Topological Strings on Local Calabi-Yau Manifolds and Instantons in Gauge Theories

LMS Durham Symposium

”Methods of Integrable Systems in Geometry”

Hiroaki Kanno (Nagoya)

An overview of the recent developments

A report of work in progress

0. Why Topological String ?

1. Exactly solvable models in string theory : Toy models for investigating various dualities, that might be compared with sine-Gordon model in 2D QFT (for soliton and boson/fermion correspondence) and Ising model in statistical physics (for phase transition)
2. Counting of instantons (BPS states) : Microscopic state counting of (extremal) black-hole, Low energy effective action of 4-dim $\mathcal{N} = 2$ supersymmetric (8 SUSY) theories (Seiberg-Witten prepotential)
3. Amusing Laboratory to enjoy and develop deep ideas in mathematics (Donaldson, Langlands, $\cdots\cdots\cdots$)

Plan of my talk — (The numbering refers to the previous slide)

1. Art of Topological Vertex

   Topological vertex as building block of topological string amplitudes on (local) toric Calabi-Yau 3-folds – Toric geometry, Link invariants (Schur functions)

2. Seiberg-Witten Prepotential

   Geometric Engineering – Asymptotic growth of Gromov-Witten invariants of local Hirzebruch surface

3. Experiments on "Non-Nef" cases

   (1) $\mathcal{O}(-p) \oplus \mathcal{O}(p-2) \to \mathbf{P}^1 \quad (p \neq 0, 1, 2)$

   (2) $K_{\mathbf{F}_n} \to \mathbf{F}_n = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(-n) \oplus \mathcal{O}_{\mathbf{P}^1})$ $(n \neq 0, 1, 2)$

1. Art of Topological Vertex

With a proposal of Topological Vertex, we have an algorithm of computing *all genus* (A-model) topological string amplitudes on (local) toric Calabi-Yau 3-fold; we can compute the partition function, that is supposed to be a generating function of (local) Gromov-Witten invariants, by a diagramatic way.

Remark :

There are no (non-trivial) compact toric Calabi-Yau manifold. A typical example of toric Calab-Yau 3-fold is K_S; the canonical bundle of toric (Fano) surface S.

The proposal (in a complete form) was made in

[M. Aganagic, A. Klemm, M. Mariño and C. Vafa, hep-th/0305132]

based on the idea of open/closed string duality, or the duality between Chern-Simons theory and Gromov-Witten theory. Some related works that led to the proposal are

[M. Aganagic, M. Mariño and C. Vafa, hep-th/0206164,

A. Iqbal, hep-th/0207114]

There is an attempt at formulating the algorithm in more rigorous manner, based on relative Gromov-Witten theory and virtual localization w.r.t. toric action;

[J. Li, C.-C. M. Liu, K. Liu and J. Zhou, math-AG/0408426]

Toric Calabi-Yau Geometry

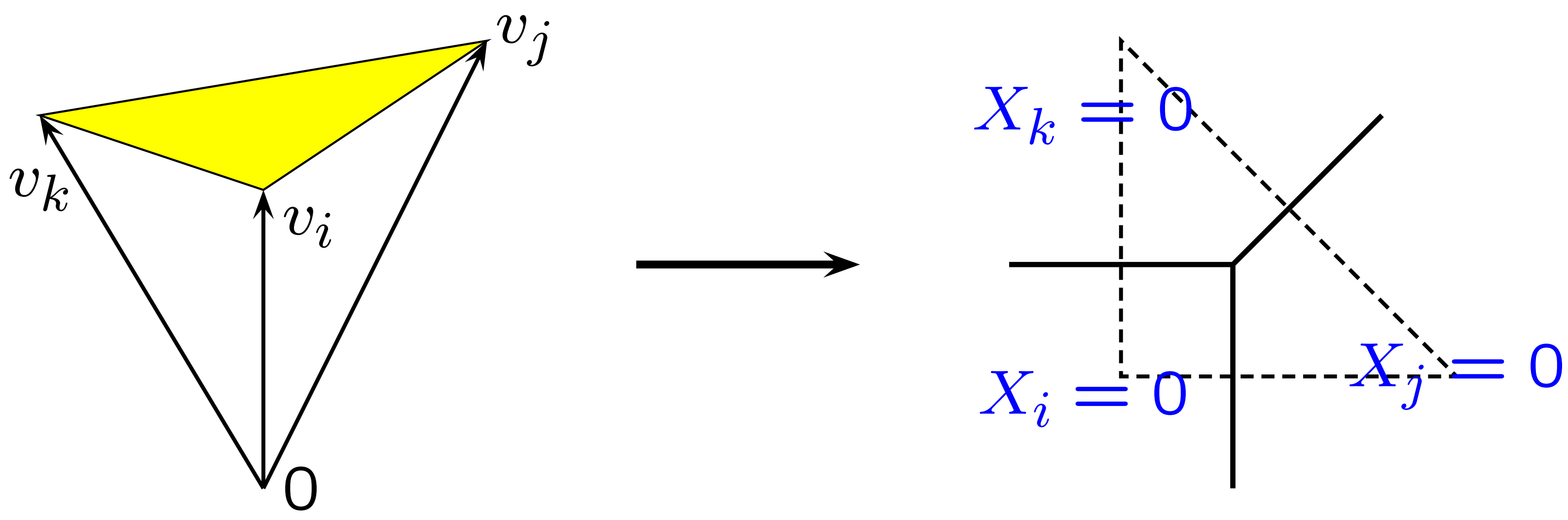
X^i $(i = 1, \cdots, k+3)$: affine coordinates
with $U(1)^k$ charges Q_i^a $(a = 1, \cdots, k)$

Moment map w.r.t. $U(1)^k$ action : $\mu(X) = \sum_{i=1}^{k+3} Q_i^a |X^i|^2$

$$CY_3(r^a) := \left\{ \sum_{i=1}^{k+3} Q_i^a |X^i|^2 = r^a \right\} \Big/ U(1)^k$$

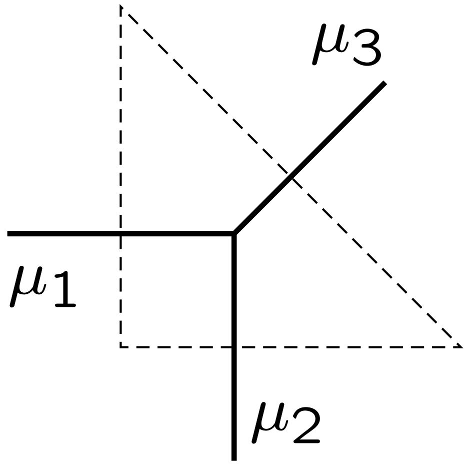
Symplectic Quotient (r^a : Kähler parameters)

Toric fan (diagram) in $\mathbb{R}^3$ is generated by $\{v_1, v_2, \cdots, v_{k+3}\}$ with $\sum_{i=1}^{k+3} Q_i^a \, v_i = 0$. Due to the Calabi-Yau condition $\sum_{i=1}^{k+3} Q_i^a = 0$, they are on the same plane (say, $z = 1$).



Building block of toric fan (diagram) representing an affine local patch $\simeq \mathbf{A}^3$ with local coordinates X^i, X^j, X^k and its dual trivalent vertex, (where the toric action degenerates at face=divisor [4-cycle], edge=curve [2-cycle] and vertex=point [0-cycle])

For each edge ($\simeq$ invariant rational curve) we assign a Young tableau (or a partition) μ_i.



$\rightarrow$ Topological Vertex

$$C_{\mu_1\mu_2\mu_3}(q), \quad q = \exp(-g_s)$$

g_s : a parameter of genus expansion

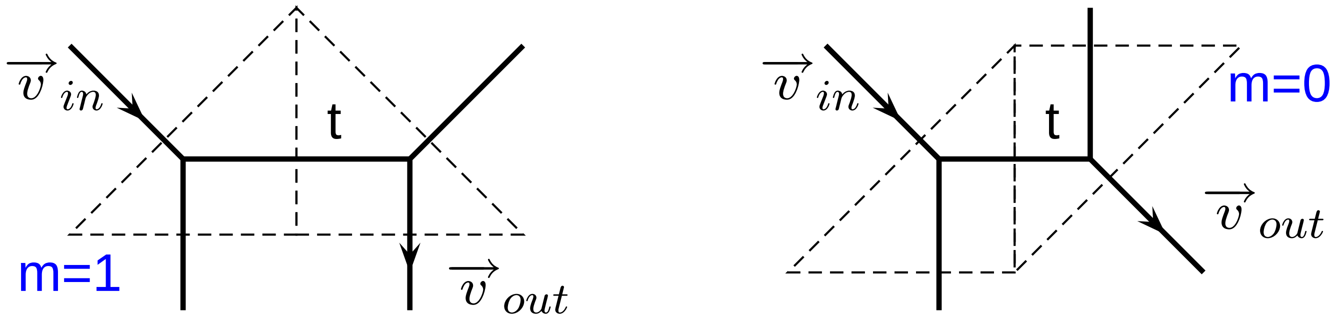
Topological string amplitude is obtained by "gluing" topological vertices $C_{\mu_1\mu_2\mu_3}(q)$, according to the gluing of affine local patches to make a given toric Calabi-Yau 3-fold.

Feynman diagram like rules

- Vertices $\Rightarrow$ trivalent coupling : $C_{\mu_1\mu_2\mu_3}(q)$
- Edges $\Rightarrow$ propagator : $(-1)^{\ell_\mu} \cdot e^{-t\cdot\ell_\mu}$

  t: the Kähler parameter of $\mathbf{P}^1$; $\ell_\mu := \sum_{(i,j)\in\mu} 1$
- Slopes of edges $\Rightarrow$ framing : $(-1)^{m\cdot\ell_\mu} \cdot q^{\frac{m}{2}\cdot\kappa_\mu}$

  $m := \overrightarrow{v}_{in} \wedge \overrightarrow{v}_{out}$; $\kappa_\mu := 2\sum_{(i,j)\in\mu}(j-i)$



$\sum_\nu C_{\mu_1\mu_2\nu}(q) \cdot e^{-t\cdot\ell_\nu} \cdot C_{\nu^t\mu_3\mu_4}(q) \cdot q^{\frac{\kappa_\nu}{2}}$ $\quad$ $\sum_\nu C_{\mu_1\mu_2\nu}(q) \cdot (-e^{-t})^{\ell_\nu} \cdot C_{\nu^t\mu_3\mu_4}(q)$

REMARKS

The gluing rule of topological vertex is different from that of 2D TQFT based on cobordisms of Riemann surfaces.

The algorithm gives all genus amplitudes ($q = \exp(-g_s)$). It is one of surprises in topological vertex formalism that a simple structure of the partition function emerges after summing up all genera.

Technical Notes

Topological Vertex is derived from the duality to the Chern-Simons theory and expressed in terms of the large N leading part of the Hopf link invariants $W_{\mu_1\mu_2}(q)$;

$$C_{\mu_1\mu_2\mu_3}(q) = q^{\frac{\kappa_{\mu_2}}{2}+\frac{\kappa_{\mu_3}}{2}} \sum_{\nu_1,\nu_2} N^{\mu_1\mu_3^t}_{\nu_1\nu_2} \frac{W_{\mu_2^t\nu_1}(q)W_{\mu_2\nu_2}(q)}{W_{\mu_2\bullet}(q)}$$

Recall that the Hilbert space of the Chern-Simons theory on $T^2 \times \mathbb{R}$ can be identified with the space of conformal blocks $\mathcal{H}(T^2)$ of WZW theory on T^2.

The Hopf link invariants $W_{PQ}(q,\lambda)$ are obtained as the normalized modular S-matrix elements on $\mathcal{H}(T^2)$;

$$\frac{S_{PQ}}{S_{\bullet\bullet}} = \frac{\sum_{w\in S_N}(-1)^w q^{-(\Lambda_P+\rho_N)\cdot w(\Lambda_Q+\rho_N)}}{\sum_{w\in S_N}(-1)^w q^{-\rho_N\cdot w(\rho_N)}}$$

$$q := \exp\left(\frac{2\pi i}{N+k}\right), \qquad \lambda := q^N$$

where the symmetric group S_N is the Weyl group, Λ_R is the highest weight of R and ρ_N is the Weyl vector.

By Weyl's character formula

$$\mathrm{ch}_R\,\xi = \frac{\sum_{w\in S_N}(-1)^w e^{(\Lambda_R+\rho_N)\cdot w(\xi)}}{\sum_{w\in S_N}(-1)^w e^{\rho_N\cdot w(\xi)}},$$

we see that $W_{PQ}(q,\lambda)$ can be written by specialization of the character, or the Schur polynomials (actually functions since we consider $N \to \infty$);

$$W_{PQ}(q,\lambda) = \mathrm{ch}_P\left(-\frac{2\pi i}{N+k}\rho_N\right)\mathrm{ch}_Q\left(-\frac{2\pi i}{N+k}(\Lambda_P+\rho_N)\right)$$
$$= \lambda^{-\frac{1}{2}(|P|+|Q|)} s_P(x_i = q^{i-\frac{1}{2}}) s_Q(x_i = q^{-\lambda_i^P+i-\frac{1}{2}})$$

This formula is proved in more general context in

[H.R.Morton and S.G. Lukac, `math.GT/0108011`]

Finally the origin of the the framing factor is the eigenvalues of the T-transformation ($T \in SL(2,\mathbb{Z})$) which is diagonal on conformal blocks.

2. Seiberg-Witten Prepotential

Seiberg-Witten prepotential $\mathcal{F}_{SW}(a,\Lambda)$ gives a non-perturbative (including instanton effects) low energy effective action of 4 dimensional $\mathcal{N}=2$ SUSY Yang-Mills theory.

Let us consider $SU(2)$ case for simplicity. Instanton expansion of $SU(2)$ SW prepotential is;

$$\mathcal{F}_{SW}(a,\Lambda) = \frac{\tau_0}{2}a^2 + \frac{a^2}{2}\left(\log\frac{a}{\Lambda} - \frac{3}{2}\right) + a^2 \sum_{k=0}^{\infty}\left(\frac{\Lambda}{a}\right)^{4k}\mathcal{F}_k$$

where the coefficients $\mathcal{F}_k$ are the "symplectic volume" $\mathcal{F}_k = \int_{\mathcal{M}_k}$ "1", where $\mathcal{M}_k$ is the moduli space of (framed) $SU(2)$ instantons on $\mathbb{R}^4$ with instanton number k.

Seiberg-Witten theory tells that the prepotential $\mathcal{F}_{SW}(a, \Lambda)$ is obtained by solving the Picard-Fuchs equation for the period integrals on $SU(2)$ Seiberg-Witten curve;

$$y^2 = (x^2 - u)^2 - 4\Lambda^4 \ ,$$

where u is the moduli parameter. (The curve degenerates at $u = \pm 2\Lambda^2$, where a massless monopole (dyon) appears.) Consider the period integral

$$a(u) := \int_\alpha \lambda_{SW} \ , \qquad a_D(u) := \int_\beta \lambda_{SW}$$

of SW differential $\lambda_{SW} = -\frac{1}{\pi}\frac{x^2 dx}{y}$.

The (rigid) special geometry implies an existence of the prepotential $\mathcal{F}_{SW}(a, \Lambda)$ that satisties

$$a_D(u) = \frac{\partial \mathcal{F}_{SW}}{\partial a}$$

Then we can proceed as follows;

Picard-Fuchs equation $\Rightarrow a = a(u), a_D = a_D(u)$

Inversion $u = u(a)$ and Integration $\Rightarrow \mathcal{F}_{SW}(a, \Lambda)$

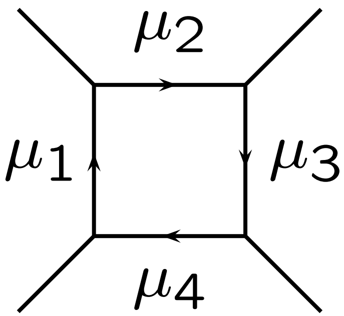
The partition function of topological srting

$$Z(t) = \exp\left(\sum_{g=0}^{\infty} g_s^{2g-2} F_g(t)\right)$$

contains the prepotential as the free energy $F_0(t)$ at genus zero. From the viewpoint of topological string Seiberg-Witten theory gives a “B-model” computation of $\mathcal{F}_{SW}$.

We can obtain the SW prepotential from the ”double scaling” limit of topological string amplitude on local Hirzebruch surface $K_{\mathbf{F}_n}$ $(n = 0, 1, 2)$, which is regarded as a “A-model” computation of $\mathcal{F}_{SW}$.

Topological string amplitude on local Hirzebruch surface $K_{\mathbf{F}_0}$



$$Z^{(\mathbf{F}_0)}_{top\ str} = \sum_{\mu_1 \cdots \mu_4} W_{\mu_4\mu_1} W_{\mu_1\mu_2} W_{\mu_2\mu_3} W_{\mu_3\mu_4} \times e^{-t_F \cdot (\ell_{\mu_1} + \ell_{\mu_3}) - t_B \cdot (\ell_{\mu_2} + \ell_{\mu_4})}$$

Recall that the Hirzebruch surface $\mathbf{F}_n$ is a $\mathbf{P}^1$ bundle over $\mathbf{P}^1$. The second homology class $H_2(\mathbf{F}_n, \mathbf{Z})$ is spanned by the two cycles B and F, where their representatives are the base $\mathbf{P}^1$ and the $\mathbf{P}^1$ fi ber, respectively. The intersection numbers of these cycles are

$$B \cdot B = -n\ , \quad F \cdot F = 0\ , \quad B \cdot F = +1\ .$$

t_B and t_F are the Kähler parameters of B and F.

Define the instanton expansion of free energy as follows ($Q_B = e^{-t_B}, Q_F = e^{-t_F}$) ;

$$\log Z_{top\,str}(Q_B, Q_F, q)$$
$$= \mathcal{F}_{one\,loop}(Q_F, q) + \mathcal{F}_{inst}(Q_B, Q_F, q)$$
$$\mathcal{F}_{inst}(Q_B, Q_F, q) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} Q_B^{nk} \mathcal{F}_k(Q_F^n, q^n)$$
$$\mathcal{F}_k(Q_F, q) = \sum_{g=0}^{\infty} \left(\sin \frac{g_s}{2} \right)^{2g-2} f_g^{(k)}(Q_F)$$

where $f_g^{(k)}$ gives the k-instanton amplitude at ”genus” g . Note that we take the expansion of the Gopakumar-Vafa type.

We list the function $f_0^{(k)}(Q \equiv Q_F)$ up to $k = 3$;

$$\begin{aligned}
f_0^{(1)} &= \frac{2}{(1-Q)^2} \\
&= 2 + 4Q + 6Q^2 + 8Q^3 + 10Q^4 + 12Q^5 \\
&\quad + 14Q^6 + 16Q^7 + 18Q^8 + \cdots \\
f_0^{(2)} &= \frac{2Q(3Q^2 + 4Q + 3)}{(1-Q)^6(Q+1)^2} \\
&= 6Q + 32Q^2 + 110Q^3 + 288Q^4 + 644Q^5 \\
&\quad + 1280Q^6 + 2340Q^7 + 4000Q^8 + \cdots \\
f_0^{(3)} &= \frac{2Q(4Q^6 + 23Q^5 + 50Q^4 + 62Q^3 + 50Q^2 + 23Q + 4)}{(1-Q)^{10}(Q^2 + Q + 1)^2} \\
&= 8Q + 110Q^2 + 756Q^3 + 3556Q^4 + 13072Q^5 \\
&\quad + 40338Q^6 + 109120Q^7 + 266266Q^8 + \cdots
\end{aligned}$$

The coefficients of the Taylor expansion in $Q_F = e^{-t_F} << 1$ (large volume region) are the G-V (or G-W) invariants.

On the otherhand, one can obtain the SW prepotential of 4D $SU(2)$ pure Yang-Mills theory by the following scaling limit;

$$Q_B = (\epsilon\Lambda)^4 \ , \quad Q_F = e^{-4\epsilon a} \ , \quad q = e^{-2\epsilon g_s}$$

with $\epsilon \longrightarrow 0$. In this limit the fiber $\mathbf{P}^1$ is collapsing, $t_F \longrightarrow 0, (1 - Q_F) \sim t_F$ and the instanton sum can be approximated by an integral (Laplace transform)

$$\sum_n N_{g,\beta} \cdot e^{-nt} \sim \int dn \ N_{g,\beta} \cdot e^{-nt}$$

Thus, it is the asymptotic growth of the Gromov-Witten invariants $N_{g,\beta}$ ($\beta = kB + nF$) as $n \to \infty$, which is relevant for the computation of $\mathcal{F}_{SW}$.

In general $f_g^{(k)}(Q_F)$ has the following structure;

$$f_g^{(k)}(Q_F) = \frac{P_g^{(k)}(Q_F)}{(1 - Q_F)^{2g+4k-2}}\,,$$

where $P_g^{(k)}(Q_F)$ is regular at $Q_F = 1$ and the asymptotic growth is governed by $P_g^{(k)}(1)$.

The terms in the topological string amplitude that survive in this limit are

$$a^2\left(\frac{\Lambda}{a}\right)^{4k}\sum_{g=0}^{\infty} g_s^{2g-2}\frac{P_g^{(k)}(1)}{2^{2g-2+8k}a^{2g}} = g_s^{-2}a^2\left(\frac{\Lambda}{a}\right)^{4k}\frac{P_0^{(k)}(1)}{2^{8k-2}}+\cdots\cdots$$

Up to sign flip at odd instanton numbers of $\mathbf{F}_1$, we obtain

an universal results independent of $n = 0, 1, 2$;

$$P_0^{(1)}(1) = 2, \ \ P_0^{(2)}(1) = 5, \ \ P_0^{(3)}(1) = 48, \cdots\cdots$$

which give the coefficients of SW prepotential

$$\mathcal{F}_1 = \frac{1}{2^5}, \quad \mathcal{F}_2 = \frac{5}{2^{14}}, \quad \mathcal{F}_3 = \frac{3}{2^{18}}, \cdots\cdots$$

More generally

Topological string $\Longrightarrow$ Nekrasov's partition function

A.Iqbal and A.-K. Kashani-Poor : `hep-th/0212279, hep-th/0306032`

T. Eguchi and H.K. : `hep-th/0310235`

Nekrasov's partition function $\Longrightarrow$ Seiberg-Witten prepotential

proved independently by three *different* approachs.

Nakajima-Yoshioka : `math.AG/0306198`

Okounkov-Nekrasov : `hep-th/0306238`

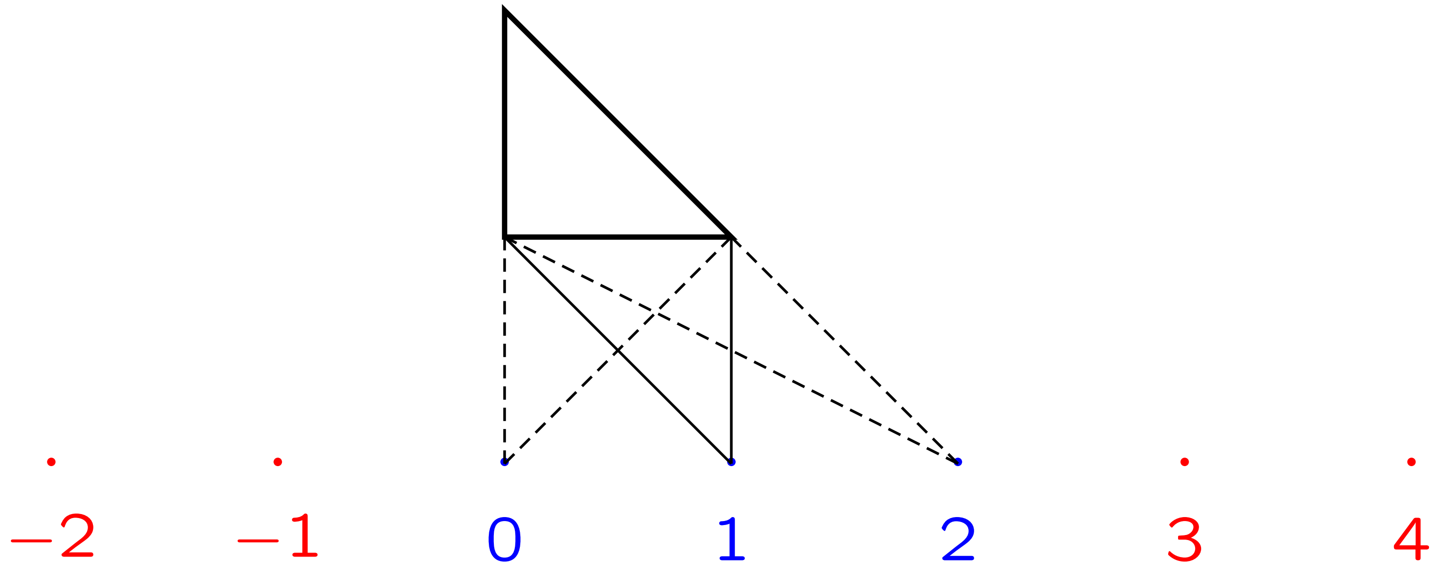
Braverman(-Etingof) : `math.AG/0401409, 0409441`

3. Comments on Non-Nef cases

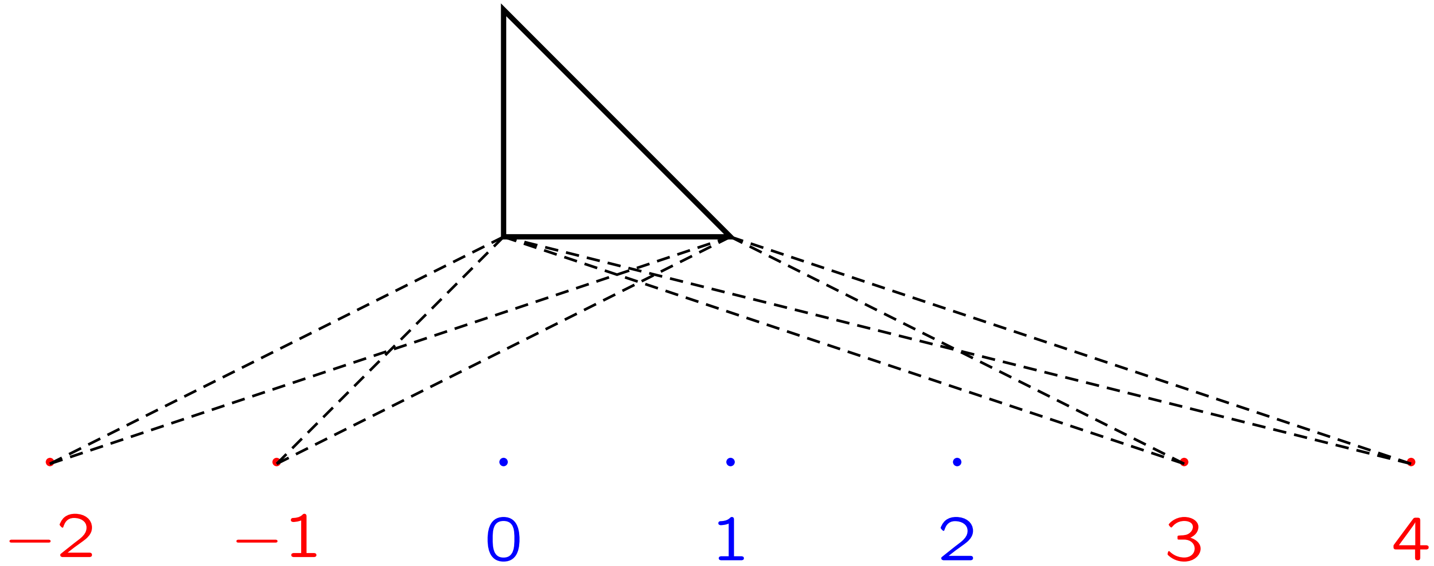
Among rational ruled surfaces (Hirzebruch surfaces) $\mathbf{F}_n$, only $\mathbf{F}_0, \mathbf{F}_1$ and $\mathbf{F}_2$ are nef. That is, for any irreducible curve C, we have $(-K_{\mathbf{F}_n}) \cdot C \geq 0$. By the adjunction formula $C \cdot C + K_S \cdot C = 2g - 2$, we see that $\mathbf{F}_n, (n \neq 0, 1, 2)$ is not nef, since the self-intersection of the base class B is $(-n)$. For non-nef cases, the toric diagram becomes concave and the dual diagram has external lines crossing each other.

For example, let us look at a simpler case of local rational curve $X_p : \mathcal{O}(-p) \oplus \mathcal{O}(p-2) \to \mathbf{P}^1$

Nef cases ($p = 0, 1, 2$)



Non-nef cases ($p \neq 0, 1, 2$)



Although, the toric diagram becomes ”ugly”, we can formally ”extrapolate” computations in terms of topological vertex.

Recently, based on the method of J function of Coates-Givental, a (B-model) computation of equivariant local Gromov-Witten invariants for non-nef local rational curve and local Hirzebruch surface is performed; [Forbes-Jinzenji math.AG/0603728]

They claim that the prepotentials of X_p for $2 < p$ are the same as that of $p = 0, 1, 2$ and for local Hirzebruch case, for example, the Gromov-Witten invariants of $\mathbf{F}_1$ and $\mathbf{F}_3$ are the same up to an appropriate shift of degree.

However, the computation by topological vertex shows rather different feature. We obtain the following partition function of topological string on $X_p : \mathcal{O}(-p) \oplus \mathcal{O}(p-2) \to \mathbf{P}^1$;

$$Z_{top\ str}^{(X_p)} = \sum_{\mu} (\mathrm{dim}_q R(\mu))^2 \, q^{\frac{(p-2)\kappa_\mu}{2}} e^{-t \cdot \ell_\mu}$$

[p dependence only appears in the framing factor $q^{\frac{(p-2)\kappa_\mu}{2}}$]

The quantum dimension $\mathrm{dim}_q R(\mu)$ is given by a specialization of the corresponding Schur function (the character);

$$\mathrm{dim}_q R(\mu) = s_\mu(q^\rho), \qquad (q^\rho : x_i = q^{i-\frac{1}{2}})$$

When $p = 0, 1, 2$ the summation over μ can be made in a closed form by using the Cauchy formula for the Schur functions

$$\sum_{\mu} s_{\mu}(x) s_{\mu}(y) = \prod_{1 \leq i,j} \left(1 - x_i y_j\right)$$

We obtain

$$Z_{top\ str}^{(X_p)} = \prod_{n=1}^{\infty} \left(1 - e^{-t} \cdot q^n\right)^{(-1)^{p-1} \cdot n}$$

and

$$\log Z_{top\ str}^{(X_p)} = \sum_{g=0}^{\infty} g_s^{2g-2} F_g(t) = \sum_{k=1}^{\infty} \frac{(-1)^p \cdot e^{-t \cdot k}}{k \sinh^2\left(\frac{k g_s}{2}\right)}$$

which leads the famous trilogarithm (multi-cover structure) at genus zero;

$$F_0(t) = (-1)^p \cdot \mathrm{Li}_3(e^{-t}) = (-1)^p \sum_{k=1}^{\infty} \frac{e^{-t\cdot k}}{k^3}$$

The claim in [Forbes-Jinzenji math.AG/0603728] is that this is valid for all p.

However, for $p \neq 0, 1, 2$, due to the appearance of the framing factor in the topological vertex formalism, we find different results in the instanton expansion;

$$Z_{top\ str}^{(X_p)} = 1 + \sum_{k=1}^{\infty} Z_k e^{-tk}$$

Gopakumar-Vafa invariants n_g^k of X_p;

$$
\begin{aligned}
n_0^1 &= (-1)^p \\
n_0^2 &= \frac{1}{4}p\,(p-2) + \frac{1}{8}\,(1-(-1)^p) \\
n_0^3 &= \frac{(-1)^p}{6}p\,(p-1)^2\,(p-2) \\
n_0^4 &= \frac{1}{12}p\,(p-1)^2\,(p-2)\left(2p^2-4p+1\right) \\
n_0^5 &= \frac{(-1)^p}{24}p\,(p-1)^2\,(p-2)\left(5p^4-20p^3+25p^2-10p+2\right) \\
\vdots & \quad \vdots
\end{aligned}
$$

In general $n_{g=0}^k$ is a polynomial in p of order $2k-2$.

Quite recently, the genue zero Gromov-Witten invariants of X_p is estimated, based on the analysis of (a one cut solution to) a corresponding matrix model; [N. Caporaso, L. Griguolo, M. Mariño, S. Pasquetti and D. Seminara, hep-th/0606120]

$$N_{0,k} = \frac{1}{k^2 k!} \frac{\big((p-1)^2 k - 1\big)!}{(p(p-2)k)!}$$

Using the Stiring's formula $n! \sim \sqrt{2\pi n} n^n e^{-n}$, one can obtain the following asmptotic growth for $p \neq 0, 1, 2$;

$$N_{0,k} \sim k^{-\frac{7}{2}} e^{k \cdot t_c}\,, \quad t_c := \log(p(p-2))^{p(p-2)}(p-1)^{2(p-1)^2}$$

(should be compared with $\quad N_{0,k} \sim k^{-3}\,, \quad t_c := 0$)

Physical implication of the difference of the asymptotic behavior is that in the corresponding matrix model, which is also related to two-dimensional (q-deformed) Yang-Mills theory on $\mathbf{P}^1$, a phase transition takes place at $t = t_c$ for $2 < p$.

How does this difference come from?

According to [Bryan and Pandharipande math.AG/0411037], in the local Gromov-Witten theory of curves, we can introduce equivariant parameters (λ_1, λ_2) for toric action on the rank two fiber of X_p.

If we take $\lambda_1 = \lambda_2$, then we have $F_0(t) = (-1)^p \cdot \mathrm{Li}_3(e^{-t})$, independent of p. But if we take "anti-diagonal" choice $\lambda_1 + \lambda_2 = 0$, then the result depends on p and there is a clear difference between $p = 0, 1, 2$ and other cases.

How about the case of local Hirzebruch surface $\mathbf{F}_n$, which was of our original interest ?

$$
\begin{aligned}
Z^{(\mathbf{F}_n)}_{top\ str} = & \sum_{\mu_1\mu_2} (K_{\mu_1\mu_2}(Q_F))^2 \cdot Q_B^{\ell_{\mu_1}+\ell_{\mu_2}} Q_F^{n\ell_{\mu_2}} \\
& \cdot (-1)^{n(\ell_{\mu_2}-\ell_{\mu_1})} q^{\frac{n}{2}(\kappa_{\mu_2}-\kappa_{\mu_1})}
\end{aligned}
$$

where $K_{\mu_1\mu_2}(Q) := \sum_\nu Q^{\ell_\nu} W_{\mu_1\nu}(q) W_{\nu\mu_2}(q)$ can be computed in a closed form by using the Schur function identities and $Q_B := e^{-t_B}, Q_F := e^{-t_F}$.

We find $f_g^{(k)}(\mathbf{F}_2) = Q_F^k \cdot f_g^{(k)}(\mathbf{F}_0)$ (independent of g), which means a simple relation $N^{\mathbf{F}_2}_{g,kB+nF} = N^{\mathbf{F}_0}_{g,kB+(n+k)F}$ between the Gromov-Witten invariants of $\mathbf{F}_0$ and $\mathbf{F}_2$.

The computation by [Forbes-Jinzenji] claims a similar relation between $\mathbf{F}_1$ and $\mathbf{F}_3$; $N^{\mathbf{F}_3}_{0,kB+nF} = N^{\mathbf{F}_1}_{0,kB+(n+k)F}$.

However, again, the topological vertex computation shows rather different results; For example,

$$
\begin{aligned}
f_0^{(2)}(\mathbf{F}_1) &= \frac{2Q^2(3Q^2+4Q+3)}{(1-Q)^6(1+Q)^2} , \\
f_0^{(2)}(\mathbf{F}_3) &= Q^{10} - 2Q^9 - Q^8 + 4Q^7 + 6Q^6 + 4Q^5 \\
&\quad + 6Q^4 + 4Q^3 - Q^2 - 2Q + 1/\,(1-Q)^6\,(1+Q)^2
\end{aligned}
$$

It is an open problem to understand the discrepancy of the Gromov-Witten invariants from the viewpoint of the equivariant Gromov-Witten theory, like the case of local curves.