THE POISSON BOUNDARY OF LAMPLIGHTER RANDOM WALKS ON TREES

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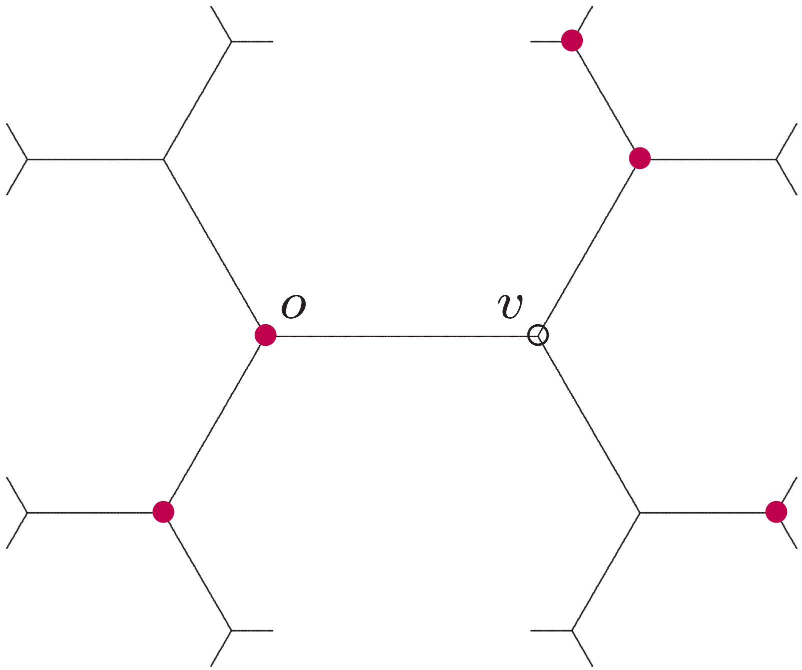
1. Introduction

$\mathbb{T}$ homogeneous tree, degree $q+1 \geq 3$.
Lamp at each vertex, states 0 (off) and 1 (on).
A Lamplighter performs random walk on $\mathbb{T}$. Makes random moves and / or changes of state of the lamp at current position (or nearby).



Configuration on X : function $\eta : X \to \{0,1\}$ with finite support

$$\mathit{supp}\,\eta = \{x \in X : \eta(x) \neq 0\}\,.$$

At each step, we have to observe the pair (η, x), where η is the current configuration of the lamps and x is the current position of the lamplighter.

Let $\mathcal{C} = \{\text{configurations}\}$.
The state space of our random process is $\mathcal{C} \times X$.

Let $p\Big((\eta, x), (\eta', x')\Big)$ be the transition probabilities of the lamplighter random walk $Z_n = (Y_n, X_n)$.

Assumed to be irreducible and space homogeneous :
$\mathcal{F}$ group of isometries of $\mathbb{T}$ that acts transitively
(e.g. free group, or affine group of $\mathbb{T}$).

$\mathcal{C}$ is also (commutative) group, pointwise addition mod 2.

$\mathcal{F}$ acts on $\mathcal{C}$ by translation

$$T_g\eta(x) = \eta(g^{-1}x)\,, \quad g \in \mathcal{F}\,,\ x \in \mathbb{T}\,.$$

Semidirect product

$$\mathcal{G} = \mathcal{C} \rtimes \mathcal{F}$$

acts transitively on our state space by

$$(\phi, g)(\eta, x) = (\phi + T_g\eta, gx)\,, \quad g \in \mathcal{F}\,,\ \phi \in \mathcal{C}\,.$$

Space homogeneity:

$$p\Big((\phi,g)(\eta,x),(\phi,g)(\eta',x')\Big) = p\Big((\eta,x),(\eta',x')\Big)$$

Basic example: (X_n) simple random walk on $\mathbb{T}$.

"Walk or switch" At each step, lamplighter tosses coin.

"Heads" – (s)he walks, lamps unchanged;

"tails" – (s)he modifies lamp at current position, does not move.

Underlying graph $\mathbb{L}$: lamplighter graph over $\mathbb{T}$. ($\to$ graph metric)

2. Behaviour at infinity

Under above assumptions: (X_n) is Markov chain on $\mathbb{T}$, factor chain

$$p(x, x') = \sum_{\eta'} p\Big((\eta, x), (\eta', x')\Big)$$

space homogeneous (invariant under $\mathcal{F}$).

Known to be transient (visits any finite set only finitely often).

$\Rightarrow$ There is a random limit configuration

$$Y_\infty = \lim_{n\to\infty} Y_n \in \widehat{\mathcal{C}} = \Big\{\zeta : X \to \{0, 1\}\Big\}$$

(not necessarily finitely supported).

$Y_\infty(x)$ is the definite state of the lamp at x.

Proposition. If (X_n) has finite first moment

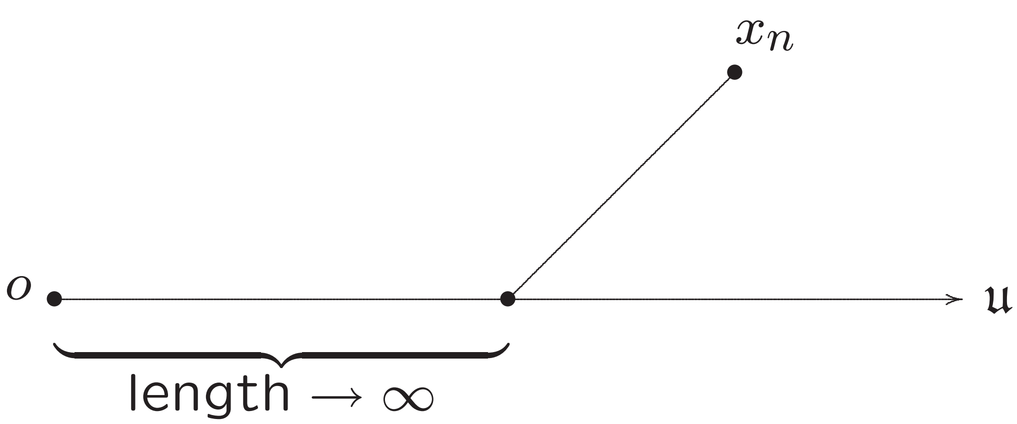
$$m_{\mathbb{T}} = \sum_x d_{\mathbb{T}}(x,x')\, p(x,x')$$

then X_n converges a.s. to a random end $X_\infty \in \partial\mathbb{T}$.

Due to [Cartwright & Soardi 1989] when $\mathcal{F}$ fixes no end (no moment condition needed), [Cartwright, Kaimanovich & Woess 1994] when $\mathcal{F}$ fixes an end.

Ends (boundary points) $\mathfrak{u} \in \partial\mathbb{T}$ are represented by geodesic rays $\mathfrak{u} = [o = x_0, x_1, \ldots]$ starting from a root $o \in \mathbb{T}$.

Convergence of $x_n \in \mathbb{T}$ to $\mathfrak{u} \in \partial\mathbb{T}$:



$\widehat{\mathbb{T}}$ end compactification of tree

$\widehat{\mathcal{C}}$ compactification of $\mathcal{C}$ (pointwise convergence)

$\Rightarrow \widehat{\mathcal{C}} \times \widehat{T}$ is natural compactification $\widehat{\mathbb{L}}$ of the lamplighter graph $\mathbb{L}$.

Theorem. [Karlsson & Woess, 2006-07] If the lamplighter random walk $Z_n = (Y_n, X_n)$ has finite first moment

$$m_{\mathbb{L}} = \sum_{(\eta',x')} d_{\mathbb{L}}\Big((\eta,x),(\eta',x')\Big)\, p\Big((\eta,x),(\eta',x')\Big)$$

then (Z_n) converges a.s. to a limit random variable

$$Z_\infty = (Y_\infty, X_\infty) \in \partial\mathbb{L} = \widehat{\mathbb{L}} \setminus \mathbb{L}\,.$$

If $X_\infty = \mathfrak{u} \in \partial\mathbb{T}$ then the limit configuration $Y_\infty = \zeta \in \widehat{\mathcal{C}}$ accumulates only at $\mathfrak{u}$.

(i.e.: $w_n \in \mathit{supp}\,\zeta$ all distinct $\Rightarrow w_n \to \mathfrak{u}$.)

3. Poisson boundary

$Z_n = (Y_n, X_n)$ is transient $\equiv$ goes off to ∞.

Topology of $\widehat{\mathbb{L}}$ provides model $\partial\mathbb{L}$ at infinity for a finer way to distinguish how Z_n goes off.

Is this the finest model ?

Preliminaries:

Set $\Pi = \bigcup_{\mathfrak{u}\in\partial\mathbb{T}} \mathcal{C}_{\mathfrak{u}} \times \{\mathfrak{u}\}$, where $\mathcal{C}_{\mathfrak{u}} = \{\zeta \in \widehat{\mathcal{C}} : \zeta \text{ accumulates only at } \mathfrak{u}\}$.

$\mathcal{C}_{\mathfrak{u}}$ is dense in $\widehat{\mathcal{C}}$. Closure of Π is subset $\widehat{\mathcal{C}} \times \partial\mathbb{T}$ of $\partial\mathbb{L}$.

$\nu_{(\eta,x)}$ distribution of Z_∞ given that lamplighter random walk starts at (η, x). Probability measure on Π (resp. $\widehat{\mathcal{C}} \times \partial\mathbb{T}$).

$\nu_{(\eta,x)}$ is the image of $\nu = \nu_{(\mathbf{0},o)}$ under group element $(\eta, g) \in \mathcal{G}$, where $g \in \mathcal{F}$ with $go = x$.

Here, action of $\mathcal{G} \ni (\eta, g)$ extends to $\widehat{\mathbb{L}}$ by

$$(\eta, g)(\zeta, \mathfrak{u}) = (\eta + T_g\zeta, g\mathfrak{u}) .$$

Leaves Borel set $\Pi \subset \widehat{\mathbb{L}}$ invariant.

Indeed, if $\zeta \in \mathcal{C}_{\mathfrak{u}}$, where $\mathfrak{u} \in \partial\mathbb{T}$, then $\eta + T_g\zeta \in \mathcal{C}_{g\mathfrak{u}}$, since adding η modifies $T_g\zeta$ only in finitely many points.

Our boundary is the probability space (Π, ν) [equivalently $(\widehat{\mathbb{L}}, \nu)$].

Is it the Poisson boundary ?
that is, the finest model of a probability space at infinity of $\mathbb{L}$ for distinguishing the possible limiting behaviour of (Z_n) ?

Various rigorous equivalent definitions, see [Kaimanovich & Vershik, 1983] (for r.w. on discrete groups).

- The space of ergodic components in the trajectory space of the random walk.

- The Martin boundary of the random walk together with the harmonic measure(s).

- Every bounded harmonic function h on $\mathbb{L}$ with respect to the transition matrix has a unique integral representation

$$h(\eta, x) = \int_{\Pi} \varphi \, d\nu_{(\eta,x)} \,, \quad \varphi \in L^{\infty}(\Pi, \nu)\,.$$

Answer (yes) relies on strip criterion of [Kaimanovich, 2000], adapted to space-homogeneous random walks by [Kaimanovich & Woess, 2002].

Lemma. [Woess, 1989; 2000] The measure on $\mathbb{L}$ defined by

$$m(\eta, x) = |\mathcal{F}_x\, o| / |\mathcal{F}_o\, x|$$

is invariant for the random walk.
(If $\mathcal{F}$ is discrete, e.g. free group, then m is counting measure.)

The m-reversal of the random walk is

$$\widehat{p}\Big((\eta, x), (\eta', x')\Big) = \frac{m(\eta', x')\; p\Big((\eta', x'), (\eta, x)\Big)}{m(\eta, x)}$$

Theorem. [Karlsson & Woess, 2006-07] (for $\mathcal{F}$ discrete),
[Sava, 2007] (for general $\mathcal{F}$)
If both the lamplighter random walk and its m-reversal have finite first moments on $\mathbb{L}$, then (Π, ν) is the Poisson boundary of the LL walk.

4. Strip criterion

We have boundaries (Π, ν) for the LL random walk and $(\widehat{\Pi} = \Pi, \widehat{\nu})$ for its m-reversal.

Proposition. Suppose that there is a measurable $\mathcal{G}$-equivariant map S assigning to $(\nu \times \widehat{\nu})$-almost every pair of points $(\beta, \widehat{\beta}) \in \Pi \times \widehat{\Pi}$ a non-empty "strip" $S(\beta, \widehat{\beta}) \subset \mathbb{L}$ such that for the ball $B(o, n)$ of radius n in the metric of $\mathbb{L}$,

$$\frac{1}{n} \log \left| S(\beta, \widehat{\beta}) \cap B(o, n) \right| \to 0 \quad \text{as} \quad n \to \infty ,$$

then (Π, ν) and $(\widehat{\Pi}, \widehat{\nu})$ are the Poisson boundaries of the LL random walk and its m-reversal, respectively.

$\mathcal{G}$-equivariant means: if $\mathfrak{g} = (\eta, g) \in \mathcal{G}$ then $\mathfrak{g} \cdot S(\beta, \widehat{\beta}) = S(\mathfrak{g}\beta, \mathfrak{g}\widehat{\beta})$.

Application of strip criterion:

μ distribution of X_∞ on $\partial\mathbb{T}$ given ; $X_0 = o$ image of ν under projection $\Pi \to \partial\mathbb{T}$.

Known. $(\partial\mathbb{T}, \mu)$ is Poisson boundary of the random walk X_n on $\mathbb{T}$.

Case 1. $\mathcal{F}$ fixes no end of $\mathbb{T}$
(in particular: $\mathcal{F}$ discrete, e.g. free)

Proposition. [Woess, 1989], argument going back to [Furstenberg, 1972].

$$\textit{supp}\,\mu = \partial\mathbb{T} \quad \text{and} \quad \mu(\mathfrak{u}) = 0 \quad \forall\, \mathfrak{u} \in \partial\mathbb{T}\,.$$

Same holds for $\widehat{\mu}$ corresponding to reversed random walk.

Now let $\beta = (\zeta, \mathfrak{u})$, $\widehat{\beta} = (\widehat{\zeta}, \widehat{\mathfrak{u}}) \in \Pi$.
By above,

$$\nu \times \widehat{\nu}\Big(\{(\beta, \widehat{\beta}) \in \Pi \times \Pi : \mathfrak{u} = \widehat{\mathfrak{u}}\}\Big) = 0 .$$

May assume $\mathfrak{u} \neq \widehat{\mathfrak{u}}$.
Let $x \in [\mathfrak{u}\,,\,\widehat{\mathfrak{u}}]$ be a vertex on the geodesic in $\mathbb{T}$ between the two ends.
Let $C(x, \mathfrak{u})$ be component of $\mathfrak{u}$ in $\mathbb{T} \setminus \{x\}$, analogously $C(x, \widehat{\mathfrak{u}})$.
Then

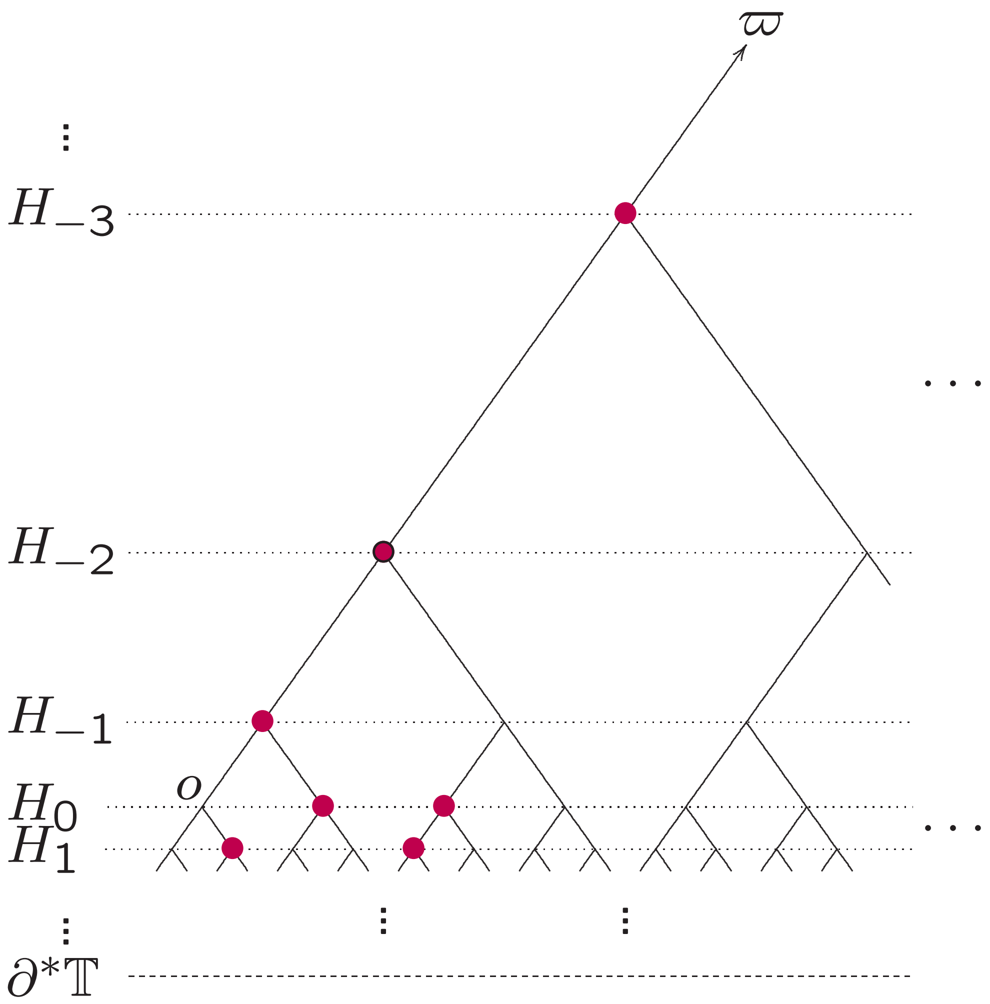
$$\eta_x = \zeta|_{C(x,\widehat{\mathfrak{u}})} + \widehat{\zeta}|_{C(x,\mathfrak{u})} \in \mathcal{C}$$

since ζ acccumulates only at $\mathfrak{u}$ and $\widehat{\zeta}$ only at $\widehat{\mathfrak{u}}$.

$$S(\beta, \widehat{\beta}) = \Big\{(\eta_x, x) : x \in [\mathfrak{u}\,,\,\widehat{\mathfrak{u}}]\Big\}$$

fulfills all requirements of strip criterion.

Case 2. $\mathcal{F}$ fixes an end ϖ of $\mathbb{T}$



Natural projection onto $\mathbb{Z}$.

For LL walk $Z_n = (Y_n, X_n)$, base walk X_n projects to random walk $\mathfrak{h}(X_n)$ on $\mathbb{Z}$.

Let a be the drift of $\mathfrak{h}(X_n)$.

The reversed random walk has drift $\hat{a} = -a$.

Proposition. [Cartwright, Kaimanovich & Woess, 1994], [Brofferio, 2004]

Suppose X_n on $\mathbb{T}$ has finite first moment.

(1) If $a > 0$ then $X_n \to X_\infty \in \partial^*\mathbb{T}$ a.s., the distribution μ of X_∞ satisfies $\mathit{supp}\,\mu = \partial^*\mathbb{T}$ and $\mu(\mathfrak{u}) = 0 \ \forall\, \mathfrak{u} \in \partial^*\mathbb{T}$.

The Poisson boundary of X_n is $(\partial^*\mathbb{T}, \mu)$.

(2) If $a \leq 0$ then $X_n \to \varpi$ a.s.

The Poisson boundary of X_n is $(\{\varpi\}, \delta_\varpi)$.

Back to application of strip criterion for LL walk

(i) If $a > 0$ then the candidate for Poisson boundary of the LL walk is more precisely

$$\Pi^* = \bigcup_{\mathfrak{u} \in \partial^* \mathbb{T}} \mathcal{C}_{\mathfrak{u}} \times \{\mathfrak{u}\}$$

with the limit measure ν (as a measure on Π^*).

The candidate for Poisson boundary of the reversed LL walk is

$$\mathcal{C}_{\varpi} \times \{\varpi\}$$

with the limit measure $\widehat{\nu}$ supported by that set.

Thus, strips $S(\beta, \widehat{\beta})$ have to be constructed for $\beta = (\zeta, \mathfrak{u})$ and $\widehat{\beta} = (\widehat{\zeta}, \varpi)$ with $\mathfrak{u} \in \partial^* \mathbb{T}$, $\zeta \in \mathcal{C}_{\mathfrak{u}}$ and $\widehat{\zeta} \in \mathcal{C}_{\varpi}$.
The construction is exactly as above.

(ii) If $a < 0$ then we just have to exchange roles of LL walk and reversed LL walk.

(iii) If $a = 0$ then both candidate boundaries are $\mathcal{C}_\varpi \times \{\varpi\}$ with the corresponding limit measures ν and $\widehat{\nu}$.

Let $\beta = (\zeta, \varpi)$ and $\widehat{\beta} = (\widehat{\zeta}, \varpi)$ with both ζ, $\widehat{\zeta} \in \mathcal{C}_\varpi$ (distinct).

Let $C^*(x, \varpi)$ be the complement of $C(x, \varpi)$. Define

$$\eta_x = \zeta|_{C^*(x,\varpi)} - \widehat{\zeta}|_{C^*(x,\varpi)}\,.$$

Then

$$S(\beta, \widehat{\beta}) = \Big\{(\eta_x, x) : \eta_x \neq \mathbf{0}\Big\}$$

fulfills all requirements, since the set of all x with $\eta_x \neq \mathbf{0}$ is a subtree of $\mathbb{T}$ with the only end ϖ. Hence it has linear growth in $\mathbb{T}$, and so has $S(\beta, \widehat{\beta})$ in $\mathbb{L}$.