TREE and LOOP AMPLITUDES in OPEN TWISTOR STRING THEORY

- Twistor String World Sheet Action with World Sheet Gauge Fields

  Classical Solutions on the Disk and Cylinder

- Canonical Quantization of the Open String and Gauge Invariance

  Gluon Vertex Operators

- n-point Tree Amplitudes

- n-point Twistor Fields One-loop Amplitudes

- Current Algebra One-loop Amplitudes

  Weierstrass P functions

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The world sheet action with Euclidean signature is

$$S = S_{YZ} + S_{\text{ghost}} + S_G \quad \text{where } S_G \text{ has } c = 28$$

$$S_{YZ} = \int d^2z \left(Y^{Iz} D_z \bar{Z}^I + Y^{I\bar{z}} D_{\bar{z}} Z^I \right)$$

with $D_\mu = \partial_\mu - iA_\mu$ and $1 \leq I \leq 8$.

The equations of motion for S_{YZ} are

$$D_{\bar{z}} Z = D_z \bar{Z} = 0, \qquad D'_z Y^z = D'_{\bar{z}} Y^{\bar{z}} = 0$$

together with the constraints $Y^{\bar{z}} Z = Y^z \bar{Z} = 0$.

The end condition on the open string

$$n_z Y^z \delta \bar{Z} = -n_{\bar{z}} Y^{\bar{z}} \delta Z$$

is satisfied by the boundary conditions

$$\bar{Z} = UZ, \qquad Y^z n_z = -U^{-1} Y^{\bar{z}} n_{\bar{z}}$$

where $U = e^{2i\alpha}$, $|U| = 1$.

The action S_{YZ} has two abelian gauge invariances

$$Y^{\bar{z}} \mapsto g^{-1} Y^{\bar{z}}, \quad Z \mapsto gZ, \quad A_{\bar{z}} \mapsto A_{\bar{z}} - i g^{-1} \partial_{\bar{z}} g \,,$$

$$Y^{z} \mapsto \bar{g}^{-1} Y^{z}, \quad \bar{Z} \mapsto \bar{g}\bar{Z}, \quad A_{z} \mapsto A_{z} - i \bar{g}^{-1} \partial_{z} \bar{g} \,.$$

For *eg.* in coordinates $A_1 = A_z + A_{\bar{z}}, \quad A_2 = i(A_z - A_{\bar{z}})$,

$$A_\mu \mapsto A_\mu + \partial_\mu \varphi + \epsilon_\mu{}^\nu \partial_\nu \psi \,,$$

where $g = e^{\psi + i\varphi}$ is in $GL(1, \mathbb{C})$ with φ and ψ pure imaginary.

$A_{\bar{z}}, A_z,$ can be thought of as components, $\mathcal{A}_{\bar{z}}, \tilde{\mathcal{A}}_z,$ of different gauge potentials, $\mathcal{A}_\mu, \tilde{\mathcal{A}}_\mu,$ associated with the transformations g, $\bar{g}$, respectively.

The gauge invariance of the theory can be used to set the potential $A_\mu = 0$.

An example of a potential on S^2, for which $A_z = \tilde{A}_z = 0$, $A_{\bar{z}} = \mathcal{A}_{\bar{z}} = 0$, is

$$\mathcal{A}_z^< = -\frac{in\bar{z}}{1+z\bar{z}}, \qquad \tilde{\mathcal{A}}_z^< = 0,$$

$$\mathcal{A}_z^> = \frac{in}{(1+z\bar{z})z}, \qquad \tilde{\mathcal{A}}_z^> = 0,$$

$$\mathcal{A}_{\bar{z}}^< = 0, \qquad \tilde{\mathcal{A}}_{\bar{z}}^< = -\frac{inz}{1+z\bar{z}},$$

$$\mathcal{A}_{\bar{z}}^> = 0, \qquad \tilde{\mathcal{A}}_{\bar{z}}^> = \frac{in}{(1+z\bar{z})\bar{z}}.$$

Then $\mathcal{A}_\mu^> - \mathcal{A}_\mu^< = -ig^{-1}\partial_\mu g, \quad \tilde{\mathcal{A}}_\mu^> - \tilde{\mathcal{A}}_\mu^< = -i\tilde{g}^{-1}\partial_\mu \tilde{g}$
for $g = z^{-n}$, $\tilde{g} = \bar{z}^{-n}$.

$Z^>(z) = z^{-n} Z^<(z), \quad \bar{Z}^>(\bar{z}) = \bar{z}^{-n} \bar{Z}^<(\bar{z})$.

Two patches: $A_\mu^> = \{z : |z| > 1 - \epsilon\}$ and $A_\mu^< = \{z : |z| < 1 + \epsilon\}$.

An example of a potential on T^2, for which $A_z = A_{\bar{z}} = 0$, is

$$\mathcal{A}_z(z,\bar{z}) = \frac{i\pi n}{\mathrm{Im}\tau}(z-\bar{z}), \qquad \tilde{\mathcal{A}}_z(z,\bar{z}) = 0,$$

$$\mathcal{A}_{\bar{z}}(z,\bar{z}) = 0, \qquad \tilde{\mathcal{A}}_{\bar{z}}(z,\bar{z}) = -\frac{i\pi n}{\mathrm{Im}\tau}(z-\bar{z}),$$

Then

$$\mathcal{A}_\mu(z+a,\bar{z}+\bar{a}) - \mathcal{A}_\mu(z,\bar{z}) = -ig_a^{-1}(\bar{z})\partial_\mu g_a(\bar{z})$$
$$\tilde{\mathcal{A}}_\mu(z+a,\bar{z}+\bar{a}) - \tilde{\mathcal{A}}_\mu(z,\bar{z}) = -i\tilde{g}_a^{-1}(z)\partial_\mu \tilde{g}_a(z)$$

for

$$g_a(z) = e^{-\frac{\pi n(a-\bar{a})}{\mathrm{Im}\tau}(z+\frac{a}{2})+i\pi n m_1 n_1 + i\eta_a},$$
$$\tilde{g}_a(\bar{z}) = e^{\frac{\pi n(a-\bar{a})}{\mathrm{Im}\tau}(\bar{z}+\frac{\bar{a}}{2})-i\pi n m_1 n_1 - i\bar{\eta}_a},$$

$$Z(z+a) = g_a(z)Z(z)$$

Many patches: $a = m_1 + n_1\tau$.

In a gauge with $A_z = A_{\bar{z}} = 0$, the equations of motion for $Z, \bar{Z}$ are $\partial_{\bar{z}} Z = \partial_z \bar{Z} = 0$, so that $Z \equiv Z(z), \bar{Z} \equiv \bar{Z}(\bar{z})$. For instanton number n:

On the disk,

$$Z^I(z) = \sum_{m=0}^{n} Z_m^I z^m$$

On the cylinder,

$$Z^I(z) = \sum_{p=0}^{n-1} c_p^I \, \theta \begin{bmatrix} \frac{1}{n}(\epsilon + 2p) \\ \epsilon' \end{bmatrix} (nz, n\tau),$$

where $1 \leq I \leq 8$.

Let $\eta_a = \pi m_1 \epsilon - \pi n_1 \epsilon'$, the translation property of Z is

$$Z(z+1) = e^{i\pi\epsilon} Z(z), \qquad Z(z+\tau) = e^{-i\pi(\epsilon' + n(2z+\tau))} Z(z),$$

the defining relations for an n-th order theta function with characteristics ϵ, ϵ'.

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\nu, \tau) = \sum_{m \in \mathbb{Z}} \exp \left\{ i\pi(m + \tfrac{1}{2}\epsilon)^2 \tau + 2\pi i (m + \tfrac{1}{2}\epsilon)\nu + \pi i m \epsilon' + \tfrac{1}{2}\pi i \epsilon \epsilon' \right\}$$

The space of n-th order theta functions is spanned by the n functions

$$\theta \begin{bmatrix} \frac{1}{n}(\epsilon + 2p) \\ \epsilon' \end{bmatrix} (nz, n\tau), \quad p = 0, 1, \ldots n-1.$$

In this gauge, we canonically quantize the Berkovits action

The field content is

	Y	Z	J^A	u	v	b	c
$U(1)$ charge	-1	1	0	0	0	0	0
conformal spin, $\mathcal{J}$	1	0	1	1	0	2	-1
central charge, c	0		28	-2		-26	

The fields $Z^I, 1 \leq I \leq 8$, comprise four boson fields, $\lambda^a, \mu^a, 1 \leq a \leq 2$, and four fermion fields $\psi^M, 1 \leq M \leq 4$.

The gauge invariance insures that the Z^I are effectively projective coordinates in the target space $\mathbb{CP}^{3|4}$.

The mode expansion for the fields with conformal spin $\mathcal{J}$ is

$$\Phi(z) = \sum \Phi_n z^{-n-\mathcal{J}}$$

The vacuum satisfies $\Phi_n|0\rangle = 0$ for $n > -\mathcal{J}$. So $Z_n^I|0\rangle = 0$ for $n \geq 1$.

$$[\![Z_m^i, Y_n^j]\!] = \delta^{ij}\delta_{m,-n}, \quad \{c_m, b_n\} = \delta_{m,-n}, \quad \{v_m, u_n\} = \delta_{m,-n},$$

$[\![,]\!]$ denote anticommutators when $i, j \geq 5$, otherwise commutators

$$[J_m^A, J_n^B] = i f^{AB}{}_C J_{m+n}^C + k m \delta_{m,-n}\delta^{AB}.$$

$$L(z) = -\sum_j :Y^j(z)\partial Z^j(z): - :u(z)\partial v(z): + 2 :\partial c(z) b(z): - :\partial b(z) c(z): + L^J(z)$$

The current associated with the abelian gauge transformation is

$$J(z) = -P(z) = -\sum_{j=1}^{8} :Y^j(z)Z^j(z): = -\sum_{j=1}^{8}\sum_m a_m^j z^{-m-1} = -\sum_m a_m z^{-m-1}$$

$$[a_m^i, Z_n^j] = -Z_{m+n}^j \delta^{ij}, \qquad X^j(z) = q_0^j + a_0^j \log z - \sum_{n\neq 0} \frac{1}{n} a_n^j z^{-n}, \qquad Z^j(z) =: e^{-X^j(z)} :$$

Gauge transformation with winding number d:

$$g(z) = z^d e^{-\sum_n f_n z^{-n}}, \quad U_g = e^{dq_0} e^{\sum_n f_{-n} a_n}, \quad U_g Z(z) U_g^{-1} = g(z) Z(z)$$

$$\langle 0|U_g V_1(z_1)V_2(z_2)\dots V_n(z_n)|0\rangle = \langle 0|e^{dq_0} V_1(z_1)V_2(z_2)\dots V_n(z_n)|0\rangle \qquad \text{Tree}$$

$$\text{tr}\left(U_g V_1(z_1)V_2(z_2)\dots V_n(z_n)w^{L_0}\right) = \text{tr}\left(e^{dq_0} e^{f_0 a_0} V_1(z_1)V_2(z_2)\dots V_n(z_n)w^{L_0}\right) \quad \text{Loop}$$

$$e^{\pm q_0} = \prod_{j=1}^{8} e^{\pm q_0^j}, \qquad Z_{n+d}^j e^{dq_0} = e^{dq_0} Z_n^j, \qquad [a_0^i, q_0^j] = \pm\delta^{ij}$$ for fermions/bosons

Scalar products

fermions:

$$\langle 0|Z_0|0\rangle = 1 = \int dZ_0 Z_0, \quad Z_0|0\rangle = e^{-q_0}|0\rangle, \quad \langle 0|e^{dq_0} Z_{-d} \ldots Z_0|0\rangle = 1$$

(Tree amplitude will vanish unless number of negative helicity modes is $d+1$).

bosons:

$$\langle 0|f(Z_0)|0\rangle = \int f(Z_0) dZ_0,$$ or, equivalently, $$\langle 0|e^{ikZ_0}|0\rangle = \delta(k)$$

$$\langle 0|e^{dq_0} \exp\left\{ i \sum_{j=0}^{d} k_j Z_{-j} \right\} |0\rangle = \prod_{j=0}^{d} \delta(k_j)$$

Physical state $|\Psi\rangle = f(Z_0)J^A_{-1}|0\rangle$

Gluon vertex operator $V(\Psi, z) = f(Z(z))J^A(z)$ describes the dependence on the mean position of the string in twistor superspace $\mathcal{Z}' = (\pi^a, \omega^a, \theta^M)$:

$$W(z) = \int \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a)\delta(k\mu^a(z) - \omega^a) \prod_{M=1}^{4} (k\psi^M(z) - \theta^M)\frac{dk}{k}$$

Multiply by polarizations: $A(\theta) = {\color{red}A_+} + \theta^1\theta^2\theta^3\theta^4{\color{blue}A_-}$

Fourier transform on ω^a, integrate over θ^M, then

$$V_-^A(z) = \int dk k^3 \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a)e^{ik\mu^a(z)\bar{\pi}_a} J^A(z)\psi^1(z)\psi^2(z)\psi^3(z)\psi^4(z)$$

and

$$V_+^A(z) = \int \frac{dk}{k} \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a)e^{ik\mu^a(z)\bar{\pi}_a} J^A(z)$$

The n-point gluon tree amplitude in instanton sector d is

$$\mathcal{A}_n^{\text{tree}} = \int \langle 0|e^{dq_0} V_{\epsilon_1}^{A_1}(z_1) V_{\epsilon_2}^{A_2}(z_2) \ldots V_{\epsilon_n}^{A_n}(z_n)|0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S$$

$d\gamma_M$ is the invariant measure on the Möbius group

$d\gamma_S$ is the invariant measure on the group of scale transformations on Z

MHV amplitudes ($d = 1$):

$$\langle 0|e^{q_0} V_-^{A_1}(z_1) V_-^{A_2}(z_2) V_+^{A_3}(z_3) \ldots V_+^{A_n}(z_n)|0\rangle$$

Because Y^I does not occur in $V^A(z)$, replace $Z^I(z)$ by $Z_0^I + zZ_{-1}^I$.

Use single trace current algebra tree amplitude $\frac{f^{A_1 A_2 \ldots A_n}}{(z_1 - z_2)(z_2 - z_3)\ldots(z_n - z_1)}$,

$$\mathcal{A}_{--+\ldots+}^{\text{tree}} = \delta^4({\pi_r}^a \bar{\pi}_{rb}) \frac{\langle 1,2\rangle^4 f^{A_1 A_2 \ldots A_n}}{\langle 1,2\rangle\langle 2,3\rangle \ldots \langle n,1\rangle}$$

Penrose spinors :

$$< rs >= \pi_r^a \pi_{ra}, \quad [rs] = \bar{\pi}_r^{\dot{a}} \bar{\pi}_{r\dot{a}}, \quad p_{ra\dot{a}} = \pi_{ra} \bar{\pi}_{r\dot{a}}.$$

Polarizations $A_1^- A_2^- A_3^+ \dots A_n^+$: $\quad \epsilon_r^+ = A_r^+ \bar{s}_{ra} \bar{\pi}_{r\dot{a}}, \quad \epsilon_r^- = A_r^- \pi_{ra} s_{r\dot{a}}$

(Vectors $s_{r\dot{a}}$ and $\bar{s}_{ra}$ defined such that $\pi_r{}^a \bar{s}_{ra} = 1$ and $\bar{\pi}_r^{\dot{a}} s_{r\dot{a}} = 1$, *eg.* $\bar{s}_{3b} \sum_{r=1}^3 \pi_r^b \bar{\pi}_r^{\dot{b}} = 0$ implies $\bar{s}_{3b} \pi_1^b = [23]/[12]$.)

$d = 0$

$$\epsilon_1^- \cdot \epsilon_2^+ \epsilon_3^+ \cdot p_1 + \epsilon_2^+ \cdot \epsilon_3^+ \epsilon_1^- \cdot p_2 + \epsilon_3^+ \cdot \epsilon_1^- \epsilon_2^+ \cdot p_3$$
$$= A_{-1} A_{+2} A_{+3} \ (\pi_1^b \bar{s}_2^a \bar{s}_{3b} \pi_{1a} \bar{\pi}^{3\dot{b}} \bar{\pi}_{2\dot{b}}) = A_{-1} A_{+2} A_{+3} \ \frac{[23]^3}{[12][31]}.$$

Path integral quantization:

$$A^{\text{tree}} = \sum_{d=1} \int DZ_I \;\; \delta((\partial_{\bar z} - iA_{\bar z})Z^I)$$
$$\cdot \int \prod_{i=1}^{n} dz_i \int D\phi_G e^{-S_G} J^{A_1}(z_1) J^{A_2}(z_2) \dots J^{A_n}(z_n)$$
$$\cdot \prod_{r=1}^{n} \frac{dk_r}{k_r} \prod_{r,a} \delta(\pi_r^a - k_r \lambda^a(z_r)) \; e^{ik_r \mu^a(z_r)\bar\pi_{ra}}$$
$$\cdot \left[A_{+r} + k_r^4 \psi^1(z_r)\psi^2(z_r)\psi^3(z_r)\psi^4(z_r) A_{-r}\right] \Big/ d\gamma_M d\gamma_S .$$

In a gauge where the potentials are zero, the path satisfies $\partial_{\bar z} Z^I = 0$.

For $d = 1$, $\quad Z^I(z) = Z_0^I + Z_1^I z$.

Replace $DZ_I \;\; \delta((\partial_{\bar z} - iA_{\bar z}^{(d=1)})Z^I)$ with $\prod_{I=1}^{8} dZ_0^I dZ_1^I$.

Loop Amplitude

- $\mathcal{A}^{\text{loop}}_{n,d} = \int \mathcal{A}^{\lambda\mu}_{n,d}\mathcal{A}^{\psi}_{n,d}\mathcal{A}^{J^A}_{n}\mathcal{A}^{\text{ghost}}\frac{df_0\, d\tau}{2\pi \text{Im}\tau}\prod_{r=1}^{n}\rho_r d\nu_r, \qquad \rho_r = e^{2\pi i\nu_r}, \quad w = e^{2\pi i\tau}$

Twistor bosonic contribution:

$$\mathcal{A}^{\lambda\mu}_{n,2} = \int \text{tr}\left(e^{2q_0}u^{a_0}\prod_{r=1}^{n}\exp\left\{ik_r\lambda^a(\rho_r)\bar{\omega}_{ra} + ik_r\mu^a(\rho_r)\bar{\pi}_{ra}\right\}w^{L_0}\right)$$
$$\times\prod_{r=1}^{n}\frac{dk_r}{k_r}\prod_{a=1}^{2}e^{-i\bar{\omega}_{ra}\pi_r{}^a}d\bar{\omega}_{ra}/d\gamma_S$$

Bosonic trace formula:

$$\text{tr}\left(e^{dq_0}u^{a_0}\prod_{j=1}^{n}e^{i\omega_j Z(\rho_j)}w^{L_0}\right) = u^{(d+1)/2}\prod_{i=1}^{d}\delta\left(\sum_{j=1}^{n}F_i^d(\hat{\rho}_j, w)\omega_j\right)$$

where $\hat{\rho}_j = u^{-\frac{1}{2}}\rho_j = e^{2\pi i\hat{\nu}_j}, \quad \hat{\nu}_j = \nu_j + if_0/4\pi, \quad u = e^{f_0}$.

where

$$F_k^d(\rho, w) = \rho^{d/2} w^{d/8-k/4}\ \theta \begin{bmatrix} 2k/d-1 \\ 0 \end{bmatrix} (-d\nu, d\tau)$$

$$F_1^2(\rho, w) = \rho\theta_3(2\nu, 2\tau), \qquad F_2^2(\rho, w) = w^{-\frac{1}{4}} \rho\theta_2(2\nu, 2\tau).$$

Expressing the second delta functions as Fourier transforms on $\tilde{\lambda}_i^a$,

$$\mathcal{A}_{n,2}^{\lambda\mu} = u^6 \int \prod_{i,a=1}^{2} \delta\left(\sum_r k_r F_i^2(\hat{\rho}_r, w)\bar{\pi}_{ra}\right)$$
$$\times \exp\left(i \sum_{r=1}^{n} \sum_{a=1}^{2} \left[\sum_{i=1}^{2} k_r \tilde{\lambda}_i^a F_i^2(\hat{\rho}_r, w)\bar{\omega}_{ra} - i\bar{\omega}_{ra}\pi_r{}^a\right]\right) \prod_{a=1}^{2} d^2\tilde{\lambda}^a \prod_{r=1}^{n} \frac{dk_r}{k_r} \prod_{a=1}^{2} d\bar{\omega}_{ra}/d\gamma_S$$

Performing the k_r integrations,

$$\mathcal{A}^{\lambda\mu}_{n,2} = \delta^4(\Sigma\pi_r\bar{\pi}_r)u^6 \int (\tilde{\lambda}^1_1\tilde{\lambda}^2_2 - \tilde{\lambda}^1_2\tilde{\lambda}^2_1)^2 \prod_{r=1}^{n} \frac{1}{{\pi_r}^1}\delta\left(\tilde{\xi}(\hat{\nu}_r,\tau){\pi_r}^1 - {\pi_r}^2\right) \prod_{a=1}^{2} d^2\tilde{\lambda}^a/d\gamma_S$$

Use the delta functions $\delta\left(\tilde{\xi}(\hat{\nu}_r,\tau){\pi_r}^1 - {\pi_r}^2\right)$ to do the integrations over ν_r,

$$\tilde{\xi}(\hat{\nu}_r,\tau) \equiv \frac{\tilde{\lambda}^2_1\xi(\hat{\nu}_r,\tau) + \tilde{\lambda}^2_2}{\tilde{\lambda}^1_1\xi(\hat{\nu}_r,\tau) + \tilde{\lambda}^1_2} = \frac{\pi^2_r}{\pi^1_r}, \qquad \text{for } \xi(\hat{\nu}_r,\tau) = \frac{F^2_1(\hat{\rho}_r,w)}{F^2_2(\hat{\rho}_r,w)} = \frac{\theta_3(2\hat{\nu}_r,2\tau)}{w^{-\frac{1}{4}}\theta_2(2\hat{\nu}_r,2\tau)} \equiv \xi_r$$

From the bilinear transformation $\xi_r \to \pi^2_r/\pi^1_r$, the invariant measure γ_S:

$$\frac{d^2\tilde{\lambda}^a}{(\tilde{\lambda}^1_1\tilde{\lambda}^2_2 - \tilde{\lambda}^1_2\tilde{\lambda}^2_1)^2} = \frac{d\xi_1 d\xi_2 d\xi_3}{(\xi_1-\xi_2)(\xi_2-\xi_3)(\xi_3-\xi_1)} d\gamma_S,$$

- $$\mathcal{A}^{\text{loop}}_{n,2} = \frac{\langle 1,2\rangle^4\delta^4(\Sigma\pi_r\bar{\pi}_r)}{\langle 1,2\rangle\langle 2,3\rangle\ldots\langle n,1\rangle} \int \frac{(\xi_3-\xi_4)}{(\xi_3-\xi_1)}\left[\prod_{r=4}^{n}\frac{(\xi_r-\xi_{r+1})}{\xi'_r}\right]\mathcal{A}^{J^A}_n \mathcal{A}^{\text{ghost}}\rho_\Pi d\nu_1 d\nu_2 d\nu_3 \frac{df_0 d\tau}{2\pi \text{Im}\tau}$$

Twistor fermionic contribution:

$$\mathcal{A}^{\psi}_{n,2} = k_1^4 k_2^4 \text{tr}\Big(e^{2q_0} u^{a_0} (-1)^{a_0} \psi^1(\rho_1)\psi^2(\rho_1)\psi^3(\rho_1)\psi^4(\rho_1)\psi^1(\rho_2)\psi^2(\rho_2)\psi^3(\rho_2)\psi^4(\rho_2) w^{L_0}\Big)$$

Ghost contribution to the loop integrand:

$$\mathcal{A}^{\text{ghost}} = \eta(\tau)^4$$

The partition function for a general fermionic "b, c" system with conformal dimensions λ and $1-\lambda$ respectively is

$$\text{tr}\Big(b_0 c_0 \omega^{L_0 - \frac{c}{24}} (-1)^F\Big) = \omega^{-\frac{c}{24}} \omega^{\frac{1}{2}\lambda(1-\lambda)} \prod_{n=1}^{\infty} (1-\omega^n)^2 = \omega^{\frac{1}{12}} \prod_{n=1}^{\infty} (1-\omega^n)^2 = \eta(\tau)^2$$

$$c = 12\lambda(1-\lambda) - 2$$

$$L(z) = -\lambda \,{}^{\times}_{\times} b(z) c'(z) {}^{\times}_{\times} + (1-\lambda) \,{}^{\times}_{\times} b'(z) c(z) {}^{\times}_{\times}$$

Current algebra tree:

$$\langle 0|J^{a_1}(z_1)J^{a_2}(z_1)\ldots J^{a_4}(z_4)|0\rangle = \frac{\sigma^{a_1a_2a_3a_4}}{(z_1-z_2)(z_2-z_3)(z_3-z_4)(z_4-z_1)} + \text{perm}$$

Current algebra loop:

$$\begin{aligned}
&\text{tr}\Big(J^a(\rho_1)J^b(\rho_2)J^c(\rho_3)J^d(\rho_4)w^{L_0}\Big)\,\rho_1\rho_2\rho_3\rho_4 \\
&= \Big\{\delta^{ab}\delta^{cd}\,\Big(k^2\chi(\tau)\,[(\chi_F^{12})^2+f(\tau)][(\chi_F^{34})^2+f(\tau)] - \chi^{(2)}(\tau)^2/\chi(\tau)\Big) + \text{perm}\Big\} \\
&+\text{tr}(J_0^aJ_0^bJ_0^cJ_0^dw^{L_0})_{\mathbf{s}} - \frac{1}{16}(\sigma^{abcd})_{\mathbf{s}} \\
&-\Big(\sigma^{abcd}\tfrac{1}{2}\chi(\tau)\,\Big\{\chi_F^{12}\chi_F^{23}\chi_F^{34}\chi_F^{41} + \frac{f(\tau)}{8\pi^2}\Big[(\zeta^{12}+\zeta^{23}+\zeta^{34}+\zeta^{41})^2 \\
&\qquad -\mathcal{P}_{12}-\mathcal{P}_{23}-\mathcal{P}_{34}-\mathcal{P}_{41}\Big]\Big\} + \text{perm}\Big)
\end{aligned}$$

where

$$\sigma^{abcd} = \text{tr}(T^aT^bT^cT^d)$$

$$\chi(\tau) = \mathrm{tr} w^{L_0}, \qquad \mathrm{tr}(J_0^a J_0^b w^{L_0}) = \delta^{ab}\chi^{(2)}(\tau), \qquad f(\tau) = \frac{\chi^{(2)}(\tau)}{k\chi(\tau)} + \frac{\theta_3''(0,\tau)}{4\pi^2\theta_3(0,\tau)}$$

$$\chi_F^{ij} = \chi_F(\nu_j - \nu_i, \tau) \quad \zeta^{ij} = \zeta(\nu_j - \nu_i, \tau),$$

$$\chi_F(\nu,\tau) = \frac{i}{2}\theta_2(0,\tau)^4\theta_4(0,\tau)^4\frac{\theta_3(\nu,\tau)}{\theta_1(\nu,\tau)}$$

$$\zeta(\nu,\tau) = \frac{\theta_1'(\nu,\tau)}{\theta_1(\nu,\tau)} - \nu\,\frac{\theta_1'''(0,\tau)}{\theta_1'(0,\tau)}$$

$$P(\nu,\tau) = -\zeta'(\nu,\tau) = -4\pi^2\chi_F^2(\nu,\tau) + \frac{\pi^2}{3}(\theta_2(0,\tau)^4 - \theta_2(0,\tau)^4).$$

Twistor string loop

$$\mathcal{A}_{4,2}^{\text{loop}} = -\frac{\langle 1,2\rangle^4 \delta^4(\Sigma \pi_r \bar{\pi}_r)}{\langle 1,2\rangle\langle 2,3\rangle\langle 3,4\rangle\langle 4,1\rangle} \frac{s}{t} \int \delta\left(\frac{(\xi_1-\xi_2)(\xi_3-\xi_4)}{(\xi_1-\xi_4)(\xi_3-\xi_2)} + \frac{s}{t}\right)$$

$$\cdot \mathcal{A}_4^{J^A} \eta(\tau)^4 \prod_{r=1}^{4} \rho_r d\nu_r \frac{df_0 d\tau}{2\pi \text{Im}\tau}$$

where

$\xi_r = \theta_3(2\hat{\nu}_r, 2\tau)/\theta_2(2\hat{\nu}_r, 2\tau)$,

$\hat{\nu}_r = \nu_r + i f_0/4\pi$.

$\frac{\langle 1,2\rangle\langle 3,4\rangle}{\langle 1,4\rangle\langle 3,2\rangle} = -\frac{s}{t}$

$\mathcal{A}_4^{J^A}$ is the current algebra loop.