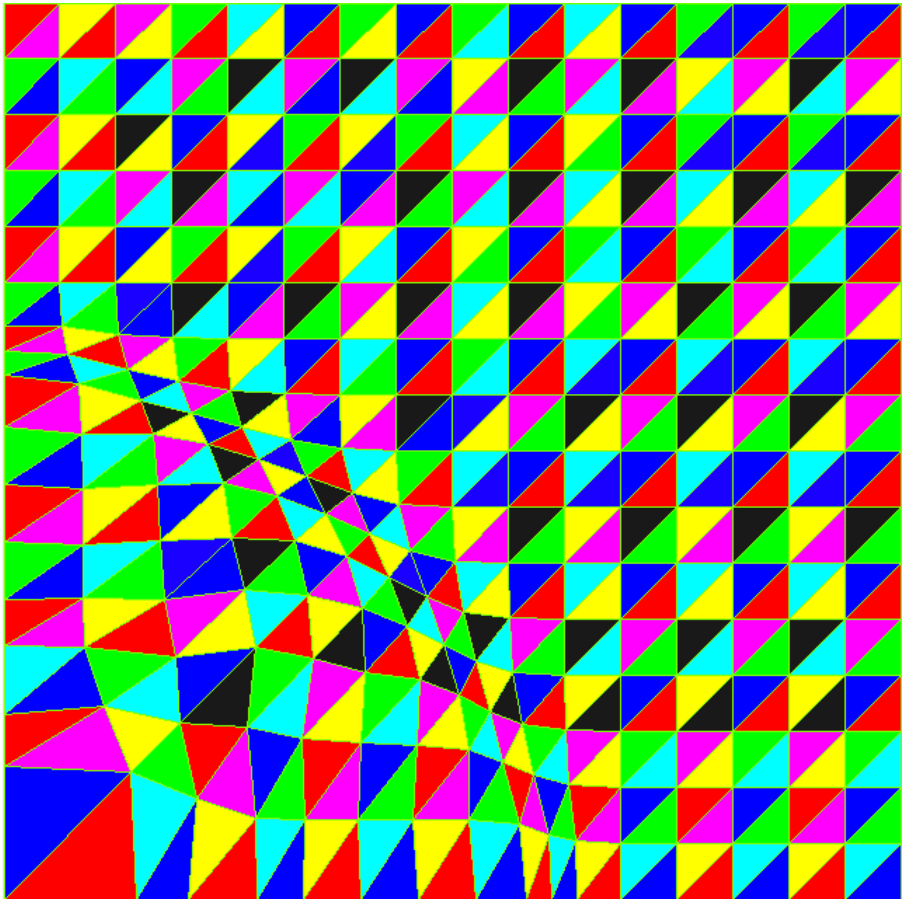


Figure 12: Moved mesh at time $t = 0.0994$.

Numerical Illustration

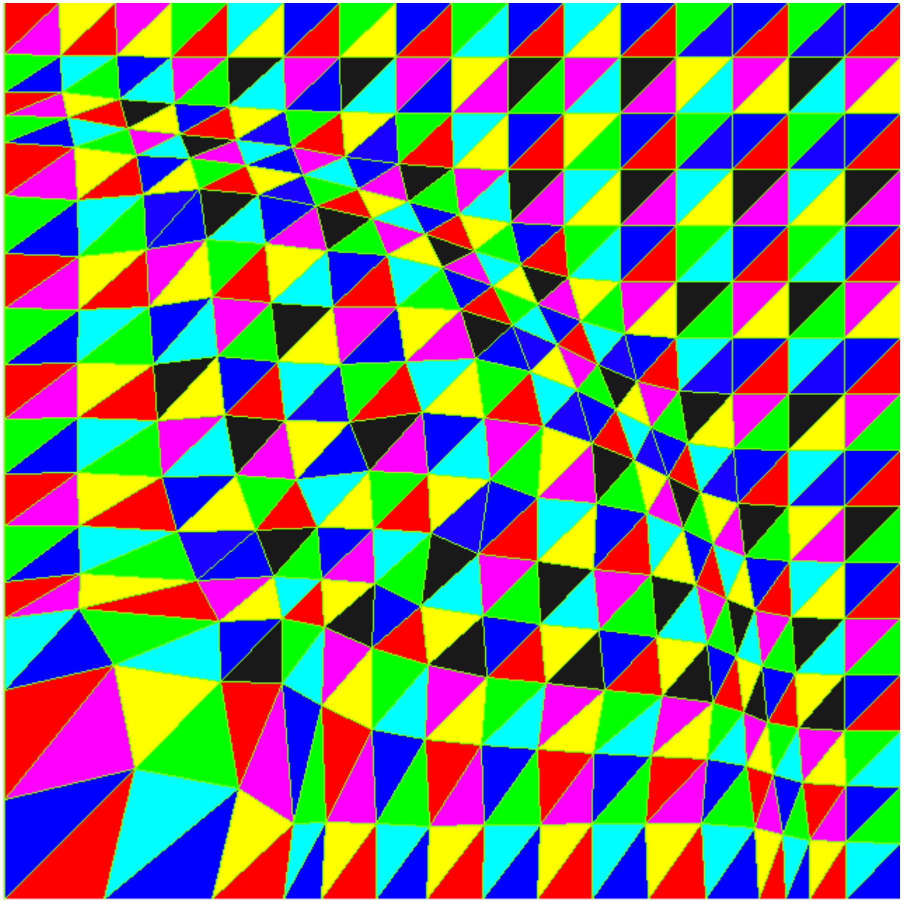


Figure 13: Moved mesh at time $t = 0.200$.

Numerical Illustration

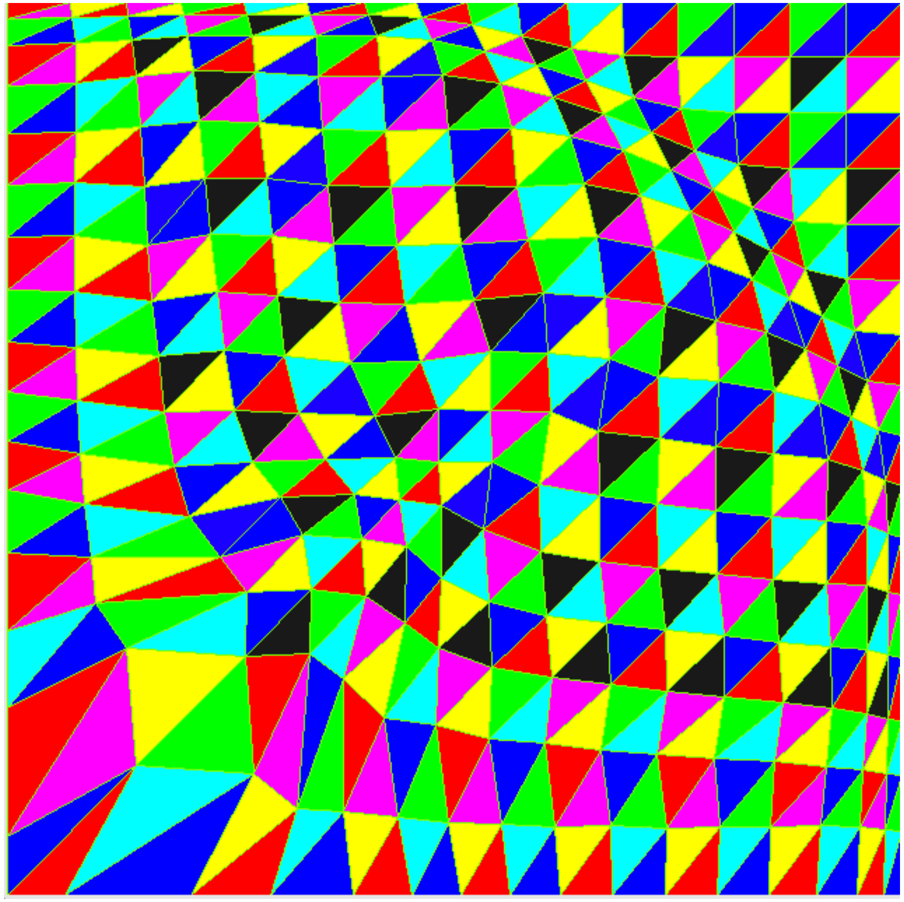


Figure 14: Moved mesh at time $t = 0.289$.

Numerical Illustration

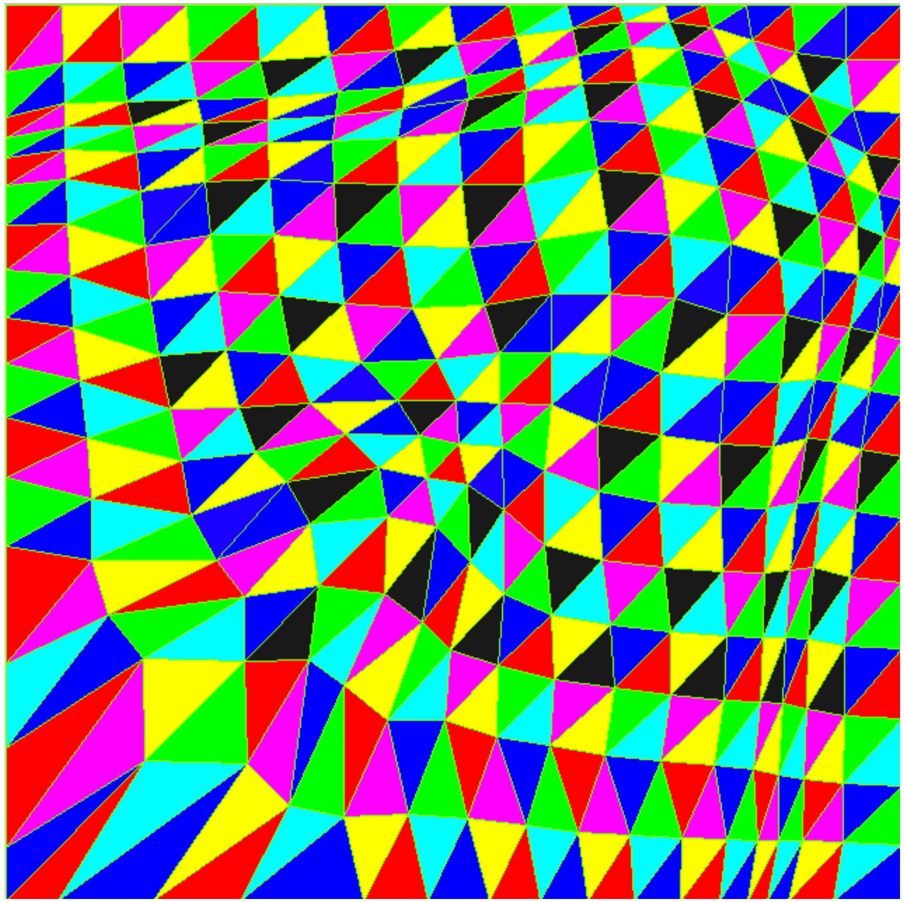


Figure 15: Moved mesh at time $t = 0.352$.

Numerical Illustration

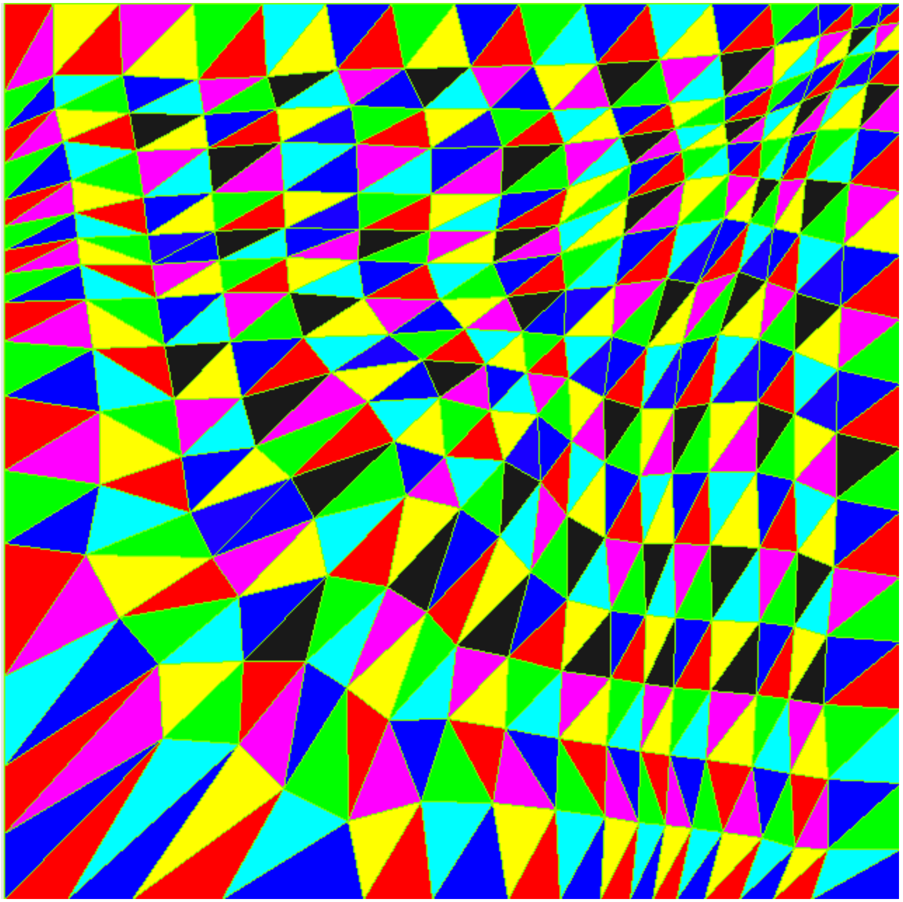


Figure 16: Moved mesh at time $t = 0.430$.

Numerical Illustration

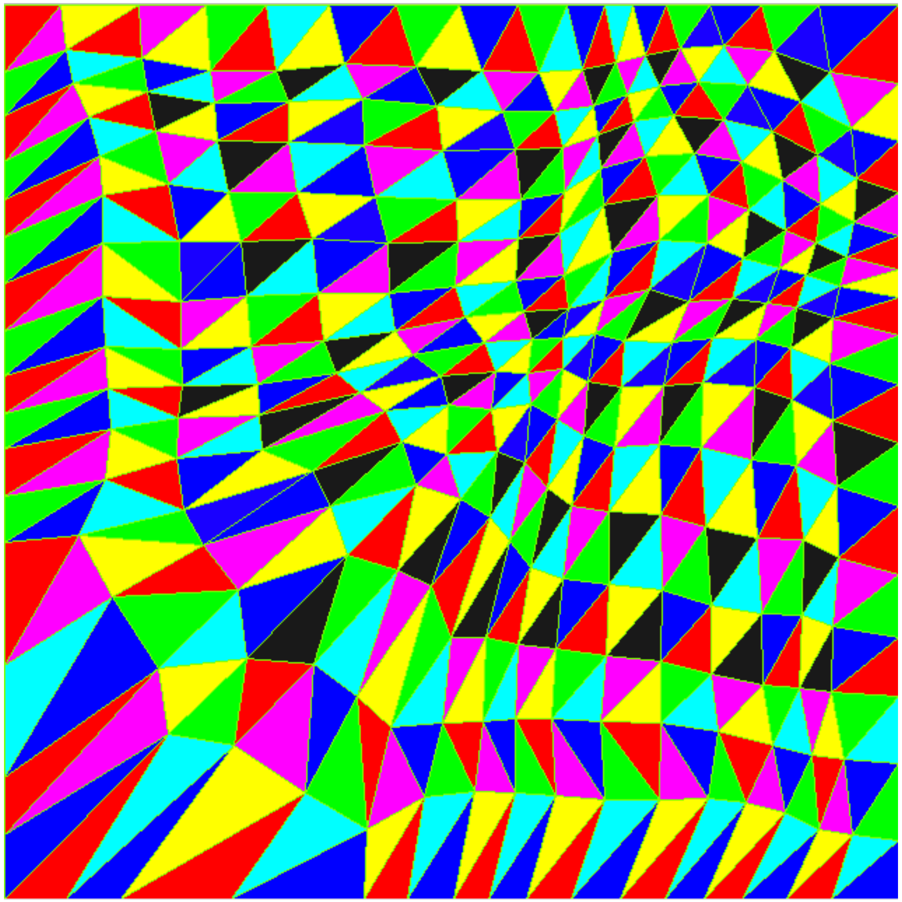


Figure 17: Moved mesh at time $t = 0.534$.

Numerical Illustration

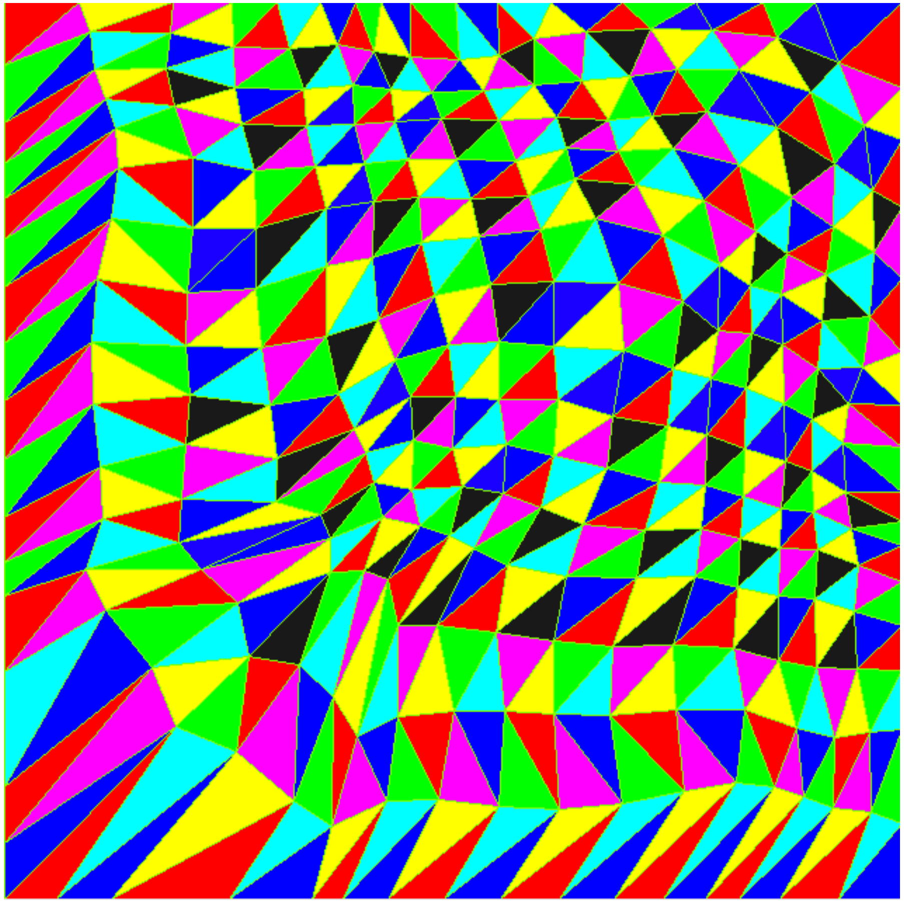


Figure 18: Moved mesh at time $t = 0.639$.

Numerical Illustration

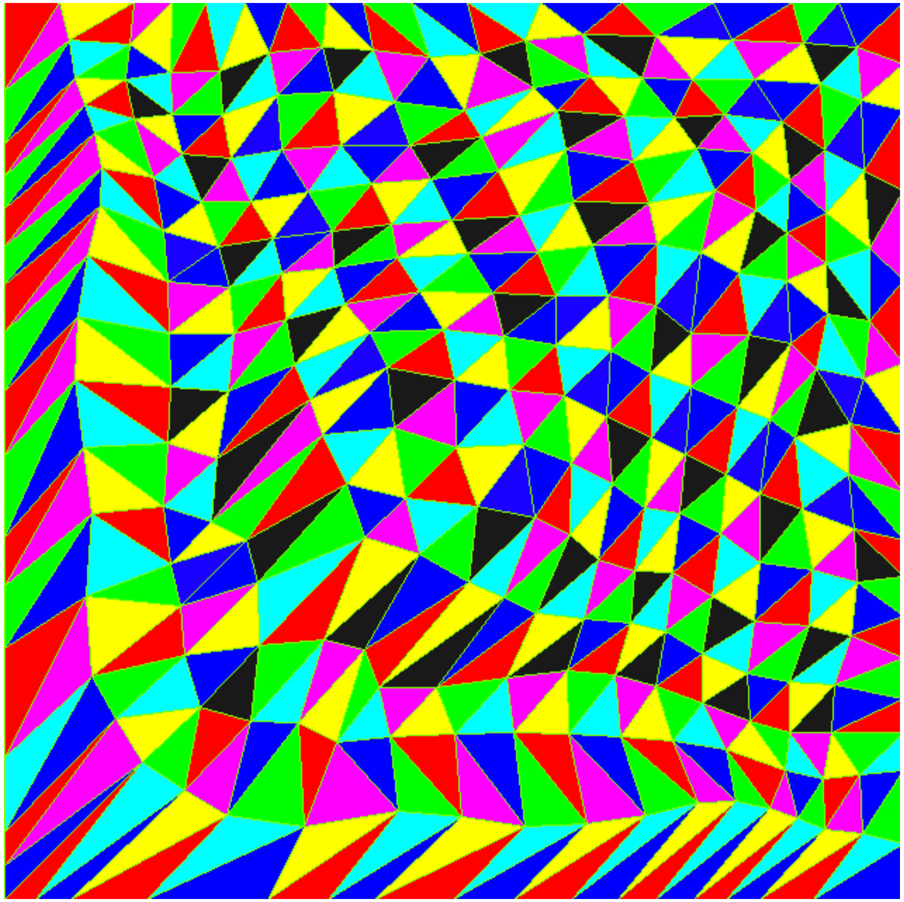


Figure 19: Moved mesh at time $t = 0.754$.

Numerical Illustration

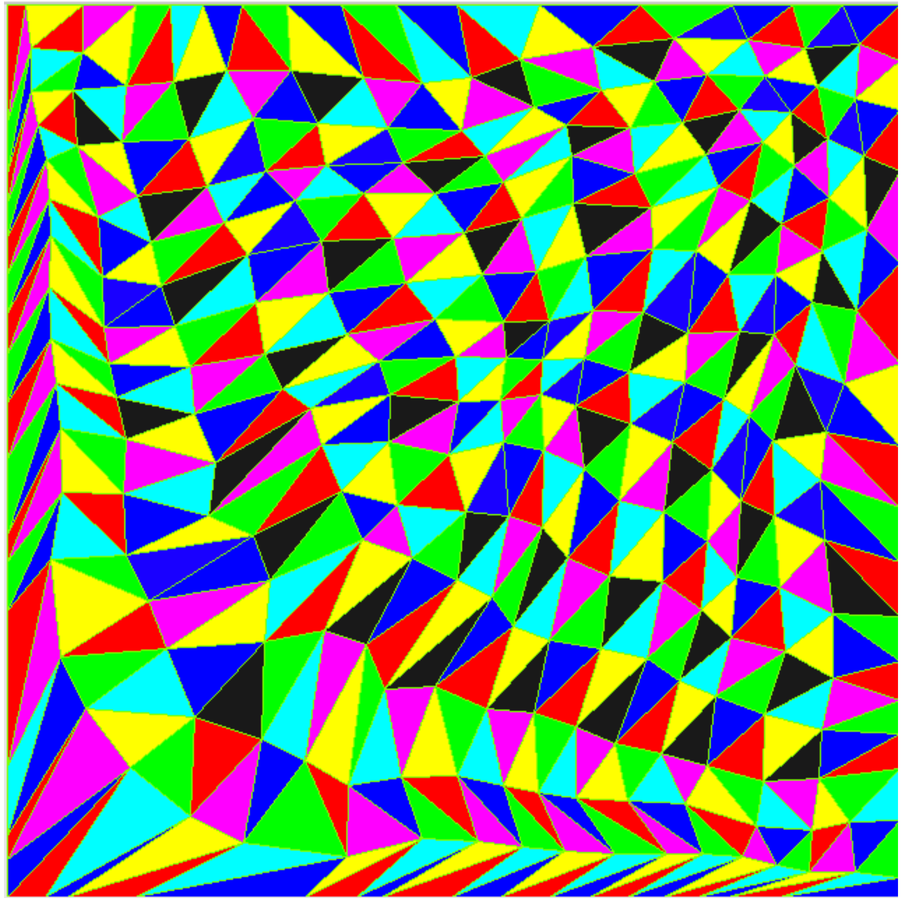


Figure 20: Moved mesh at time $t = 0.857$.

Numerical Illustration

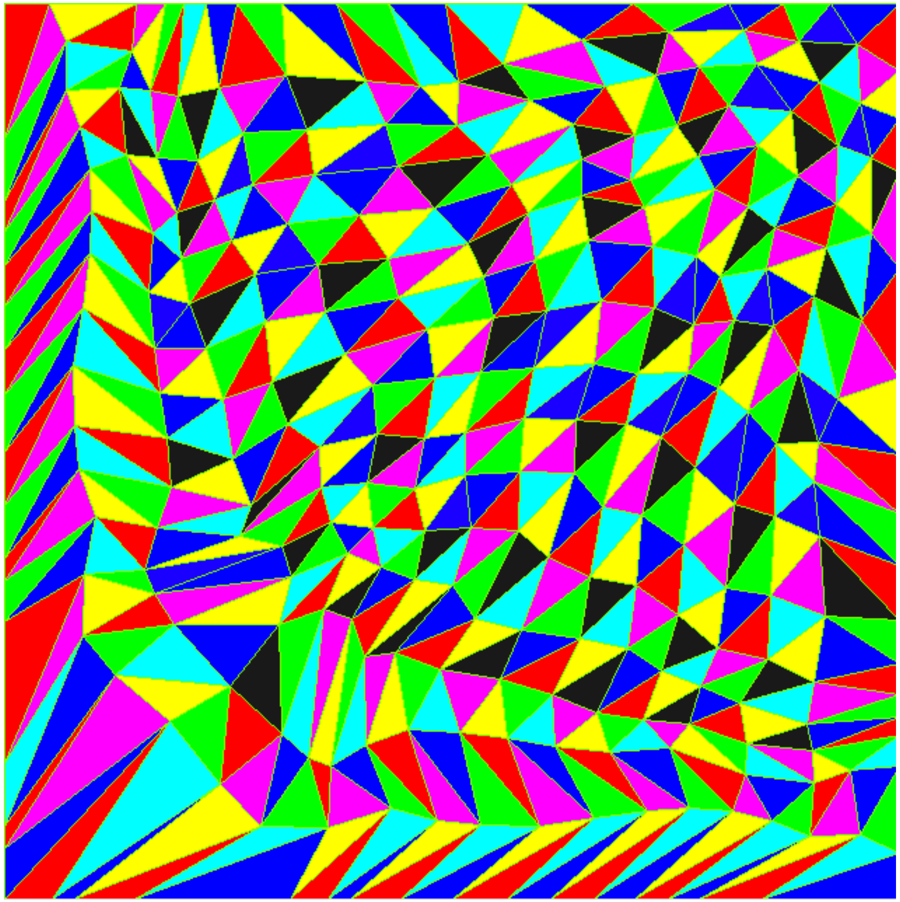


Figure 21: Moved mesh at time $t = 0.969$.

Numerical Illustration: Symmetry

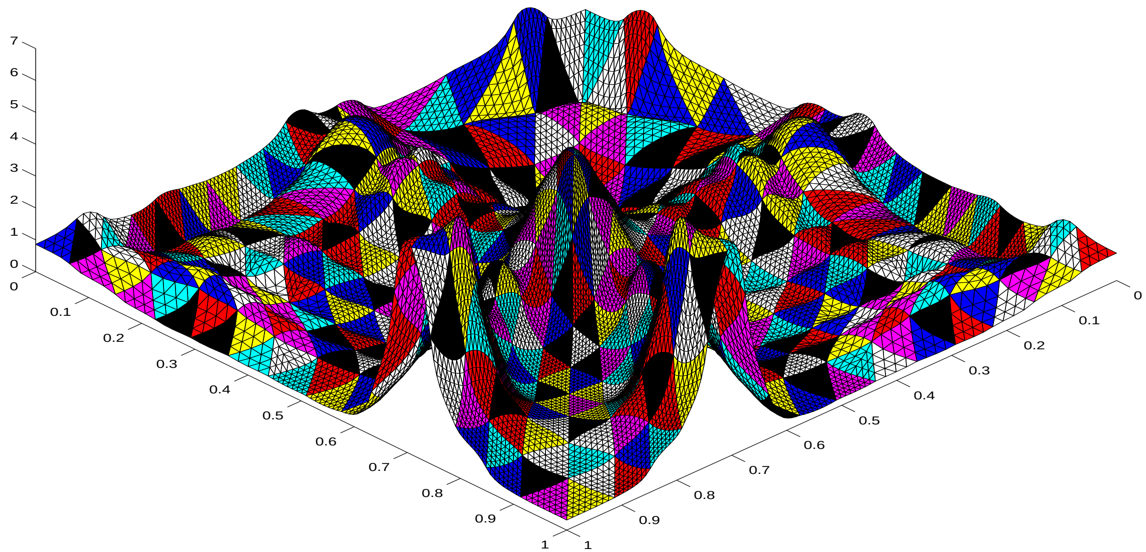


Figure 22: Recovered pressure (rotated) at time $t = 0.534$.

Conclusions

- We have proposed new conservative finite volume schemes (for Lagrangian hydrodynamics).
- They are based on standard integral form of the conservation laws and utilize non-oscillatory (TV based) function recovery.
- The function recovery procedures seem to be able to replace traditionally used “artificial viscosity” and limiters.
- The local mesh refinement used in the function recovery is essential and needs further study for efficiency. It can easily destroy symmetry.
- The most expensive part in the computation is the constrained minimization with quadratic inequality constraints. To speed it up, we may need a multilevel procedure (not as straightforward due to the quadratic inequalities).
- The monotone Gauss–Seidel in the pressure recovery has provable mesh–independent convergence (there are no inequalities).
- Extension to higher order integral moments is feasible. This will lead to new DG (discontinuous Galerkin) schemes.