Periodic Monopoles

- G-connection A on M_3, curvature F.

Higgs field Φ, $D\Phi = d\Phi + [A, \Phi]$.

Monopole eqns $D\Phi = -*F$ — B

Integrable

- Nahm Transform: solutions of B on M (with BCs) correspond to solutions of

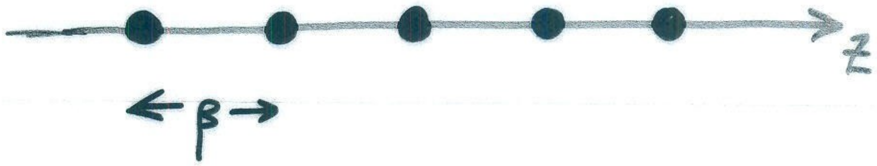
  - ✓ Nahm eqns $\frac{d}{ds} T_j = \frac{1}{2} \varepsilon_{jkl} [T_k, T_l]$ on $\mathbb{R}$ — $M = \mathbb{R}^3$
  - ✓ Hitchin eqns on $\mathbb{R} \times S^1$ if $M = \mathbb{R}^2 \times S^1$
  - ? Monopole eqns on $\mathbb{R} \times T^2$ if $M = \mathbb{R} \times T^2$

Interchanges RANK and CHARGE.

size of G — topology

- Example: $M = \mathbb{R}^3$, $G = SU(2)$, collinear string of N monopoles with spacing β.

[BC $|\Phi|^2 \to 1$ as $r \to \infty$ fixes monopole size]



Nahm data: $T_j(s) = f_j(s)\, \Sigma_j$

* $\{\Sigma_1, \Sigma_2, \Sigma_3\}$ N-dim irrep of $su(2)$

* $f_j(s)$ Jacobi elliptics, parameter $\sim \frac{\beta}{\beta+1}$.

- Pictures in Dunne & Khemani, J. Phys. A (2005)
- Not axially-symmetric (unless $\beta = 0$)
- No obvious $N \to \infty$ limit...
- Asymptotically, field is $U(1)$ Dirac monopole, with

$-\frac{1}{2r}$

$$\Phi = 1 - \frac{1}{2} \sum_{n=1}^{N} \left[x^2 + y^2 + (z - \beta n)^2\right]^{-1/2}.$$

But $\sum_{-\infty}^{\infty}$ diverges.

- In $U(1)$ case (linear), can regularize:

$$\Phi = c - \frac{1}{2r} - \frac{1}{2}\sum_{n\neq 0}\left[\frac{1}{\sqrt{\rho^2+(z-\beta n)^2}} - \frac{1}{\beta|n|}\right]$$

String of Dirac monopoles along z-axis, $\Phi \sim \frac{1}{\beta}\log\rho$ as $\rho\to\infty$. $\boxed{\rho^2 = x^2+y^2}$

- $SU(2)$ case: (Φ, A_j) smooth, 2π-periodic, with $|\Phi| \sim \frac{k}{2\pi}\log\rho$, $|D\Phi| \sim O(\frac{1}{\rho})$ as $\rho\to\infty$.

Topology: on torus $\rho = \text{const}$, $\Phi \rightsquigarrow$ line bundle (+ve eigenspace) $\rightsquigarrow$ Chern # k.

String of k-monopoles localized on z-axis.

- Cherkis & Kapustin (CMP 2001):
  - * Interpretation ... branes & susy YM
  - * Full analysis of Nahm Transform

Nahm data: solution of $U(k)$ Hitchin eqns on $\mathbb{R}\times S^1$ (with suitable BCs).

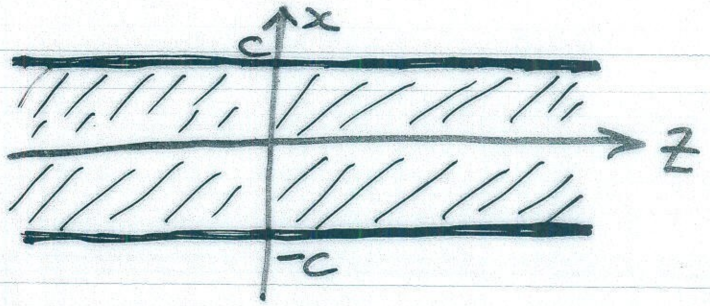
Case $k=1$: solve explicitly, parameter $C>0$

$C \sim$ (monopole size)/(z-period).

- (Φ, A_j) not explicit ... but can get approx picture via Nahm Transform.

For $0 < C \ll 1$, get 

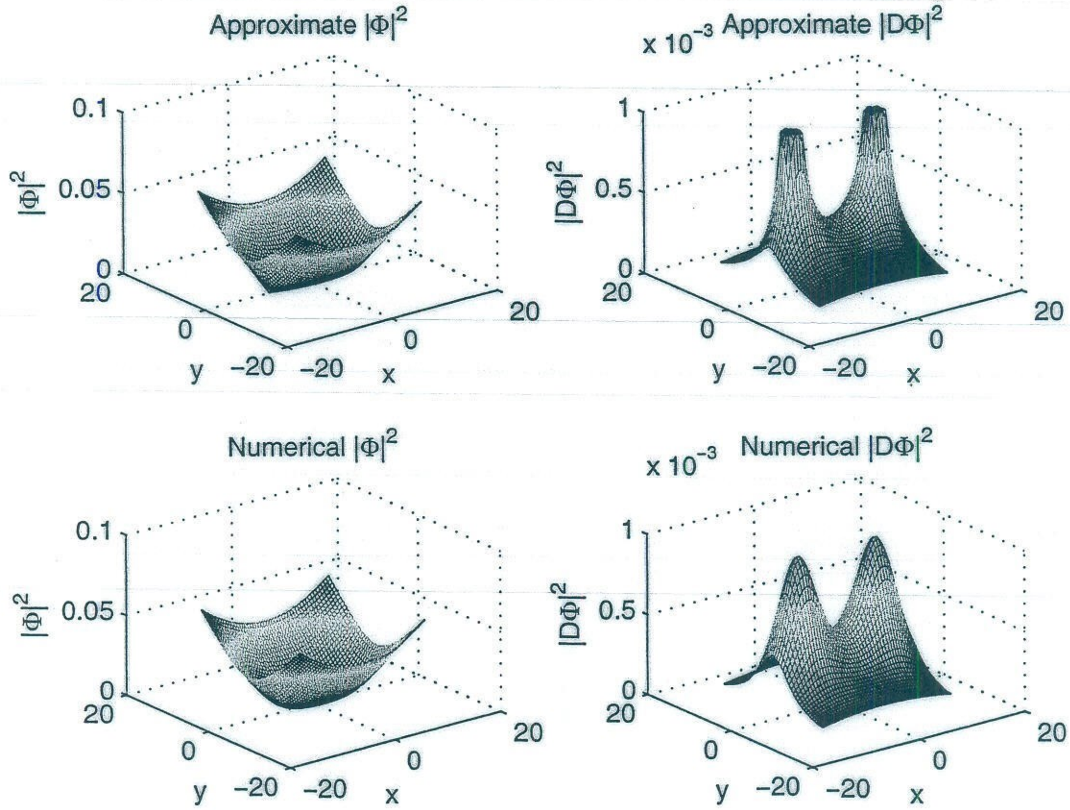
For $C \gg 1$, get strip, width $2C$



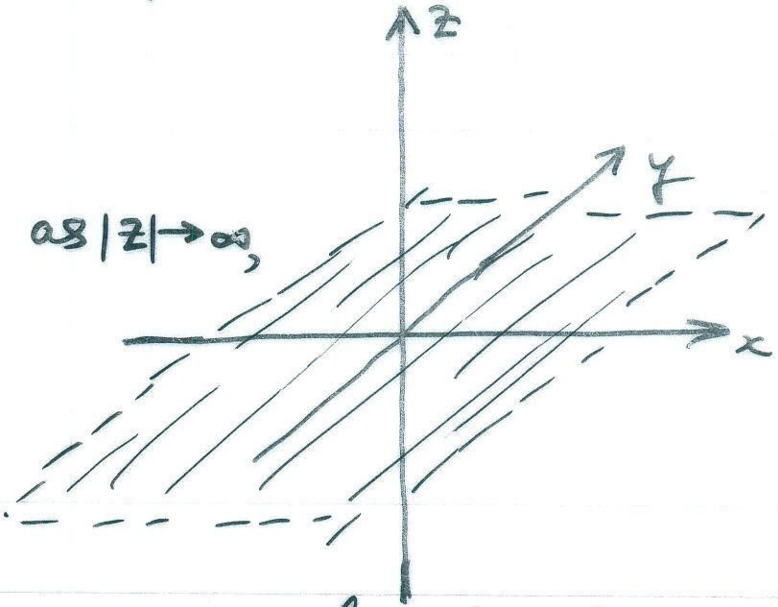
* $\Phi \sim 0$ on strip
* $D\Phi$ peaked on edges

- Can one understand the $N \to \infty$ process?

C=8


Approximate |Φ|^2
0.1
0.05
0
|Φ|^2
20
0
y
-20
-20
0
20
x
x 10^-3 Approximate |DΦ|^2
1
0.5
0
|DΦ|^2
20
0
y
-20
-20
0
20
x
Numerical |Φ|^2
0.1
0.05
0
|Φ|^2
20
0
y
-20
-20
0
20
x
x 10^-3 Numerical |DΦ|^2
1
0.5
0
|DΦ|^2
20
0
y
-20
-20
0
20
x

- Doubly-periodic: monopole sheet.



* Period 1 in x & y.
* Expect $|\Phi / z| \sim$ const as $|z| \to \infty$,

  $|D\Phi| \sim$ const.

* U(1) solution:

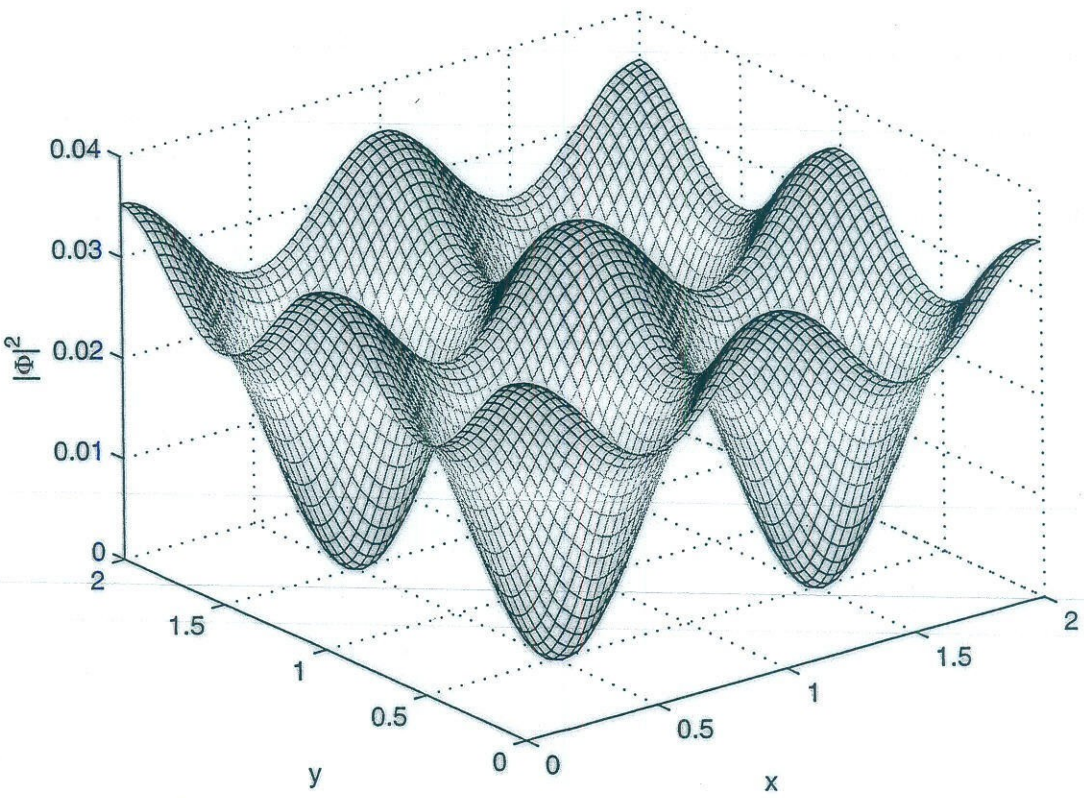
$$\left\{ \begin{array}{l} \underline{A}\text{: homogeneous connection on line bundle} \\ \quad \text{over } T^2\text{, Chern \# } N \\ \Phi = -2\pi i N z \end{array} \right.$$

Nahm transform of this is (essentially) the same field, on dual $\mathbb{R} \times T^2$.

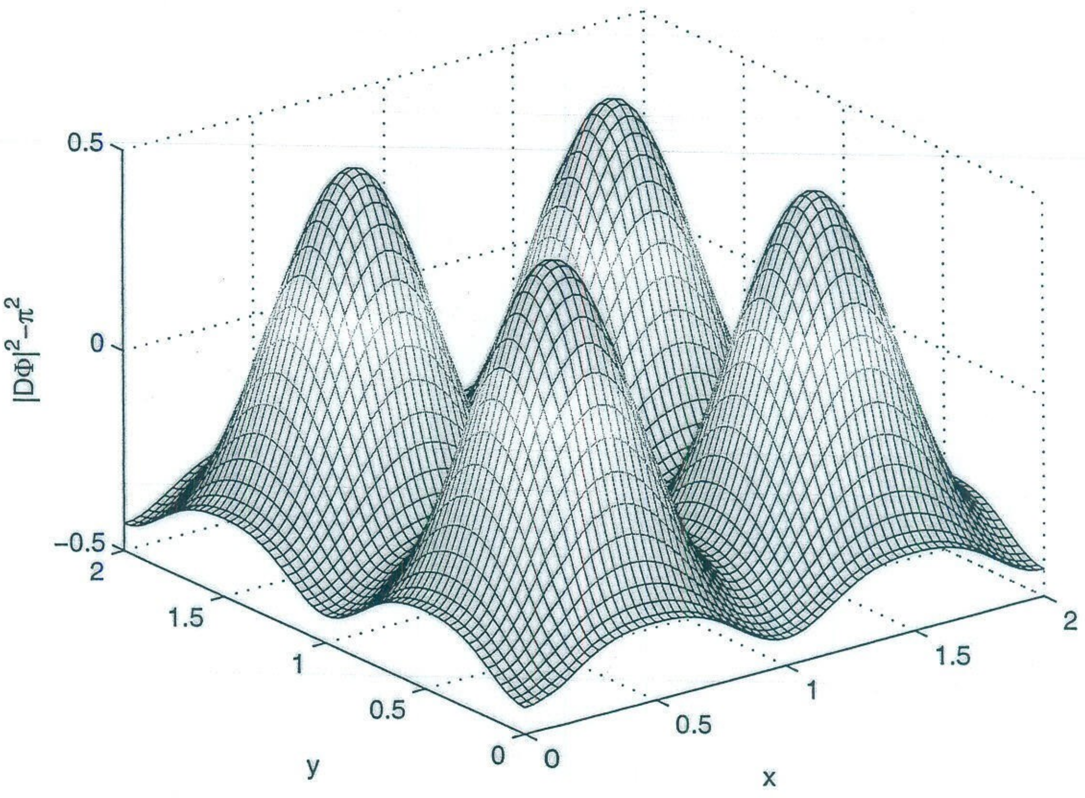
* SU(2) case: from $\Phi\big|_z$ as $z \to \pm\infty$

  get $N_{\pm} \in \mathbb{Z}$.

* Embedding of the U(1) solution in SU(2) has $N_+ = N_- = N$. Perturbations dim $4N$, localized around $z = 0$.
* What is moduli space? What if $N_+ \neq N_-$? Nahm transform?



0.04
0.03
0.02
0.01
0
$|\Phi|^2$
2
1.5
1
0.5
0
y
0
0.5
1
1.5
2
x



0.5
0
−0.5
$|D\Phi|^2-\pi^2$
2
1.5
1
0.5
0
y
0
0.5
1
1.5
2
x