Limit laws for one-dimensional random walks in random environment in the Kesten-Kozlov-Spitzer regime

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Joint work with N. Enriquez and O. Zindy

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At each site $x \in \mathbb{Z}$, we choose $\omega_x \in (0,1)$ independently with the same law μ: we denote by

$$P = \mu^{\otimes \mathbb{Z}}$$

the law of the environment.

The law of the Markov chain (X_n) in environment (ω_x) is given by

$$P_\omega\left(X_{n+1} = x+1 \mid X_n = x\right) = \omega_x,$$

$$P_\omega\left(X_{n+1} = x-1 \mid X_n = x\right) = 1 - \omega_x.$$

We denote by

$$\mathbb{P}(\cdot) = E(P_\omega(\cdot))$$

the annealed law.

Beta case: $\mu \sim$Beta(α, β), i.e. has density

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\ x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}_{(0,1)}(x)$$

For $i \in \mathbb{Z}$ we set

$$\rho_i = \frac{1-\omega_i}{\omega_i}.$$

Theorem 1 *(Solomon, 75)*
i) If $E(\log(\rho_0)) < 0$, then

$$\lim_{n\to\infty} X_n = +\infty, \quad \mathbb{P} \text{ p.s.}$$

If $E(\log(\rho_0)) = 0$ then X_n is recurrent (Sinaï's walk)

ii) If $E(\rho_0) < 1$ then

$$\lim_{n\to\infty} \frac{X_n}{n} = v_\infty = \frac{1-E(\rho_0)}{1+E(\rho_0)}, \quad \mathbb{P} \text{ p.s.}$$

If $\mu \sim$Beta(α, β):

$$E(\log \rho_0) < 0 \text{ iff } \alpha > \beta.$$

$$E(\rho_0) < 1 \text{ iff } \alpha > \beta + 1$$

$$v_\infty = \frac{\alpha-\beta-1}{\alpha+\beta-1}$$

Finer behaviour is related to the function $s \mapsto E\left(\rho_0^s\right)$

and to the solution of the equation

$$E(\rho_0^\kappa) = 1.$$

Let $\tau(n) = \inf\{k,\ X_k = n\}$.

Theorem 2 *(Kesten-Kozlov-Spitzer, 75) Suppose the distribution of* $\log(\rho_0)$ *is non-arithmetic and there exists* $\kappa > 0$ *such that*

$$E(\rho_0^\kappa) = 1, \quad E(\rho_0^\kappa \log^+(\rho_0)) < \infty.$$

i) If $\kappa < 1$ *then*

$$\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow{law} L_\kappa$$

where L_κ *is a positive stable law with index* κ*.*
ii) For $\kappa = 1$

$$\frac{\tau(n)}{n \log n} \xrightarrow{law} c$$

where c *is a positive constant.*
iii) Fluctuation for $1 \leq \kappa < 2$*.*

For $\kappa < 1$, L_κ is determined by its Laplace transform

$$e^{-d\lambda^\kappa}$$

Question: what is the value of d?
KKS's proof is based on the tail estimate of the variable

$$R = \rho_1 + \rho_1\rho_2 + \cdots + \rho_1 \cdots \rho_n + \cdots$$

$$P(R \geq t) \sim C_K t^{-\kappa}$$

Theorem 3 *(Enriquez, S., Zindy) For $\kappa < 1$*

$$d = 2^\kappa \frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E(\rho_0^\kappa \log \rho_0)$$

i.e.

$$\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow{law} 2 \left(\frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E(\rho_0^\kappa \log \rho_0) \right)^{\frac{1}{\kappa}} \mathcal{S}_\kappa^{ca}$$

where $\mathcal{S}_\kappa^{ca}$ is the normalized stable distribution with index κ and Laplace transform $e^{-\lambda^\kappa}$.

$$\frac{X_n}{n^\kappa} \xrightarrow{law} \frac{\sin(\pi\kappa)}{2^\kappa \pi\kappa^2 C_K^2 E(\rho_0^\kappa \log \rho_0)} \left(\frac{1}{\mathcal{S}_\kappa^{ca}} \right)^{\frac{1}{\kappa}}$$

Corollary 1 (The Beta case): *If* $\mu \sim$*Beta*(α, β), *then*

$$\kappa = \alpha - \beta$$

$$d = \frac{\pi 2^{\alpha-\beta}}{\sin(\pi(\alpha-\beta))} \cdot \frac{\psi(\alpha) - \psi(\beta)}{B(\alpha, \beta)^2}$$

where $\psi(z) = (\log \Gamma(z))'$ *is the dilogarithm and* $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$.

This comes from a result of Chamayou et Letac

Proposition 1 *(Chamayou, Letac)*
In the Beta case

$$R \stackrel{law}{=} \frac{1 - W}{W}$$

where $W \sim$*Beta*$(\alpha - \beta, \beta)$.

This implies $C_K = \frac{1}{(\alpha-\beta)B(\alpha,\beta)}$

Remark: The r.v. R has the following probabilistic expression

$$R = E_\omega(\sharp\{\text{crossings } 1 \to 0\})$$

In general, we derived a probabilistic representation of C_K and d, which is easy to evaluate numerically.

The case $\kappa = 1$

In the case $\kappa = 1$, Kesten's constant is always explicit (this is due to Goldie) and equal to $C_K = 1/E(\rho_0 \log(\rho_0))$. We strongly believe that the constant c in the limit should be equal to $c = \frac{1}{2C_K^2 E(\rho_0 \log \rho_0)}$, which would give the remarkably simple result

$$X_n/(\frac{n}{\log n}) \to E(\rho_0 \log(\rho_0))/2$$

in probability. We still have small technical problems to treat this critical case!

Sinai's potential

$$V_x(\omega) = \begin{cases} \sum_{k=1}^{x} \log \rho_k, & x \geq 0 \\ V_0 = 0 & \\ V_x(\omega) = \sum_{k=x+1}^{0} \log \rho_k, & x \leq 0 \end{cases}$$

$E(\log(\rho_0)) < 0$ implies that $\lim_\infty V_k = -\infty$. The random walk X_n descends along the potential (V_x).

For example if $a < b < c$

$$P_\omega^b(\tau(a) < \tau(c)) = \frac{\sum_{k=b}^{c-1} e^{V_k}}{\sum_{k=a}^{c-1} e^{V_k}}$$

and if $c > 0$

$$E_\omega^0(\tau(c)) = \sum_{\substack{i \leq c-1 \\ 0 \leq j \leq c-1,\, j \geq i}} e^{V_j - V_i}.$$

Weak descending ladder epochs : $e_0 = 0$

$$e_{i+1} = \inf\{k > e_i,\ V_k \leq V_{e_i}\}$$

The height of the excursion

$$H_i = \sup\{V_k - V_{e_i},\ k \in [e_i, e_{i+1}]\}.$$

$(H_i)_{i \geq 0}$ is an iid sequence and (Iglehart)

$$P(H_0 \geq h) \sim C_I e^{-\kappa h},$$

where

$$C_I = \frac{(1 - E(e^{\kappa V(e_1)}))^2}{\kappa E(\rho_0^\kappa \log(\rho_0)) E(e_1)}.$$

This is related to the renewal theory and to the tail of the absolute maximum $S = \max\{V_k,\ k \geq 0\}$

$$P(S \geq h) \sim C_F e^{-\kappa h},$$

where

$$\frac{C_I}{C_F} = 1 - E(e^{\kappa V(e_1)})$$

We consider $\tau(e_n)$ the time to cross n excursions and

$$\mathbb{E}(e^{-\lambda\frac{\tau(e_n)}{n^{1/\kappa}}})$$

We have

$$\tau(e_n) = \tau(0, e_1) + \tau(e_1, e_2) + \cdots + \tau(e_{n-1}, e_n).$$

The time $\tau(e_i, e_{i+1})$ is roughly of order e^{H_i}, hence

$$\mathbb{P}(\tau(e_i, e_{i+1}) \geq t) \asymp t^{-\kappa}$$

and $\tau(e_i, e_i + 1)$ has a heavy tail for $\kappa \leq 1$. Hence, only a few count, well separated, and we can consider them as independent.
This leads to

$$\mathbb{E}\left(e^{-\lambda\frac{\tau(e_1)}{n^{1/\kappa}}} \mid V_k \geq 0 \; \forall k \leq 0\right)^n$$

since $P(\cdot \mid V_k \geq 0 \; \forall k \leq 0)$ is the shape of the potential around a ladder time e_i.

The RW (X_n) tries a geometric number of times N to reach the level e_1 and finally succeed

$$\tau(e_1) = F_1 + \cdots + F_N + S$$

When H is large the parameter of N is close to 1 : we can neglect S and consider $\tau(e_1)$ as a geometric r.v. with parameter $E_\omega(\tau(e_1))$.

$$E_\omega(\tau(e_1)) \sim E_\omega(F) E_\omega(N) \sim 2e^H M_1 M_2,$$

where

$$M_1 = \sum_{k=-\infty}^{T_H} e^{-V_k}, \quad M_2 = \sum_{k=0}^{e_1} e^{V_k - H},$$

where $T_H = \inf\{k \geq 0, V_k = H\}$ is the first time where the level H is reached. This analysis leads to

$$E\left(\int_0^\infty e^{-u} e^{-2\frac{u\lambda}{n^{1/\kappa} e^H M_1 M_2}} du |\ V_k \geq 0\ \forall k \leq 0\right)^n$$

$$= \left(1 - \int_0^\infty e^{-u}(1 - H(\frac{u\lambda}{n^{1/\kappa}}))du\right)^n$$

where

$$H(v) = E\left(e^{-2vZ} | \, V_k \geq 0 \; \forall k \leq 0\right)$$

and

$$Z = e^{H} M_1 M_2$$

By Tauber theorem, this leads to understand the tail of Z. Since only the large H count we can freely condition by the event $\{H = S\}$, where $S = \max\{V_k, \; \forall k \geq 0\}$ and we set

$$\mathcal{I} = \{H = S\} \cap \{V_k \geq 0 \; \forall k \leq 0\}.$$

we also replace M_2 by the full sum

$$M_2 = \sum_{k=0}^{\infty} e^{V_k - H}.$$

Under $P(\cdot|\mathcal{I})$ we have the symmetry

$$(V_k) \stackrel{law}{=} (H - V_{T_H - k})$$

and

$$M_1 \stackrel{law}{=} M_2$$

In fact, for large H, M_1, M_2 and H are asymptotically independent (this is obtained by a delicate almost coupling argument).

$$P(e^H M_1 M_2 \geq t \mid \mathcal{I}) \sim C_I E(M_1^{-\kappa})^2$$

To understand the asymptotic shape of M_1, we use the Girsanov transform: let

$$\tilde{\mu}(d\rho) = \rho^\kappa \mu(d\rho)$$

and $\tilde{P}$ the associated law on the environment. When H is large, the path $(V_0, \cdots, V_{T_H})$ behaves like (V_k) under $\tilde{P}(\cdot | V_k > 0 \; \forall k > 0)$. Hence, we have

$$P(Z \geq t \mid \mathcal{I}) \sim C_Z t^{-\kappa}$$

where

$$C_Z = C_I E(M^{-\kappa})^2$$

where

$$M = \sum_{-\infty}^{\infty} e^{-V_k}$$

and where $(V_k)_{k \leq 0}$ is distributed under $P(\cdot | V_k \geq 0 \; \forall k \leq 0)$ and $(V_k)_{k \geq 0}$ is distributed under $\tilde{P}(\cdot | V_k > 0 \; \forall k > 0)$.

This gives an expression of the limit law in terms of the $E(M^{-\kappa})$ and the constant C_I (which is explicit).

Remark that the Kesten r.v.

$$\begin{aligned} R &= 1 + \rho_1 + \rho_1\rho_2 + \cdots \\ &= e^S M_2 \end{aligned}$$

Using similar arguments we can get

$$P(R \geq t) \sim C_K t^{-\kappa}$$

where

$$C_K = C_F E(M^{-\kappa})$$

This relates the constant C_Z to C_K and gives the result.