BRANCHING BROWNIAN MOTION: THE PERILS OF A QUADRATIC POTENTIAL

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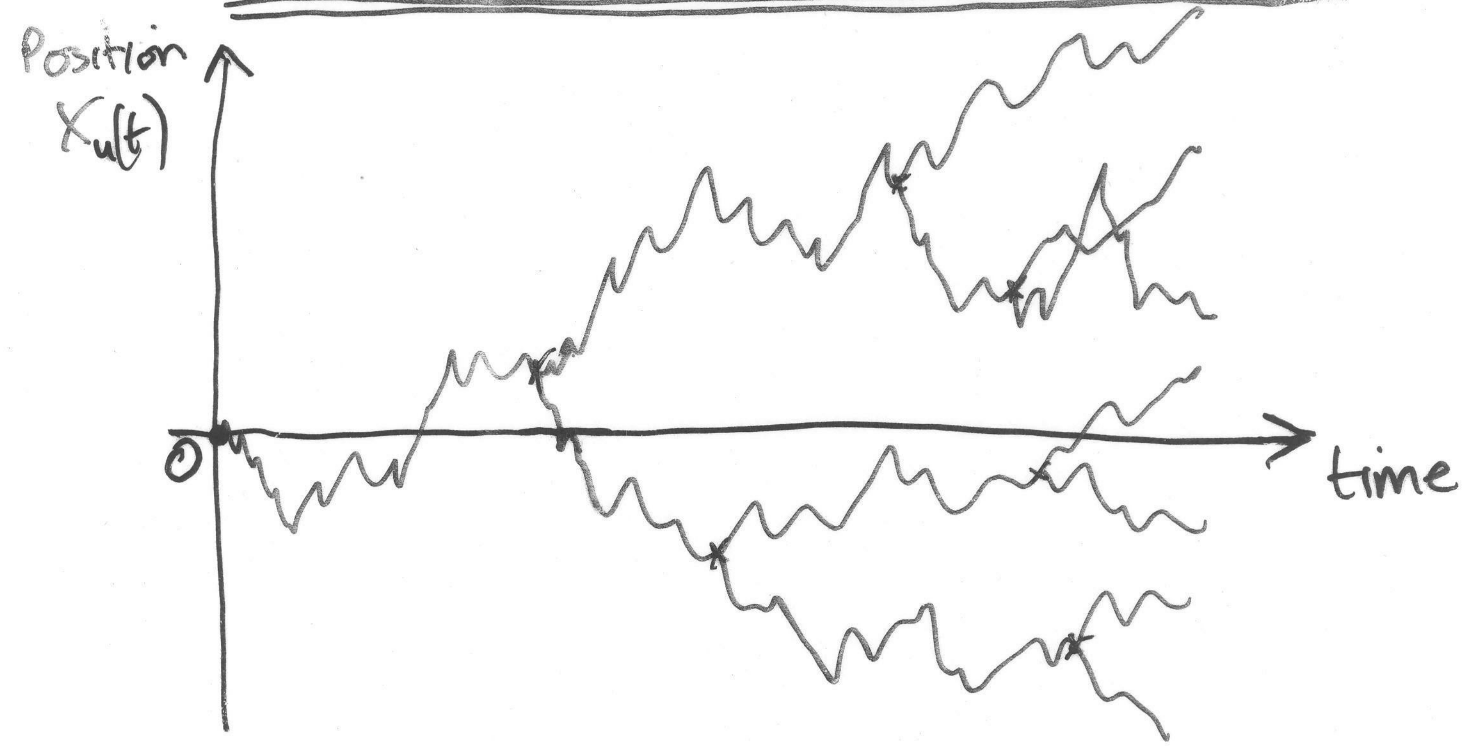
WORK IN PROGRESS !

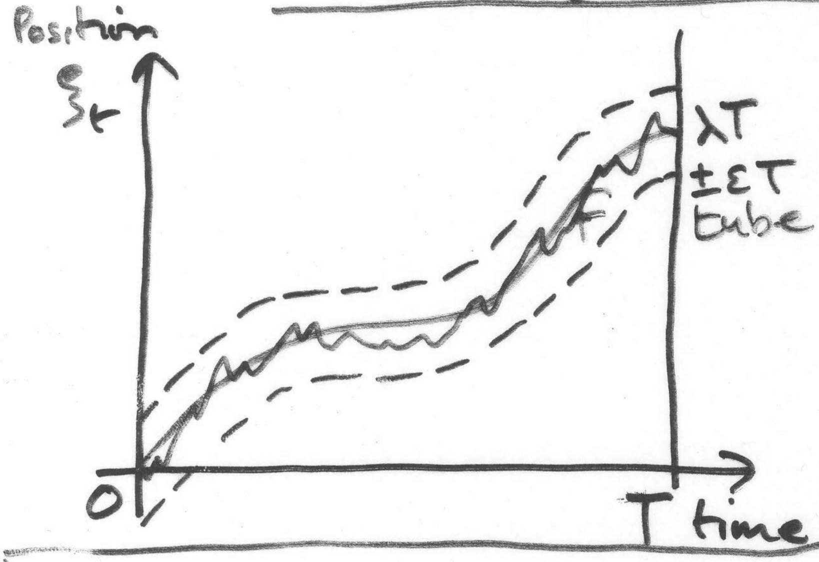
BRANCHING BROWNIAN MOTION (BBM)



- Branching rate $\beta(x)$
  - STANDARD BBM, $\beta(x) \equiv r$
  - BBM WITH QUADRATIC POTENTIAL, $\beta(x) = \beta x^2$ $(\beta > 0)$
- Binary splitting, indep. BMs from birth position
- P^x probs. from single initial particle at x
- N_t set of individuals at time t, Ulam–Harris labels eg. $\emptyset 21$ is first child of second child of original
- $\mathbb{X}_t := \{X_u(t) : u \in N_t\}$ BBM configuration at time t
- $X_u(s)$ is spatial position of unique ancestor of $u \in N_t$ that is alive at time $s \leq t$

QUESTIONS: • Probability particles follow 'difficult' paths?
• Numbers of particles following 'easy' paths?

Path Large Deviations for BM



$(\xi_t)_{t \geq 0}$ BM under $\widehat{P}$

$f : [0,T] \to \mathbb{R}$

$f(0) = 0, \ f(T) = \lambda T$

$$\widehat{P}\left(\begin{array}{l} \xi \text{ follows 'close'} \\ \text{to path on } [0,T] \end{array} \right) \sim e^{-\frac{1}{2}\int_0^T f'(s)^2 ds}$$

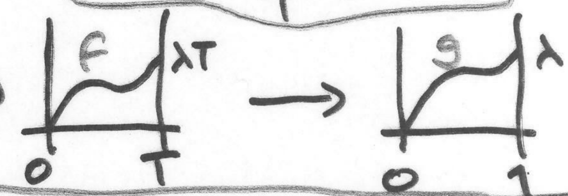
SCALING

$$\xi^T(s) := \frac{\xi_{sT}}{T}$$

$g : [0,1] \to \mathbb{R}$

$g(s) = \frac{1}{T} f(sT)$

$C_0[0,1]$ cts f^ns on $[0,1]$, $f(0)=0$



$$I(g) := \begin{cases} \frac{1}{2}\int_0^1 g'(s)^2 ds & \text{when } g \text{ abs. cts \& } g' \text{ sq. integ} \\ +\infty & \text{otherwise} \end{cases}$$

SCHILDER'S THEOREM

IF C closed subset of $C_0[0,1]$

$$\limsup_{T \to \infty} \frac{1}{T} \ln \widehat{P}(\xi^T \in C) \leq -\inf_{g \in C} I(g)$$

IF V is open subset of $C_0[0,1]$

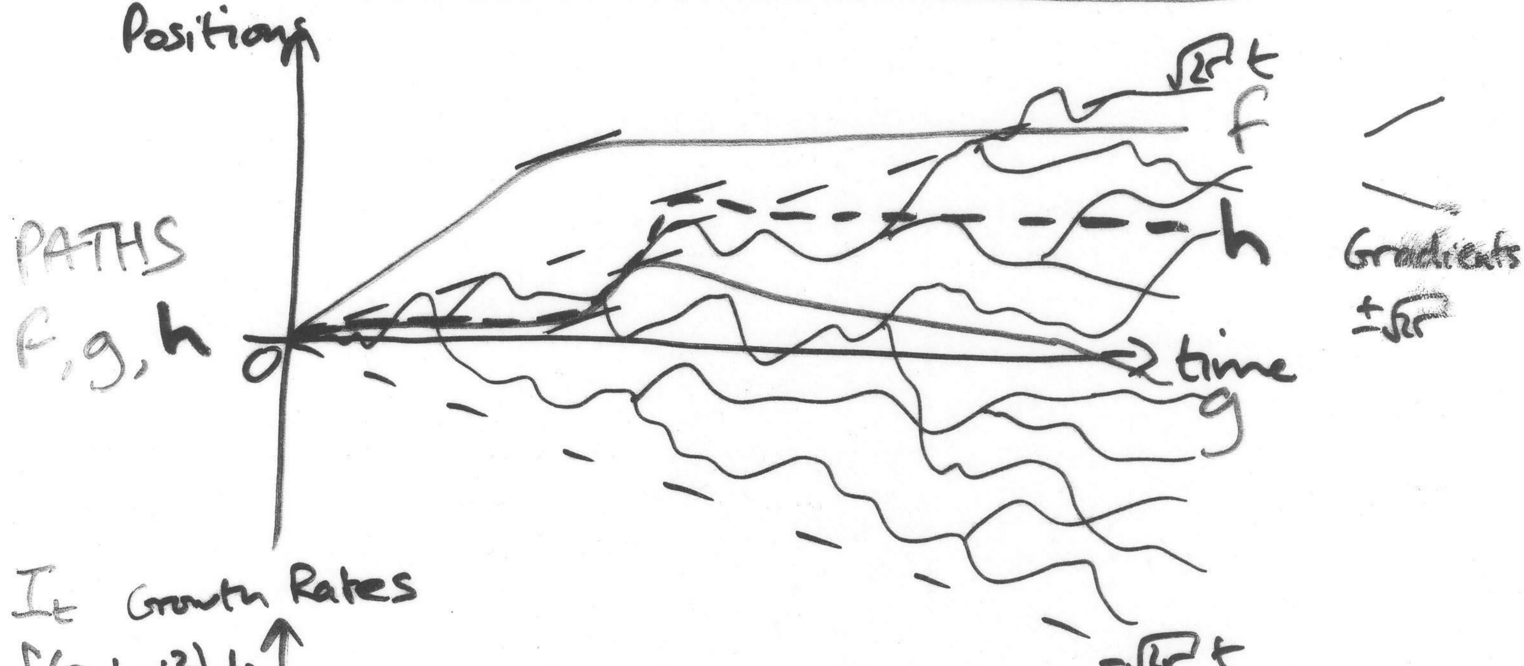
$$\liminf_{T \to \infty} \frac{1}{T} \ln \widehat{P}(\xi^T \in V) \geq -\inf_{g \in V} I(g)$$

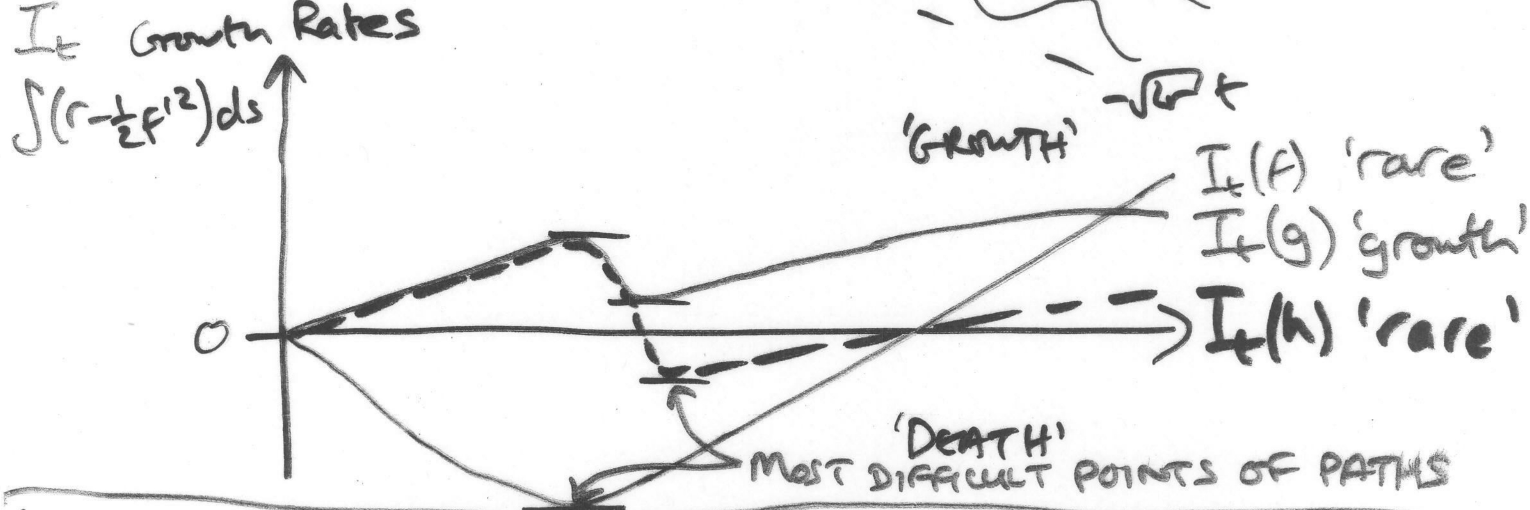
EXPECTATION GUESS : STANDARD BBM

$\beta(x) \equiv r$

$$\mathbb{E}\left(\#\text{ follow 'close' to } f\right) = e^{rt}\,\mathbb{P}\left(\begin{array}{c}\text{'typical' particle}\\ \text{follows 'close' to } f\end{array}\right)$$

$$\sim \exp\left\{\int_0^t \left(r - \tfrac{1}{2} f'(s)^2\right) ds\right\} =: \exp\left\{I_t(f)\right\}$$





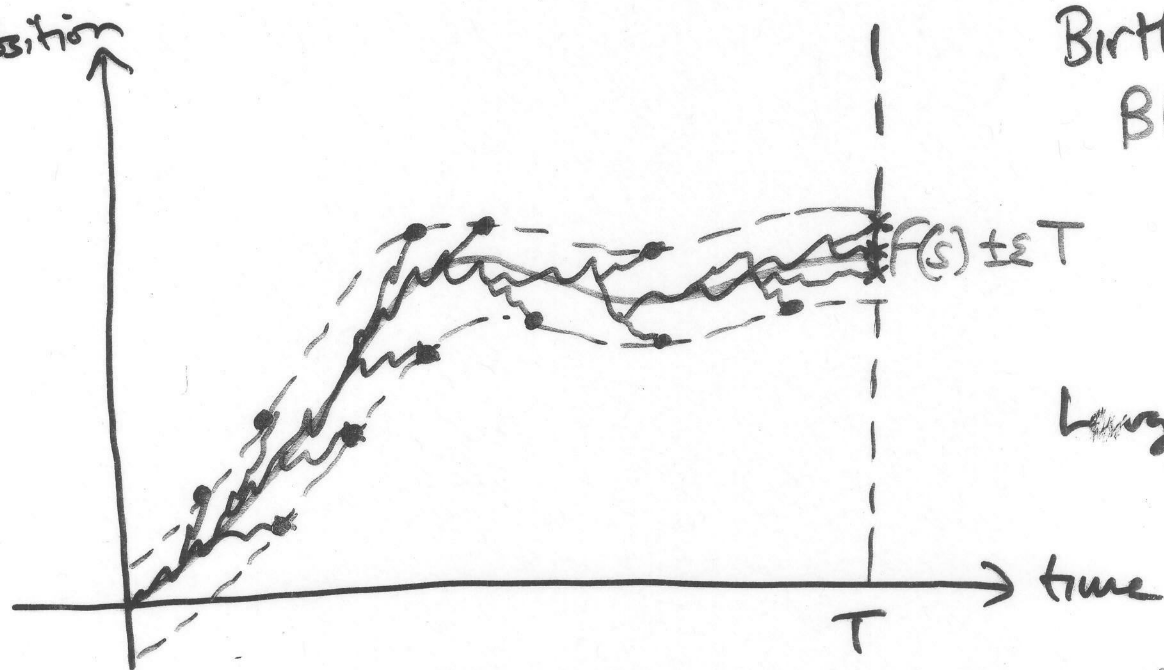
'RARE' PATHS If $\inf_{w \in [0,t]} \int_0^w \left(r - \frac{1}{2} f'(s)^2\right) ds < 0$, guess

$$\mathbb{P}(\text{particle follows } f) \sim \exp\left\{\inf_{w \in [0,t]} \int_0^w \left(r - \tfrac{1}{2} f'(s)^2\right) ds\right\}$$

'GROWTH' PATHS If $\inf_{w \in [0,t]} \int_0^w \left(r - \frac{1}{2} f'(s)^2\right) ds > 0$, guess

$$\ln \#\{\text{following close to } f\} \sim \int_0^t \left(r - \tfrac{1}{2} f'(s)^2\right) ds$$

A TIME-DEPENDENT BIRTH-DEATH APPROXIMATION



Birth rate $B(x)$

Large time T

How many particles in BBM stay 'close' to path f?

- Kill particles when <u>hit</u> side of tube around f
- <u>Approximate</u> by <u>time dependent B-D process</u> on $[0,T]$
  - birth rate $\lambda(s) := B(f(s))$
  - death rate $\mu(s) := \frac{1}{2} f'(s)^2$

Let $M(s)$ be # offspring alive at time s

→ <u>EXACT</u> solutions for population growth:

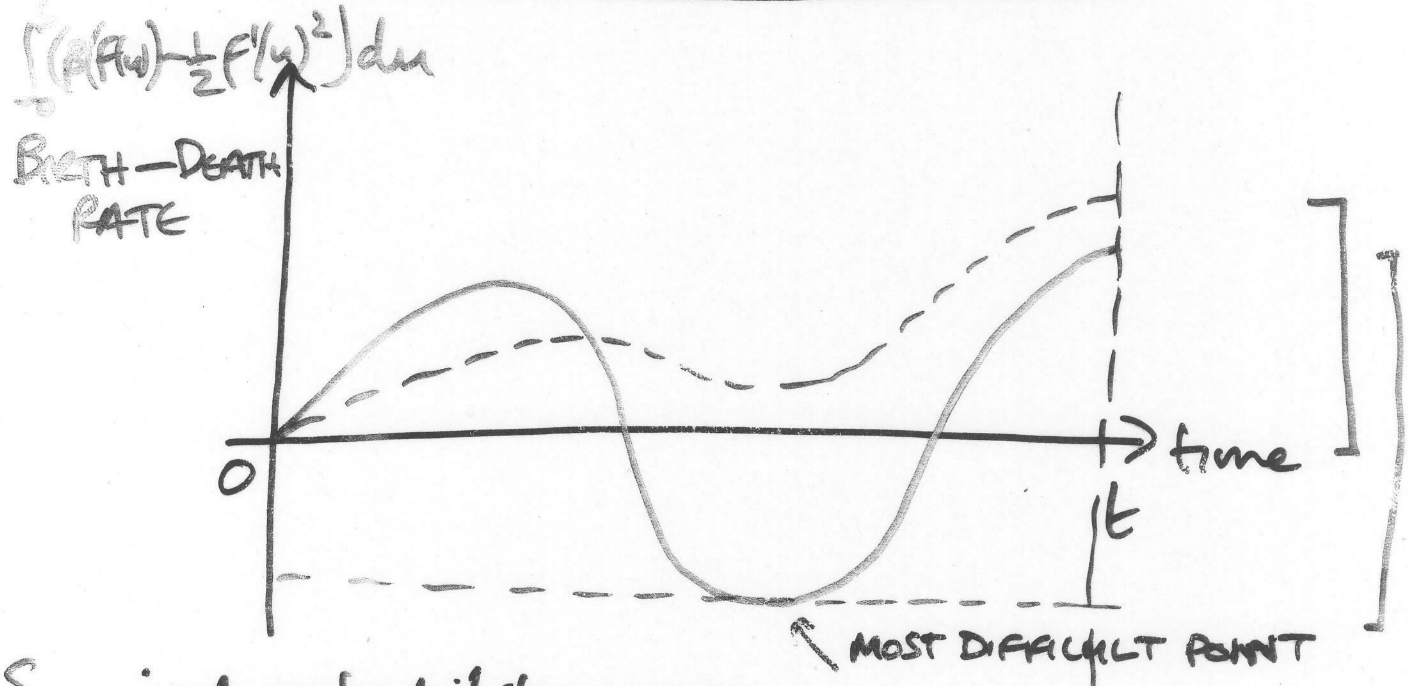
e.g. $\nu(t) := \int_0^t \{\lambda(s) - \mu(s)\} ds$ $\boxed{\text{"effective birth rate" = TOTAL BIRTH LESS TOTAL DEATH RATE}}$

$$W_t := e^{\nu(t)}\left(1 + \int_0^t \mu(s) e^{-\nu(s)} ds\right), \quad U_t := 1 - \frac{1}{e^{\nu(t)} W_t}; \quad V_t := 1 - \frac{1}{W_t}$$

$$\mathbb{P}(M_t = 0) = U_t, \quad \mathbb{P}(M_t = n) = (1 - U_t)(1 - V_t) V_t^{n-1} \quad n = 1, 2, \ldots$$

$$\mathbb{E} M_t = e^{\nu(t)}, \quad \mathbb{E}(M_t \mid M_t \geq 1) = W_t, \quad \frac{M_s}{\mathbb{E} M_s} \text{ cgt mg}, \ldots$$

Some B-D approx guesses



Survival probability guess

$$\mathbb{P}\left(\begin{array}{l}\#\text{particles staying near}\\ \text{path } f \text{ over } [0,t]\end{array} > 0\right) \sim \frac{1}{1+\int_0^t \frac{1}{2}f'(s)^2 e^{-\int_0^s(\beta(f(u))-\frac{1}{2}f'(u)^2)\,du}\,ds}$$

then,
$$\boxed{\ln \mathbb{P}(\#\{---\}>0) \sim -\sup_{s\in[0,t]} \int_0^s \left(\tfrac{1}{2}f'(u)^2-\beta(f(u))\right)du}$$

Conditional on survival, guess

$$\#\left\{\begin{array}{l}\text{particles closely following}\\ \text{path } f \text{ over } [0,t]\end{array}\right\} \sim e^{\int_0^t\{\beta(f(s))-\frac{1}{2}f'(s)^2\}\,ds} \Big/ \left(1+\int_0^t \tfrac{1}{2}f'(s)^2 e^{-\int_0^s\{\beta(f(u))-\frac{1}{2}f'(u)^2\}\,du}\,ds\right)$$

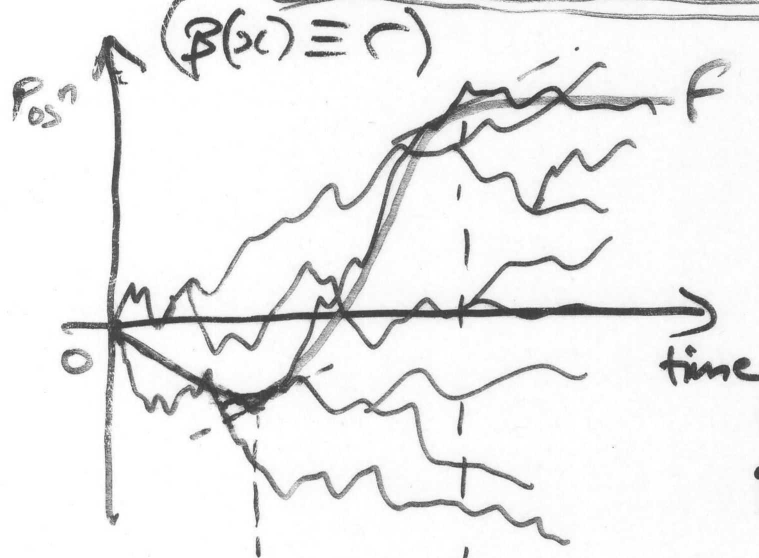
$$= 1+\int_0^t \beta(f(s))\, e^{\int_s^t(\beta(f(u))-\frac{1}{2}f'(u)^2)\,du}\,ds$$

then,
$$\boxed{\ln \#\{---\} \sim \sup_{s\in[0,t]} \int_s^t \left\{\beta(f(u))-\tfrac{1}{2}f'(u)^2\right\}du}$$

Problems:

- Precise results?
- Scaling? OR - fixed f on all $\mathbb{R}$?
- Nhd of f? - Conditions on $\beta(x)$?

PATHS IN STANDARD BBM



For $T \geq 0$, $u \in N_T$, $X_u : [0,T] \to \mathbb{R}$
define $X_u^T : [0,1] \to \mathbb{R}$ by

$$\boxed{X_u^T(s) := \frac{X_u(sT)}{T}}$$

If $B \subset C[0,1]$, for $w \in [0,1]$ let
$B_w \subset C[0,w]$ where $x \in B \Rightarrow x|_{[0,w]} \in B_w$

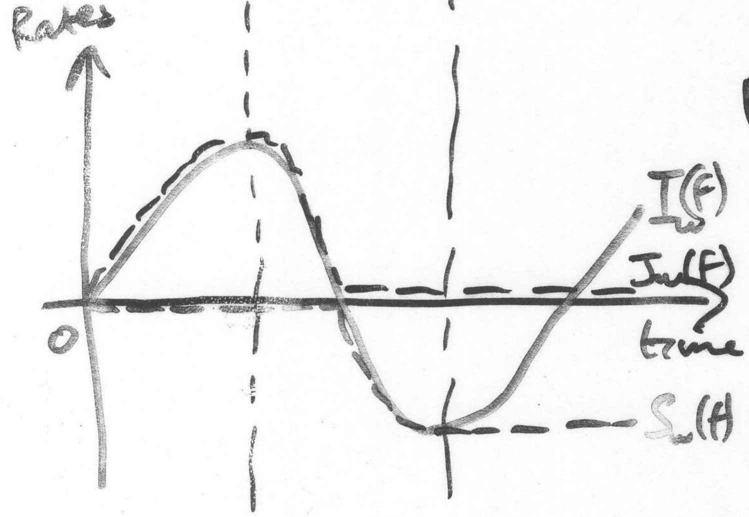
$$N_w^T(B) := \left|\left\{u \in N_{wT} : X_u^T|_{[0,w]} \in B_w\right\}\right|$$

For $f \in C[0,1]$,

$$I_w(f) := \int_0^w \left(r - \tfrac{1}{2} f'(s)^2\right) ds$$

$$S_w(f) := \inf_{s \in [0,w]} I_s(f) \in (-\infty, 0]$$

$$J_w(f) := \begin{cases} I_w(f) & \text{if } S_w(f) = 0 \\ 0 & \text{if } S_w(f) < 0 \end{cases}$$



THEOREM 1 (Lee (1992), Hardy & H. (2006)) For $w \in [0,1]$,
if $C \subset C[0,1]$ closed,

$$\boxed{\limsup_{T\to\infty} \frac{1}{T} \ln \mathbb{P}\left(N_w^T(C) > 0\right) \leq -\inf_{g \in C} S_w(g)}$$

if $V \subset C[0,1]$ open,

$$\boxed{\liminf_{T\to\infty} \frac{1}{T} \ln \mathbb{P}\left(N_w^T(V) > 0\right) \geq -\inf_{g \in V} S_w(g)}$$

THEOREM 2 (G-H (1998)) For $w \in [0,1]$, (c.f. Strassen's Law)
if $C \subset C[0,1]$ closed,

$$\boxed{\limsup_{T\to\infty} \frac{\ln N_w^T(C)}{T} \leq \sup_{g \in C} J_w(g)}$$

if $V \subset C[0,1]$ open,

$$\boxed{\liminf \frac{\ln N_w^T(V)}{T} \geq \sup_{g \in V} J_w(g)}$$

Explosions! $\beta(x) = \beta x^p$ some $p>0$:

$p>2$ $\quad N(t) = +\infty \quad \forall t > T(\omega)$

$p=2$ $\quad N(t) < \infty \ (\forall t)$ but $\mathbb{E}N(t) = +\infty \ (\forall t > \hat{t})$

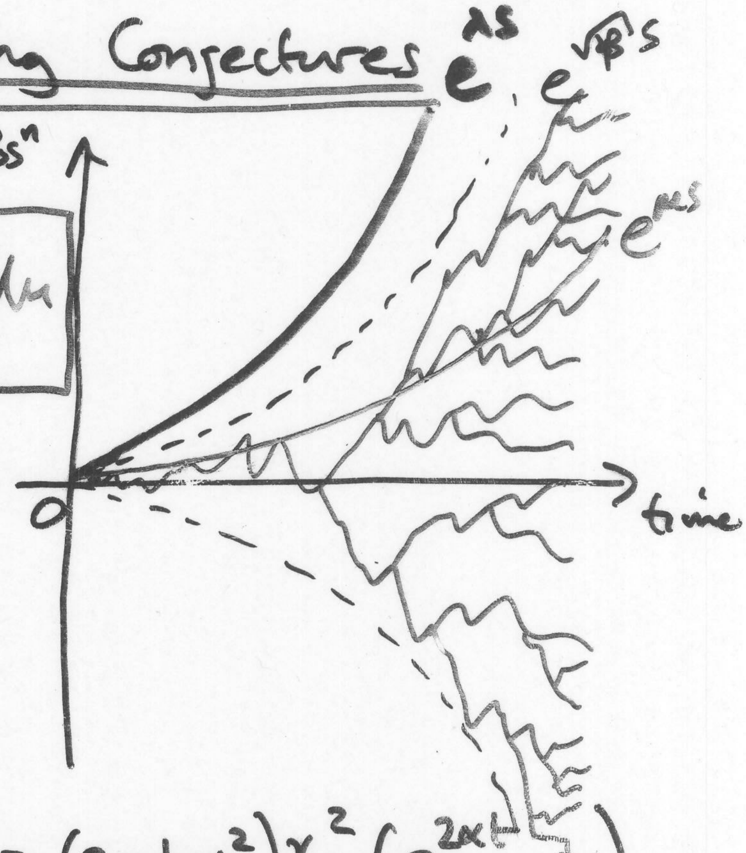
$p<2$ $\quad$ ... okay!

Quadratic: $\mathbb{E}\#$'s blows up. [McKean – Itô]

Quadratic Breeding Conjectures

$\beta(x) = \beta x^2 \quad (\beta > 0)$

$$\boxed{I_s(f) := \int_0^s \left\{\beta f(u)^2 - \tfrac{1}{2} f'(u)^2\right\} du}$$



Interface:

$I_s(f) = 0 \quad \forall s \in [0,t]$

$$\Leftrightarrow \boxed{f(s) = x_0 e^{\sqrt{2\beta}\, s}}$$

Exponential paths:

$$\boxed{f(s) = x_0 e^{\alpha s}}, \quad I_t(f) = \left(\beta - \tfrac{1}{2}\alpha^2\right)\frac{x_0^2}{2\alpha}\left(e^{2\alpha t} - 1\right)$$

for $\alpha := \lambda > \sqrt{2\beta}$: 'RARE' PATHS

$$\boxed{\ln \mathbb{P}\left(\begin{array}{c}\text{survival 'along'}\\ \text{path } f\end{array}\right) \sim -\frac{x_0^2}{4\lambda}\left(\lambda^2 - 2\beta\right)e^{2\lambda t}}$$

VERY unlikely!

for $\alpha := \mu < \sqrt{2\beta}$: 'GROWTH' PATHS

$$\boxed{\ln \#\left\{\begin{array}{c}\text{particles 'near'}\\ f \text{ over } [0,t]\end{array}\right\} \sim \frac{x_0^2}{4\mu}\left(2\beta - \mu^2\right)e^{2\mu t}}$$

MASSIVE GROWTH – best near $\mu = \sqrt{\beta} - \varepsilon$

Right-most particle:

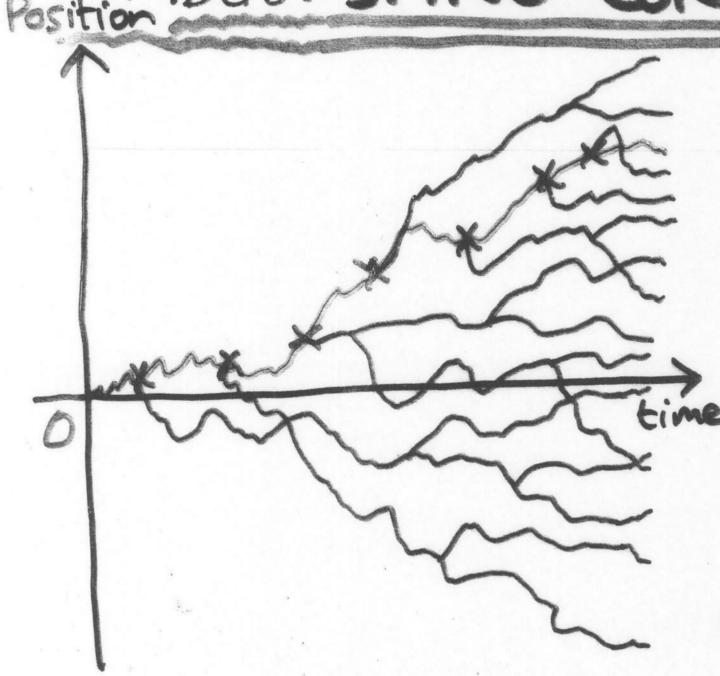
$$\boxed{\frac{\ln R_t}{t} \to \sqrt{2\beta} \quad \text{a.s.}}$$

CONJECTURE ??

Possible scaling:

$$f_t(s) = \exp\left\{t\, g\left(\tfrac{s}{t}\right)\right\} \ ?$$

KEY IDEAS: SPINE CONSTRUCTION



'SPINE' ξ – subset of nodes of single ∞-line descent from initial ancestor

$\xi_t = X_u(t)$ where $u \in \xi \cap N_t$

P^x original BBM $(\mathcal{F}_t)_{t \geq 0}$

$\tilde{P}^x$ BBM + 'spine' $(\tilde{\mathcal{F}}_t)_{t \geq 0}$

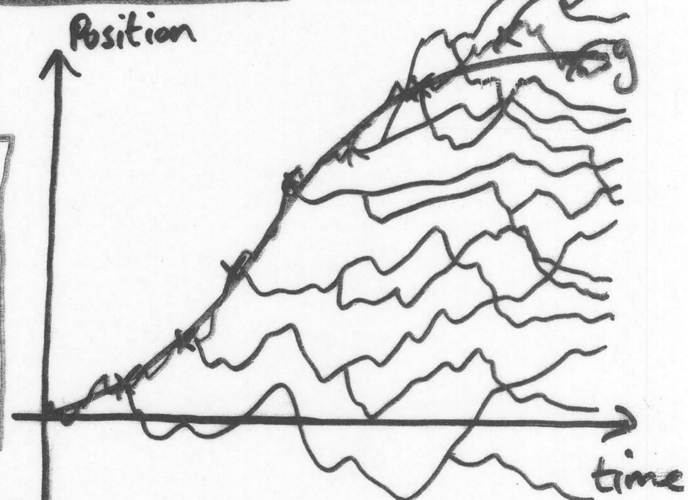
– uniform choice each fission

Under $\tilde{P}^x$, ξ_t is BM started at x

SPINE CHANGE OF MEASURE

Define $\tilde{Q}^x$ on $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ by

$$\boxed{\left.\frac{d\tilde{Q}^x}{d\tilde{P}^x}\right|_{\tilde{\mathcal{F}}_t} = e^{\int_0^t g'(s)\,d\xi_s - \frac{1}{2}\int_0^t g'(s)^2 ds} \times e^{-\int_0^t \beta(\xi_s)ds}\, 2^{n_t}}$$



Under $\tilde{Q}$, can construct BBM by:

- Spine diffusion $(\xi_s)_{s\geq 0}$ starts at x & $\boxed{\xi_t - g(t) \text{ is a } \tilde{Q}\text{-BM}}$
- spine undergoes fission at accelerated rate $2\beta(\xi_s)$
- at fission, choose offspring at random to continue spine other particles initiate independent P-BBMs

'ADDITIVE' MARTINGALES

Define $Q := \tilde{Q}|_{\mathcal{F}_\infty}$, then

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = Z_g(t) := \sum_{u \in N_t} e^{\int_0^t g'(s)\,dX_u(s) - \frac{1}{2}\int_0^t g'(s)^2 ds - \int_0^t \beta(X_u(s))ds}$$

c.f. Chauvin & Rouault (1988); Lyons, Pemantle, Peres (1995); ...
Kyprianou (2004); Harris & H. (2006); ...

Rough ideas for 'proofs'

'Rare' Paths

$$\ln P(\exists \text{ particle 'close' } f) \sim \inf_{s\in[0,t]} \int_0^t \left\{\beta(f(s)) - \tfrac{1}{2} f'(s)^2\right\} ds$$

- Upper bound

$$P(\exists \text{ particle 'close' } f) \le E(\#\text{following } f) \quad \checkmark$$

- Lower bound

$$P(\exists \text{ particle 'close' } f) = Q\left(A; \frac{1}{Z_f}\right)$$

$$\ge \tilde{Q}\left(A; \frac{1}{\underbrace{\tilde{Q}(Z_f \mid \mathcal{G}_\infty)}_{\text{SPINE DECOMPOSITION}}}\right) \quad \checkmark$$

Agrees with BD heuristics

? $\limsup t^{-1} \ln R_t \le \sqrt{2\beta}$, P-a.s.

'Easy' paths

- Convergence criteria for Z_f
  - When Z_f UI P-mg with $Z_f(\infty) > 0$ a.s. P, P & Q agree on a.s. events

    $\therefore$ $\exists$ particle 'close' f under P-a.s.
    (since spine is 'close' f under Q-a.s.)

    $$\boxed{\liminf t^{-1} \ln R_t \ge \sqrt{2\beta} \quad \text{a.s. } P} \quad \checkmark$$

  - $Z_f \approx \sum I\{\text{near } f\} e^{-(\sim)}$

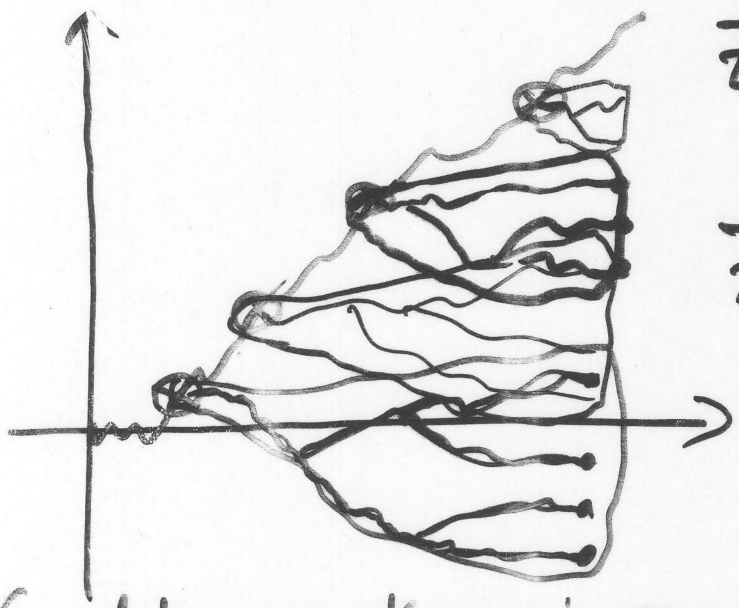
    so $\boxed{\#\{\text{near } f\} \approx Z_f e^{+(\sim)}}$

GROWTH RATE 'ALONG' f

Agrees with BD heuristics

— A LOT left to make precise!!!

SPINE DECOMPOSITION



$$Z_f(t) = \sum_{u \in N_t} e^{\int_0^t f' dX_u - \frac{1}{2}\int_0^t f'^2 ds - \int_0^t \beta(X_u) ds}$$

$Z_f(t) =$ "spine contribution"

$+ \sum_{u < \xi_t}$ "contribution from sub-tree started at u on spine"

Condition on knowing spine's motion, fission times (& # child

Recall, $Z_f(t)$ is P-mg, so average subtree contribution is value at the time of birth S_u

$$e^{\int_0^{S_u} f'(s) d\xi_s - \frac{1}{2}\int_0^{S_u} f'(s)^2 ds - \int_0^{S_u} \beta(\xi_s) ds}$$

subtrees behave as under P

Then,

SPINE DECOMPOSITION

$$\boxed{\tilde{Q}\left(Z_f(t) \middle| \tilde{\mathcal{G}}_\infty\right) = e^{\int_0^t f' d\xi_s - \frac{1}{2}\int_0^t f'^2 ds - \int_0^t \beta(\xi_s) ds} + \sum_{u < \xi_t} e^{\int_0^{S_u} f' d\xi_s - \frac{1}{2}\int_0^{S_u} f'^2 ds - \int_0^{S_u} \beta(\xi_s) ds}}$$

BM under $\tilde{Q}$

but, under $\tilde{Q}$, $\xi_s = B_s + f(s)$ then

$$\tilde{Q}\left(Z_f(t) \middle| \tilde{\mathcal{G}}_\infty\right) = e^{\int_0^t \{\frac{1}{2} f'(s)^2 - \beta(\xi_s)\} ds + \int_0^t f' dB_s} + \sum_{u < \xi_t} e^{\int_0^{S_u} (\frac{1}{2} f'(s)^2 - \beta(\xi_s)) ds + \int_0^{S_u} f' dB_s}$$